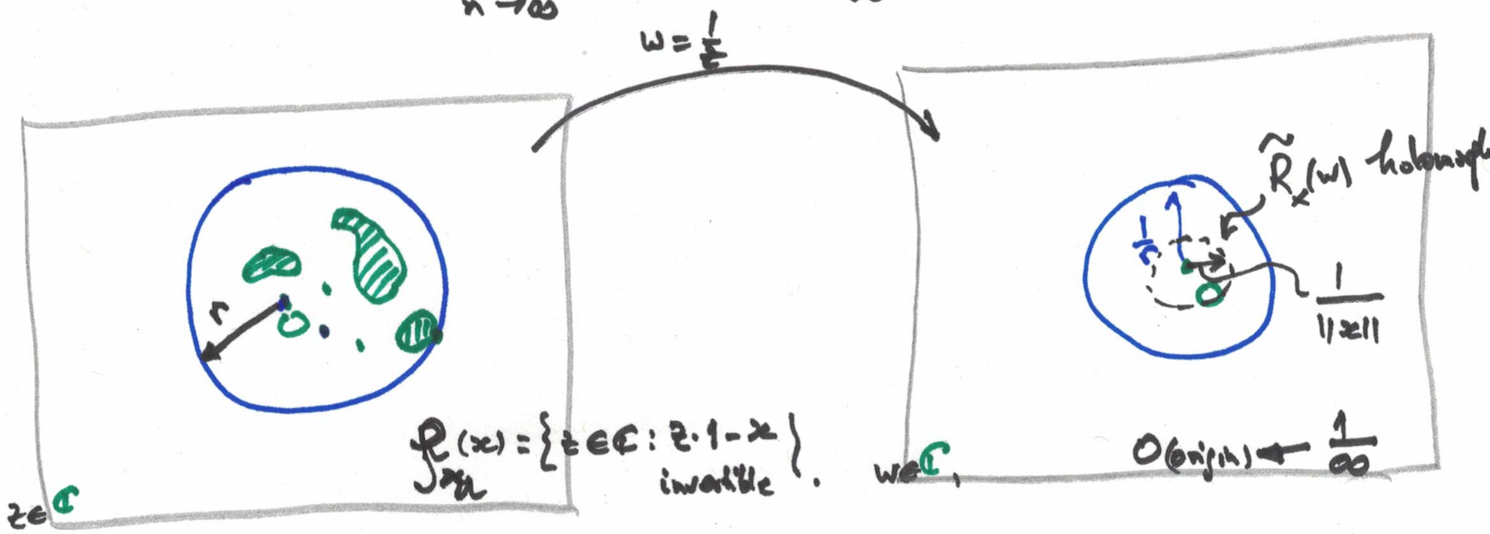
Gelfand Formula — Proof part 2:

Remain to prove: $\lim\limits_{n\to\infty} \|x^n\|^{1/n} \le r_{\mathcal{U}}(x)$.



$\sigma_{\mathcal{U}}(x) \longrightarrow U_r = \{z \in \mathbb{C} : |z| \le r\} \supseteq \sigma_{\mathcal{U}}(x)$. But. $\forall \varepsilon > 0$.

$U_{r-\varepsilon} \not\supseteq \sigma_{\mathcal{U}}(x)$

$z \longmapsto w = \frac{1}{z}$.

$z \in \mathbb{C} \dashrightarrow R_x(z) = (z\cdot 1 - x)^{-1} = \frac{1}{z}\left(1 - \frac{1}{z}x\right)^{-1} \rightarrow \tilde{R}_x(w) = w\cdot(1 - w\cdot x)^{-1}$

holomorphic for $|z| > r$ $\qquad$ holomorphic for $|w| < \frac{1}{r}$

$\Updownarrow$: $\tilde{R}_x(0) = 0$.

Radius of convergence:

$\tilde{R}_x(w) = \sum\limits_{n\ge 0} a_n \cdot w^n$. $\longrightarrow$ $\dfrac{1}{\text{Radius of Convergence}} = \overline{\lim\limits_{n\to\infty}} \|a_n\|^{1/n}$.

$a_n \in \mathcal{U}$, $w \in \mathbb{C}$
$|w| <$ Radius of Convergence.

(Hadamard Formula.
Raabe / n^{th} Root)
n^{th} root test.

For $w \in \mathbb{C}$, $|w| < \dfrac{1}{\|x\|}$

$$\tilde{R}_x(w) = w \cdot (1 - w \cdot x)^{-1} = w \cdot \sum_{n=0}^{\infty} (wx)^n =$$

$$= \sum_{n=0}^{\infty} x^{n} \cdot w^{n+1} = \sum_{n=1}^{\infty} x^{n-1} \cdot w^n$$

$$\Rightarrow a_n = x^{n-1}$$

$\longrightarrow$ By n^{th} root test $\Rightarrow$ Power Series Converges for $w \in \mathbb{C}$,

$$|w| < \text{Radius of Convergence} = \frac{1}{\lim\limits_{n\to\infty} \|x^n\|^{1/n}}.$$

Note: $\lim\limits_{n\to\infty} \|x^{n-1}\|^{1/n} = \lim\limits_{n\to\infty} \left(\|x^{n-1}\|^{\frac{1}{n-1}}\right)^{\frac{n-1}{n}} = \lim\limits_{n\to\infty} \|x^n\|^{1/n}$

$z = \dfrac{1}{w} \Rightarrow$ ~~[illegible]~~ $R_x(z)$ is holomorphic

for any $z \in \mathbb{C}$, $|z| > \lim\limits_{n\to\infty} \|x^n\|^{1/n}$

and $|z| = \lim\limits_{n\to\infty} \|x^n\|^{1/n}$ is

on the boundary of holomorphy.

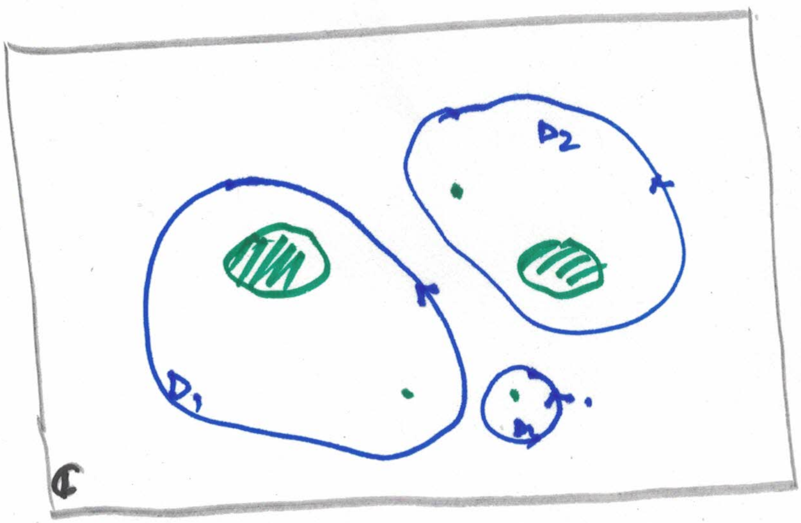
Holomorphic Calculus in Banach Algebras

Let $\mathfrak{U}$ be a Banach algebra with a unit.

Let $x \in \mathfrak{U}$ and let $\sigma_{\mathfrak{U}}(x)$ denote its spectrum and $\rho_{\mathfrak{U}}(x)$ denote its resolvent set.

Let $D \subset \mathbb{C}$ be an open set s.t. $\sigma_{\mathfrak{U}}(x) \subset D$.

<u>Assume</u> ∂D (boundary of D) is "Jordan rectifiable". (sufficiently "nice").



$\sigma_{\mathfrak{U}}(x)$.

$D \dashrightarrow \partial D = \partial D_1 \cup \partial D_2 \cup \partial D_3$

$\; \| \;$

$D_1 \cup D_2 \cup D_3$.

<u>Definition</u> Let $f : D \to \mathbb{C}$ be a holomorphic function. We shall extend the definition of f from D to $\mathfrak{U}$:

$$f(x) := \frac{1}{2\pi i} \oint_{\partial D} f(z) \cdot R_x(z)\, dz = \frac{1}{2\pi i} \oint_{\partial D} (z \cdot 1 - x)^{-1} \cdot f(z)\, dz.$$

$x \in \mathfrak{U}$.

<u>Claim:</u> ($f(x) \in \mathfrak{U}$) This is a well-defined object.

<u>Proposition 1</u>. $\mathcal{U}$ is a Banach algebra with identity.

$x \in \mathcal{U}$. Let P be a polynomial, $P(t) = a_n t^n + a_{n-1} t^{n-1} + \dots + a_0$.

Then $\sigma(P(x)) = P(\sigma(x))$,

where: $P(\sigma(x)) = \{ P(\lambda), \lambda \in \sigma(x) \}$, $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 \cdot 1$

<u>Proof</u>.

Let $P(t) = a_n \cdot (t - b_1) \cdot (t - b_2) \cdot \dots \cdot (t - b_n)$, $b_1 \dots b_n \in \mathbb{C}$.

Let $y = P(x) = a_n x^n + \dots + a_0 \cdot 1 = a_n \cdot (x - b_1 \cdot 1) \cdot \dots \cdot (x - b_n \cdot 1)$

1) Let $\lambda \in \sigma(x)$: $x - \lambda \cdot 1$ is not invertible.

$$y - P(\lambda) \cdot 1 = a_n \cdot x^n + \dots + a_1 \cdot x + \cancel{a_0 \cdot 1} - a_n \cdot \lambda^n \cdot 1 - \dots - a_1 \cdot \lambda \cdot 1 - \cancel{a_0}$$

$$= a_n \underbrace{(x^n - \lambda^n \cdot 1)}_{(x - \lambda \cdot 1)(x^{n-1} + x^{n-2} \cdot \lambda + \dots + \lambda^{n-1} \cdot 1) = (x^{n-1} + \dots + \lambda^{n-1} \cdot 1) \cdot (x - \lambda \cdot 1)} + \dots + a_1 \cdot (x - \lambda \cdot 1) = (x - \lambda \cdot 1) \cdot (a_n(\dots) + a_{n-1}(\dots) + \dots)$$

$$= (x - \lambda \cdot 1) \cdot Q(x) = Q(x) \cdot (x - \lambda \cdot 1).$$

<u>Claim</u>: $y - P(\lambda) \cdot 1$ is not invertible.

Assume it is invertible. Let $u = (y - P(\lambda) \cdot 1)^{-1}$:

$$(x - \lambda \cdot 1) \underbrace{Q(x) \cdot u}_{v} = u \cdot (x - \lambda \cdot 1) \, Q(x) = 1.$$

$(x - \lambda \cdot 1) v = 1$ $\quad \overbrace{(u \cdot Q(x))}^{w} \cdot (x - \lambda \cdot 1) = 1$ $\Rightarrow$ $(x - \lambda \cdot 1) v = 1$ $\Rightarrow x - \lambda \cdot 1$ inve

Contradiction with assumption. $x - \lambda \cdot 1$ is not invertible.

$P(\lambda) \in \sigma(y)$: $\quad P(\sigma(x)) \subset \sigma(P(x))$.

ii). let $\mu \in \sigma(\underbrace{P(x)}_{y})$: $\quad y - \mu \cdot 1$ is not invertible.

$(a_n \neq 0)$

let $\underbrace{R(t)}_{\text{polynomial.}} = P(t) - \mu \cdot 1 = a_n \cdot (t - c_1) \cdot (t - c_2) \cdot \ldots \cdot (t - c_n)$

$\{c_1, c_2, \ldots c_n\}$: zeros of $R(t)$.

In particular: $\mu = P(c_1) = P(c_2) = \ldots = P(c_n)$.

We assumed: $R(x)$ is not invertible.

$\quad \parallel$

$a_n (x - c_1 \cdot 1)(x - c_2 1) \cdot \ldots \cdot (x - c_n 1)$ not invertible.

$\Leftrightarrow (x - c_1 1) \cdot \ldots \cdot (x - c_n 1)$ is not invertible.

Claim: At least one k, $x - c_k \cdot 1$ is not invertible.

Why: otherwise, all $(x - c_k)$ invertible $\Rightarrow \prod_{k=1}^{n} (x - c_k)$ invertible.

$\Rightarrow R(x)$ invertible.

Hence. $\exists\, k \in [n]$: $x - c_k \cdot 1$ not invertible $\Rightarrow c_k \in \sigma(x)$.

And $\mu = P(c_k)$. $\rightarrow \mu \in P(\sigma(x))$.

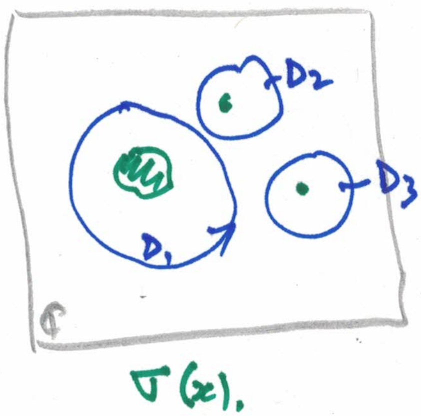
Thus: $\sigma(P(x)) \subset P(\sigma(x))$

$\blacksquare$

Construction of. $\frac{1}{2\pi i} \oint_\Gamma R_x(z) \cdot f(z)\, dz$.

Take $\ell \in \mathcal{U}'$, a bounded linear functional.

Let. $F(\ell) = \frac{1}{2\pi i} \oint_\Gamma \ell(R_x(z))\, f(z)\, dz. = \frac{1}{2\pi i} \oint_\Gamma \ell\left((z \cdot 1 - x)^{-1}\right) f(z)\, dz.$



$D = \bigsqcup_k D_k$

$\partial D = \bigcup_k \partial D_k =: \Gamma$

$\dashrightarrow \exists\, r_0 > 0.$

$\forall z \in \Gamma,\ \forall \lambda \in \sigma(x).$

$|z - \lambda| \geq r_0.$

$\Downarrow$

$\|R_x(z)\| \leq \frac{1}{r_0}.$

$F(\ell) \in \mathbb{C}$, well defined.

1) $F(a_1 \ell_1 + a_2 \ell_2) = a_1 F(\ell_1) + a_2 F(\ell_2)$.

2). $|F(\ell)| \leq \frac{1}{2\pi} \int_\Gamma |\ell(R_x(z))|\, |f(z)|\, dz \leq \frac{\|\ell\|}{2\pi r_0} \|f\|_\infty \cdot \mathrm{length}(\Gamma)$

$\Longrightarrow$. F is a bounded. linear functional over $\mathcal{U}'$.

Technical. Assumption : $\mathcal{U}'' = \mathcal{U}$ ($\mathcal{U}$ is a reflexive space).

$\Rightarrow F \in \mathcal{U}$.

[With a bit of care ---> You can show $F \in \mathcal{U}$].