L21

Holomorphic Calculus (2)

Recall: $\mathcal{U}$ is a Banach algebra with identity.

$x \in \mathcal{U}$, $D \subset \mathbb{C}$ open set, s.t. $\sigma_{\mathcal{U}}(x) \subset D$.

Let $f : D \to \mathbb{C}$ be a holomorphic function.

Claim:

$$\underbrace{\frac{1}{2\pi i} \oint_{\Gamma = \partial D} (z \cdot 1 - x)^{-1} f(z) dz}\ \in \mathcal{U}$$

Why: should be understood (constructed) as a Riemann integral.

$\Gamma = \partial D \longrightarrow$ construct a partition: $\Pi = \{ z_k, z \in [N] \} \subset \partial D$.

$$\frac{1}{2\pi i} \oint_{\Gamma} (z I - x)^{-1} f(z) dz. = \frac{1}{2\pi i} \lim_{|\Pi| \downarrow 0} \sum_{k=1}^{N} (z_k \cdot 1 - x)^{-1} f(z_k) \cdot \text{length}(z_k).$$

Theorem. Let $\mathcal{U}$ be a Banach algebra with identity and $x \in \mathcal{U}$. Let $\mathcal{F}(x)$ denote the family of all functions f analytic in an open neighborhood N_f of $\sigma(x) = \sigma_{\mathcal{U}}(x)$.

(a) For $f \in \mathcal{F}(x)$, let Γ_1 and Γ_2 be two chains in $\mathbb{C}$ that satisfy the following:

$$\Gamma_1 = \sum_{\gamma \in S_1} n(\gamma) \cdot \gamma \quad , \quad \Gamma_2 = \sum_{\gamma \in S_2} n(\gamma) \cdot \gamma$$

where $\gamma : [0,1] \to \mathbb{C}$, $\gamma(0) = \gamma(1)$, $n(\gamma) \in \mathbb{Z}$ such that:

(i) $\forall z \in \mathbb{C} \setminus N_f$, $\quad \mathrm{Ind}(\Gamma_1, z) = 0 = \mathrm{Ind}(\Gamma_2, z)$.

(ii). $\forall z \in \sigma(x)$, $\quad \mathrm{Ind}(\Gamma_1, z) = 1 = \mathrm{Ind}(\Gamma_2, z)$.

(iii) $\forall z \in N_f \setminus (\sigma(x) \cup \mathrm{Ran}(\Gamma_1))$, $\mathrm{Ind}(\Gamma_1, z) \in \{0, 1\}$.

$\forall z \in N_f \setminus (\sigma(x) \cup \mathrm{Ran}(\Gamma_2))$, $\mathrm{Ind}(\Gamma_2, z) \in \{0, 1\}$.

Then:
$$\frac{1}{2\pi i} \oint_{\Gamma_1} (z \cdot 1 - x)^{-1} f(z)\, dz = \frac{1}{2\pi i} \oint_{\Gamma_2} (z \cdot 1 - x)^{-1} f(z)\, dz.$$

(b). Assume f_1, f_2 are two holomorphic functions on open neighborhoods N_{f_1}, N_{f_2} of the spectrum $\sigma(x)$. Assume Γ ~~and Γ_x~~ is a chain that satisfies conditions (i), (ii), (iii) at part a. ~~Then~~ with $\mathrm{Ran}(\Gamma) \subset N_{f_1} \cap N_{f_2}$. Assume further that $f_1(z) = f_2(z)$, for all $z \in N_{f_1} \cap N_{f_2}$. Then:

$$\frac{1}{2\pi i} \oint_\Gamma (z \cdot 1 - x)^{-1} f_1(z)\, dz = \frac{1}{2\pi i} \oint_\Gamma (z \cdot 1 - x)^{-1} f_2(z)\, dz.$$

(c). Assume. $f, g \in \mathcal{F}(x)$. Let $f \cdot g \in \mathcal{F}(x)$ defined by. $(f \cdot g)(z) = f(z) \cdot g(z)$, on. $N_{f \cdot g} = N_f \cap N_g$. Then:

$$\frac{1}{2\pi i} \oint_\Gamma (z \cdot 1 - x)^{-1} (f \cdot g)(z)\, dz = \left(\frac{1}{2\pi i} \oint_\Gamma (z \cdot 1 - x)^{-1} f(z)\, dz\right) \cdot \left(\frac{1}{2\pi i} \oint_\Gamma (z \cdot 1 - x)^{-1} g(z)\, dz\right)$$

In other words:

$$(f \cdot g)(x) = f(x). \cdot g(z)$$

where Γ is a chain, $\mathrm{Ran}\, \Gamma \subset N_f \cap N_g$, that satisfies (i), (ii), (iii) at part a (d). If $(f_n)_{n\geq 1}$, $f_\infty \in \mathcal{F}(x)$ with $U \subset N_{f_n}$, $\forall n \geq 1$, $U \subset N_{f_\infty}$ and $f_n \to f_\infty$ uniformly on compact subsets of the open set U with $\sigma(x) \subset U$, then:

$$\lim_{n\to\infty} \| f_n(x) - f_\infty(x) \| = 0 .$$

(e). If $f(z) = \sum_{n \geq 0} a_n z^n$, $a_n \in \mathbb{C}$, has radius of convergence $r > r_{\mathcal{U}}(x)$, then:

$$f(x) = \sum_{n \geq 0} a_n x^n , \quad f(x) \in \mathcal{U} .$$

(f) If $\lambda \in \rho(x)$ and $f(z) = \dfrac{1}{z-\lambda}$, then

$$f(x) = (x - \lambda \cdot 1)^{-1} = -R_x(\lambda)$$

(g) If $G : \mathcal{F}(x) \to \mathcal{U}$ is an algebra homomorphism, i.e. $G(a\cdot f + b \cdot g) = a\, G(f) + b \cdot G(g)$; $G(f \cdot g) = G(f) \cdot G(g)$, and: $G(p_1) = x$, where $p_1(z) = z$, and obeys (d):

$f_n \to f$ uniformly on compact subsets of $U \overset{\text{open}}{\subset} \bigcap_{n\geq 1} N_{f_n} \cap N_{f_\infty}$

then $\|G(f_n) - G(f)\| \to 0$

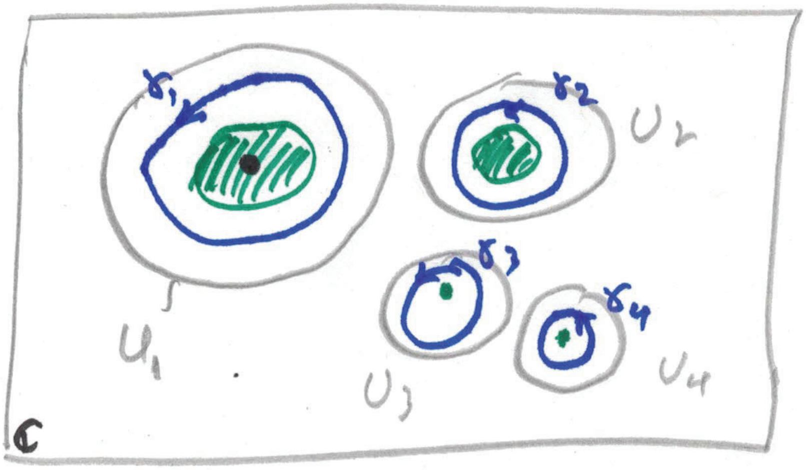
Then:

$$G(f) = f(x).$$

i.e.

$$G(f) = \frac{1}{2\pi i} \oint_{\Gamma} (z\cdot 1 - x)^{-1} f(z)\,dz.$$

Remarks:



non-empty compact (spectrum)

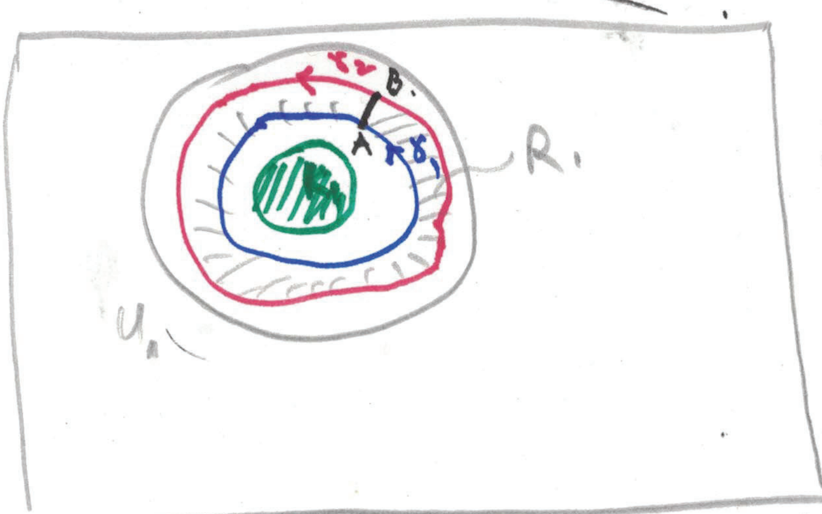
$$\sigma(x) = \underbrace{K_1 \cup K_2 \cup K_3 \cup K_4}_{\text{example}}.$$

$$N_f = \underbrace{U_1 \cup U_2 \cup U_3 \cup U_4}_{\text{example}}.$$

open set, where $f : N_f \to \mathbb{C}$ is holomorphic.

$$\Gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4.$$

Proof: Part (a):



Want:

$$\oint_{\gamma_1} (z\cdot 1 - x)^{-1} f(z)\,dz = \oint_{\gamma_2} (z\cdot 1 - x)^{-1} f(z)\,dz$$

Create: $\gamma = (-\gamma_1) \cup [AB] \cup \gamma_2 \cup [BA] = \partial R$ → open connected $\subset \mathbb{C}$.

$f\big|_R : R \to \mathbb{C}$ is holomorphic., R open connected., $\ell \in \mathcal{U}'$

$$\ell\left(\oint_{\partial R} (z\cdot 1 - x)^{-1} f(z)\, dz\right) = \oint_{\partial R} \underbrace{\ell\left(R_x(z)\right)}_{\text{holomorphic on } R} f(z)\, dz = 0.$$

$R \subset \rho(x)$ (resolvent set).

$$\forall \ell \in \mathcal{U}' \Rightarrow \underbrace{\frac{1}{2\pi i} \oint_{\partial R} (z\cdot 1 - x)^{-1} f(z)\, dz = 0.}$$

$$\underbrace{-\frac{1}{2\pi i} \oint_{\gamma_1} (z\cdot 1 - x)^{-1} f(z)\, dz} + \cancel{\int_{[AB]} \ldots} + \underbrace{\frac{1}{2\pi i} \int_{\gamma_2} (z\cdot 1 - x)^{-1} f(z)\, dz} + \cancel{\int_{[BA]} \ldots} = 0$$

$$\Rightarrow \frac{1}{2\pi i} \oint_{\gamma_1} (z\cdot 1 - x)^{-1} f(z)\, dz = \frac{1}{2\pi i} \oint_{\gamma_2} (z\cdot 1 - x)^{-1} f(z)\, dz.$$

Part (b) $\to$ similar.

(c)

Take f, g as in the theorem.

$$f(x) = \frac{1}{2\pi i} \oint_{\Gamma} (z \cdot 1 - x)^{-1} f(z)\, dz, \qquad g(x) = \frac{1}{2\pi i} \oint_{\tilde{\Gamma}} (w \cdot 1 - x)^{-1} g(w)\, dw$$

$$f(x) \cdot g(x) = \frac{1}{(2\pi i)^2} \oint_{z \in \Gamma} \oint_{w \in \tilde{\Gamma}} f(z)\, g(w) \underbrace{(z \cdot 1 - x)^{-1}}_{R_x(z)} \cdot \underbrace{(w \cdot 1 - x)^{-1}}_{R_x(w)} dz\, dw. =$$

$$(w \cdot 1 - x)^{-1} - (z \cdot 1 - x)^{-1} = (z \cdot 1 - x)^{-1} \left[(z \cdot 1 - x) - (w \cdot 1 - x) \right] (w \cdot 1 - x)^{-1} =$$

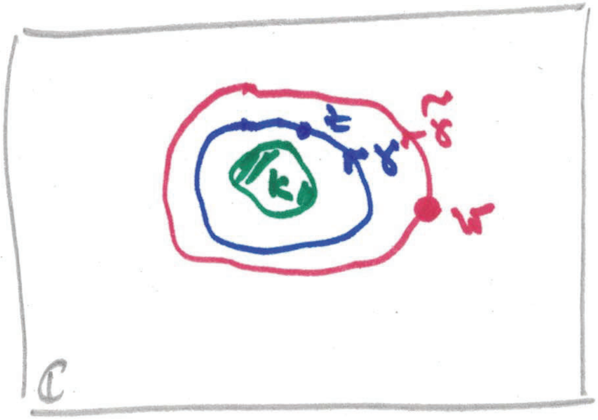
$$= (z - w) \cdot (z \cdot 1 - x)^{-1} \cdot (w \cdot 1 - x)^{-1}$$

$$= \frac{1}{(2\pi i)^2} \iint_{\tilde{\Gamma} \times \Gamma} \frac{f(z) \cdot g(w)}{z - w} (w \cdot 1 - x)^{-1}\, dz\, dw - \frac{1}{(2\pi i)^2} \iint_{\tilde{\Gamma} \times \Gamma} \frac{f(z)\, g(w)}{z - w} (z \cdot 1 - x)^{-1} dz\, dw$$

$$= \frac{1}{2\pi i} \oint_{\tilde{\Gamma}} \left(\frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{z - w}\, dz \right) g(w) \cdot (w \cdot 1 - x)^{-1}\, dw \; -$$

$$- \frac{1}{2\pi i} \oint_{\Gamma} \left(\frac{1}{2\pi i} \oint_{\tilde{\Gamma}} \frac{g(w)}{z - w}\, dw \right) f(z) \cdot (z \cdot 1 - x)^{-1}\, dz$$

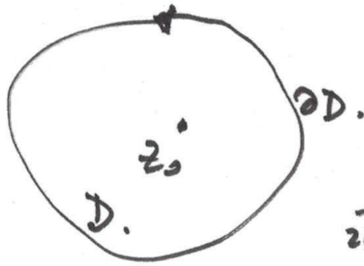
How to choose Γ and $\tilde{\Gamma}$:



$\Gamma \subset \operatorname{int}(\tilde{\Gamma})$:

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{f(z)}{z-w}\,dz = 0.$$

$$\frac{1}{2\pi i}\int_{\tilde{\Gamma}} \frac{g(w)}{z-w}\,dw = -g(z).$$



$$\frac{1}{2\pi i}\int_{\partial D} \frac{\varphi(z)}{z-z_0}\,dz = \varphi(z_0).$$

$$\Rightarrow f(x)\cdot g(x) = -(-1)\frac{1}{2\pi i}\oint_{\Gamma} g(z)f(z)(z\cdot 1-x)^{-1}\,dz = (f\cdot g)(x)$$

(d). $f_n \to f_\infty$ uniformly:

$$\|f_n(x) - f_\infty(x)\| = \left\| \frac{1}{2\pi i}\oint_{\Gamma}(z\cdot 1-x)^{-1} f_n(z)\,dz - \frac{1}{2\pi i}\oint_{\Gamma}(z\cdot 1-x)^{-1} f_\infty(z)\,dz \right\| \leq$$

$$\leq \frac{1}{2\pi}\oint_{\Gamma}\|(z\cdot 1-x)^{-1}\|\cdot \underbrace{|f_n(z)-f_\infty(z)|}_{\leq \|f_n-f\|_\infty}\,dz \leq \frac{\operatorname{length}(\Gamma)}{2\pi}\|f_n - f_\infty\|_\infty \cdot \sup_{z\in\Gamma}\|(z\cdot 1-x)^{-1}\|$$

$$\mathrm{dist}(\Gamma, \sigma(x)) = d_0 > 0. \Rightarrow \forall z \in \Gamma :$$

$$|z - \lambda| \geq d_0, \quad \forall \lambda \in \sigma(x).$$

$$\Rightarrow \underbrace{\|(z \cdot 1 - x)^{-1}\|}_{\text{uniformly bounded on } \Gamma} \cancel{\leq \frac{1}{d_0}}$$

See below :

$$\Rightarrow \lim_{n \to \infty} \|f_n(x) - f_\infty(x)\| = 0.$$

<u>Correction :</u>

Need : $\sup_{z \in \Gamma} \|(z \cdot 1 - x)^{-1}\| < \infty.$

For each $z \in \Gamma \subset \rho(x)$, $(z \cdot 1 - x)^{-1}$ is bounded,

and $d(\Gamma, \sigma(x)) > 0 \Rightarrow z \longmapsto (z \cdot 1 - x)^{-1}$ is <u>continuous</u> (even holomorphic) on Γ.

Hence $z \longmapsto \|(z \cdot 1 - x)^{-1}\|$ is continuous on the compact set Γ,

it follows: $\sup_{z \in \Gamma} \|(z \cdot 1 - x)^{-1}\| < \infty$