L23

Fredholm Alternative for Compact Operators

Theorem [Fredholm Alternative]. Assume H is a Hilbert space and $T: H \to H$ a compact operator. Then exactly one alternative holds true:

(1) $1-T$ is invertible.

or

(2) $\ker(1-T) \neq \{0\}$.

Remarks.

① $1-T$ is invertible $\Leftrightarrow$ $\forall y \in H \ \exists!\, x \in H$ s.t. $(1-T)x = y$

$\ker(1-T) \neq \{0\} \Leftrightarrow \exists f \neq 0, f \in H$ s.t. $(1-T)f = 0$. $\Leftrightarrow$

$\Leftrightarrow$ If $y, x \in H$ s.t. $(1-T)x = y$, then x is not a unique solution for $(1-T)x = y$.

An ~~alternative~~ equivalent way of stating the Fredholm alternative:

$\underbrace{\text{Uniqueness}}$ of solution of $(1-T)x = y \implies$ ~~Uniqueness~~ $\underbrace{\text{Existence}}$ of solution.

② Assume $A \in \mathbb{C}^{n\times n}$. square matrix.

The following are equivalent:

(i) A is invertible.

(ii). $\ker(A) = \{0\} \Leftrightarrow A: \mathbb{C}^n \to \mathbb{C}^n$ is injective.

(iii) $\mathrm{Ran}(A) = \mathbb{C}^n \Leftrightarrow A: \mathbb{C}^n \to \mathbb{C}^n$ is surjective

Examples: 1. Integral Equations.

Given $g: I \to \mathbb{R}$, $I \subset \mathbb{R}$. and $k: I \times I \to \mathbb{R}$.

Find $f: I \to \mathbb{R}$ s.t.

$$f(x) + \int_I k(x,y) \cdot f(y)\, dy = g(x), \quad \forall x \in I.$$

2. Riesz - Schauder Theorem:

If $T: H \to H$ is a compact operator, then $\sigma(T)$ is a discrete set having no limit points except perhaps $\lambda = 0$. Further, any nonzero $\lambda \in \sigma(T)$ is an eigenvalue of finite multiplicity.

Proof / Sketch [Riesz Schauder]:

If $z \cdot 1 - T$ invertible $\Leftrightarrow$ $z \in \rho(T)$.
$(z \in \mathbb{C})$.

Take $z \neq 0$, $z \cdot 1 - T$ invertible $\Leftrightarrow$ $1 - \frac{1}{z} \cdot T$ invertible.

$\underbrace{z \cdot 1 - T \text{ not invertible}}_{\Leftrightarrow\, z \in \sigma(T).}$ $\Leftrightarrow$ $\underbrace{1 - \underbrace{\tfrac{1}{z} \cdot T}_{\text{compact op.}} \text{ not invertible.}}$

By Fredholm: $\exists f \neq 0$ $f \in \ker\left(1 - \frac{1}{z}\right)T$

$\Rightarrow$ z is an eigenvalue.

$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda \text{ eigenvalue of } T\} \cup \{0\}$. possibly.

$\Leftrightarrow \exists f \neq 0,\ Tf = \lambda \cdot f$

$\Updownarrow$
$\dim H < \infty$ or $\dim H = +\infty$.

We have seen that $\dim \ker(T - \lambda \cdot 1) < \infty$

Proof of Fredholm Alternative: ← To be proved.

> **Proposition:** If T is a compact operator, then $1-T$ has closed range.

Assume T is compact. If $1-T$ is invertible then nothing left to prove.

Assume $1-T$ is <u>not</u> invertible.

Let $A: H \to H$ a finite rank operator s.t. $\| T - A \| < \frac{1}{2}$.

$\Downarrow$

$1-(T-A)$ invertible.

Let $S = A \cdot (1-T+A)^{-1}$

$\longrightarrow$ S is a finite rank operator.

$\left[(1-(T-A))^{-1} = \sum_{n \geq 0} (T-A)^n \right]$

Note:

$$(1-S) \cdot \underbrace{(1-T+A)}_{\text{invertible.}} = 1-T+A-A = 1-T$$

$\Rightarrow$ 1). $\operatorname{Ran}(1-T) = \operatorname{Ran}(1-S)$.

2). $\ker(1-T) \neq \{0\}$ iff. $\ker(1-S) \neq \{0\}$.

3). $(1-T)$ not invertible $\Longleftrightarrow$ $1-S$ not invertible.

Plan:

To be shown.

$1-T$ not invertible $\Rightarrow$ $1-S$ not invertible $\Rightarrow$ $\ker(1-S) \neq \{0\}$ $\Rightarrow$

$\Rightarrow \ker(1-T) \neq \{0\}$

Separately: $[1-S$ has closed range$]$.

Lemma. Assume $S : H \to H$ is a finite rank operator.

Then $1-S$ has closed range.

Proof.

S finite rank $\to$ $$S = \sum_{k=1}^{N} \langle \cdot, u_k \rangle v_k$$

s.t. $\{v_1 \dots v_N\}$ orthonormal set.

$\{u_1 \dots u_N\}$ orthogonal set

Let $E = \mathrm{span}\{u_1 \dots u_N, v_1 \dots v_N\}$, $\dim E \leq 2N < \infty$.

$$\begin{array}{l|l} S(E) \subset E & (1-S)(E) \subset E \\ S(E^\perp) = 0. & (1-S)(E^\perp) \subset E^\perp \end{array} \Rightarrow E, E^\perp \text{ are invariant subspaces of } 1-S$$

$$1-S = (1-S)\big|_E \oplus (1-S)\big|_{E^\perp} = (1-S)\big|_E \oplus 1\big|_{E^\perp}$$

$$\left(1-S \sim \begin{bmatrix} 1-S|_E & 0 \\ 0 & 1 \end{bmatrix} \right).$$

orthogonal decomp.

$$\mathrm{Ran}(1-S) = \left(\mathrm{Ran}\,(1-S)\big|_E\right) \oplus E^\perp.$$

But: $(1-S)\big|_E : E \to E$ is finite dim. linear problem.

$\mathrm{Ran}\,(1-S)\big|_E = (1-S)(E) \to$ closed subspace.

$\Rightarrow$ $\mathrm{Ran}\,(1-S)\big|_E \oplus E^\perp$ is closed. ☐

<u>Proof of Proposition</u>:

$$\mathrm{Ran}(1-T) = \mathrm{Ran}(1-S) \xrightarrow{\text{by Lemma}} \mathrm{Ran}(1-T) \text{ is closed.} \quad \blacksquare$$

Recall: $$S = \sum_{k=1}^{N} \langle \cdot, u_k \rangle v_k, \qquad E = \mathrm{span}\{u_1 \ldots u_N, v_1 \ldots v_N\}.$$

We obtained:

$$1-S = (1-S)\big|_E \oplus 1\big|_{E^\perp}$$

<u>Now</u>: $1-S$ <u>not invertible</u>. $\Longrightarrow (1-S)\big|_E$ <u>not invertible</u>.

$$(1-S)\big|_E : E \to E. \qquad \Big\Updownarrow$$

$$(1-S)\big|_E : E \to E \text{ not injective}.$$

$\Downarrow$

$$\exists x \neq 0, x \in E \text{ s.t. } (1-S)(x) = 0. \Leftrightarrow x = S\,x$$

$$x = Sx = \sum_{k=1}^{N} \langle x, u_k \rangle v_k$$

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_N v_N, \quad c_1 \ldots c_N \in \mathbb{C}$$

$$\underline{\sum_{k=1}^{N} c_k v_k} = \sum_{k=1}^{N} \Big(\underbrace{\sum_{j=1}^{N} \langle v_j, u_k \rangle c_j}_{} \Big) v_k$$

$$\Rightarrow \cancel{c_k}: \begin{bmatrix} \langle v_1, u_1 \rangle & \cdots & \langle v_N, u_1 \rangle \\ & & \\ \langle v_1, u_N \rangle & \cdots & \langle v_N, u_N \rangle \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix} \Rightarrow \underline{\det\left(I_N - \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}\right) = \cdot}$$

$1-S|_E$ not injective $\Rightarrow$ $\ker(1-S) \neq \{0\}$.

$\Rightarrow$ $\ker(1-T) \neq \{0\}$.

Ends Fredholm.

Riesz-Schauder : show : $\sigma(T)$ is discrete. with no accumulation point except 0.

$\{ \underset{\in \mathbb{C}}{z} : 1 - z\cdot T$ not invertible $\}$. discrete with no accumulation point.

$z_0 \cdot A$ finite rank. $\| z\cdot T - z_0 A \| < \frac{1}{2}$

$$S = z_0 A \cdot (1 - z\cdot T + z_0 \cdot A)^{-1} \dashrightarrow \text{finite rank.}$$

$$(1-S)\cdot(1 - z\cdot T + z_0 A) = \underbrace{1 - z\cdot T}_{\text{not invertible}}$$

$(1-S)$ not invertible.

dep on z

$$\det\left[I - \begin{bmatrix} \langle v_1, u_1 \rangle & \dots & \langle v_N, u_1 \rangle \\ \vdots & & \\ \langle v_1, u_N \rangle & \dots & \langle v_N, u_N \rangle \end{bmatrix} \right] = 0. \iff \text{Meromorphic. } ~~\text{Holom.}~~ |z| = 0$$

dep on

$$A = \sum_k \langle \cdot, u_k \rangle v_k \dashrightarrow S = \sum_{k=1}^{N} \langle \cdot, \bar{z}_0 (1 - zT + z_0 A)^{-1*} u_k \rangle V_k$$