C^*-algebras

Assume H is a Hilbert space and let $B(H)$ denote the algebra of bounded operators on H, with usual operator norm $\|\cdot\|$.
(induced norm).

Definition. A subset $C \subset B(H)$ is called a C^*-algebra if:

1) C is a subalgebra with identity of $B(H)$: $(C, +, \cdot)$ (is an algebra). composition: $T_1 \circ T_2$

2) If. $T \in C$ then $T^* \in C$ (taking adjoint is an internal operation).

3) C is closed w.r.t. operator norm induced topology.

$\Leftrightarrow$ If. $(T_n)_{n \geq 1}$ is a sequence of operators converging in operator norm in $B(H)$ then $S = \lim_{n\to\infty} T_n \in C$.

If. $T_n \in C$, $S \in B(H)$, s.t. $\lim_{n\to\infty} \|T_n - S\| = 0$, then

Definition. A subset $W \subset B(H)$ is called a W^*-algebra if:

1) W is a subalgebra with identity of $B(H)$.

2) If $T \in W$ then $T^* \in W$

3). W is closed w.r.t. strong operator topology:

If. $(T_n)_{n\geq 1}$ and S are operators in $B(H)$, $T_n \in W, \forall n \geq 1$.
and. $\forall x \in H$, $\lim_{n\to\infty} \|(T_n - S)x\| = 0$
Then $S \in W$.

Remark: 1) Any W^*-algebra is a C^*-algebra.

2). Any C^*-algebra is a Banach algebra.

3). Abstract theory $\longrightarrow$ Abstract C^*-algebra: Banach algebra + involution $(*)$.

$*: B \to B$.

B^*-algebra. (same thing).

$(a_1 A_1 + a_2 A_2)^* = \bar{a}_1 A_1^* + \bar{a}_2 A_2^*$.

$(A_1 \cdot A_2)^* = A_2^* \cdot A_1^*$

$\|A^*\| = \|A\|$

$\to \|A^* A\| = \|A\|^2$

For us: we focus on (concrete) (non-abstract) C^*-algebras.

$\Updownarrow$

Algebra of bounded operators on a Hilbert space

The Square Root Lemma.

Definition. Assume $A \in B(H)$.

1. A is said positive or strictly positive, and $A > 0$ as notation, if:

   i) $A = A^*$

   ii) $\forall x \in H, x \neq 0$, $\langle Ax, x \rangle > 0$.

2. A is said positive semidefinite (or non-negative), and we write $A \geq 0$, if

   i) $A = A^*$

   ii) $\forall x \in H$, $\langle Ax, x \rangle \geq 0$.

Remark: If H is a complex vector space with scalar product, then (ii) $\Rightarrow$ (i).

Lemma 1. Assume $A \geq 0$. For every $x, y \in H$, define:

$$\langle\langle x, y\rangle\rangle = \langle Ax, y\rangle.$$

Then $\langle\langle \cdot, \cdot\rangle\rangle$ is a semi-scalar product on H, i.e.: i). $\langle\langle x, x\rangle\rangle \geq 0$ $\forall x \in H$. $\langle\langle 0, 0\rangle\rangle = 0$.

ii). $\langle\langle x, y\rangle\rangle = \overline{\langle\langle y, x\rangle\rangle}$

iii). $\forall x, y, z \in H$, $c_1, c_2 \in \mathbb{C}$, $\langle\langle c_1 x + c_2 y, z\rangle\rangle = c_1 \langle\langle x, z\rangle\rangle + c_2 \langle\langle y, z\rangle\rangle$.

Furthermore, the C-S inequality holds:

$$\forall x, y \in H, \quad |\langle\langle x, y\rangle\rangle|^2 \leq \langle\langle x, x\rangle\rangle \cdot \langle\langle y, y\rangle\rangle.$$

Lemma 2. If $A \in B(H)$ and $A \geq 0$ then:

$$\|A\| = \sup_{\|x\|=1} \langle Ax, x\rangle.$$

Proof.

1. Let $a = \sup\limits_{\|x\|=1} \langle Ax, x\rangle$.

$$a = \sup_{\|x\|=1} \langle Ax, x\rangle \overset{\text{by C-S}}{\leq} \sup_{\|x\|=1} (\|Ax\| \cdot \|x\|) = \sup_{\|x\|=1} \|Ax\| = \|A\|.$$

2. $$\|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{\substack{\|x\|=1 \\ \|y\|=1}} |\langle Ax, y\rangle| \overset{\text{by Lemma 1}}{\leq} \sup_{\substack{\|x\|=1 \\ \|y\|=1}} \sqrt{\langle Ax, x\rangle \cdot \langle Ay, y\rangle}$$

$$= \sqrt{\sup_{\|x\|=1} \langle Ax, x\rangle} \cdot \sqrt{\sup_{\|y\|=1} \langle Ay, y\rangle} = \sqrt{a}\sqrt{a} = a.$$

(1). & (2) $\Rightarrow$ $a = \|A\|$.

<u>Lemma 3</u>. If $A \in B(H)$, $A \geq 0$ and $\|A\| \leq 1$ then:

(1) $1 - A \geq 0$

(2) $\|1 - A\| \leq 1$

<u>Pf</u>.

Follows from Lemma 2, e.g.:

$$\langle (1-A)x, x \rangle = \|x\|^2 - \underbrace{\langle Ax, x \rangle}_{\leq \|A\| \cdot \|x\|^2 = \|x\|^2} \geq \|x\|^2 - \|x\|^2 \geq 0.$$

$\Rightarrow 1 - A \geq 0.$

$\langle Ax, x \rangle \geq 0$.

$$\Rightarrow \langle (1-A)x, x \rangle \leq \|x\|^2 \rightarrow \|1 - A\| = \sup_{\|x\|=1} \langle (1-A)x, x \rangle \leq 1.$$

<u>Lemma 4</u>. Let $f: \mathbb{D} \to \mathbb{C}$, $\mathbb{D} = \overline{B_1(0)} = U_1(0) = \{z \in \mathbb{C} : |z| \leq 1\}$

$f(z) = \sqrt{1 - z}$, s.t. $f(0) = 1$. <u>and</u>. $f: B_1(0) \to \mathbb{C}$ is analytic.

Then: $f(z) = 1 - \sum_{n=1}^{\infty} c_n z^n$, where $c_n \geq 0$, $\sum_{n=1}^{\infty} c_n = 1$

and the series converges <u>absolutely</u> for every $z \in \mathbb{D}$, and <u>uniformly</u> on $\mathbb{D}$.

<u>Remark</u>: 1) f is analytic on $\operatorname{int}(\mathbb{D}) = B_1(0)$.

2) f is continuous on. <u>$\mathbb{D}$</u>.

Pf:

Expand. $z \longmapsto \sqrt{1-z}$ in Taylor series around 0.

$$f^{(k)}(z) = \left(-\tfrac{1}{2}\right)\cdot\left(\tfrac{1}{2}\right)\cdot\left(\tfrac{3}{2}\right)\cdot\ldots\cdot\left(\tfrac{2k-3}{2}\right)\cdot (1-z)^{\frac{1}{2}-k}.$$

$$f^{(k)}(0) = -\frac{(2k-2)!}{(k-1)!\, 2^{2k-1}}$$

Set $c_n = -\dfrac{f^{(n)}(0)}{n!} = \dfrac{(2n-2)!}{(n-1)!\, n!\, 2^{2n-1}} > 0.$

$c_0 = f(0) = 1.$

Then: $f(z) = 1 + \sum\limits_{n\geq 1} \dfrac{f^{(n)}(0)}{n!} z^n = 1 - \sum\limits_{n=1}^{\infty} c_n z^n$

convergence radius: $r = \lim\limits_{n\to\infty} \left|\dfrac{c_n}{c_{n+1}}\right| = \ldots = 1.$

<u>Absolute convergence</u>: $\sum\limits_{n=1}^{\infty} |-c_n z^n| \leq \sum\limits_{n=1}^{\infty} |c_n| = \lim\limits_{t\nearrow 1} \sum\limits_{n=1}^{\infty} c_n t^n = \lim\limits_{t\nearrow 1} [1 - f(t)] =$

For $|z| \leq 1$.

$$= \lim_{t\nearrow 1} \left[1 - \sqrt{1-t}\right] = 1.$$

<u>Uniform Conv</u>: on $\mathbb{D}$.

$$\left|\sum_{n=N}^{\infty} c_n z^n\right| \leq \sum_{n=N}^{\infty} |c_n| \xrightarrow{N\to\infty} 0$$

$\forall |z| \leq 1$

<u>Theorem</u> [The Square Root Lemma]. Let $A \in B(H)$, $A \geq 0$.
Then there exists a unique $B \in B(H)$, $B \geq 0$ so that $B^2 = A$.

Furthermore; 1) $B = \lim_{N\to\infty} p_N(A)$. in operator norm, for some sequence of polynomials $(p_N)_{N\geq 1}$

where each p_N is a polynomial of degree N.

2) If $C \in B(H)$ and. $AC = CA$ then $BC = CB$.

<u>Remark</u>.

1) If A belongs to some C^*-algebra $\mathcal{C}$ then $B \in \mathcal{C}$.
In other words, $A \mapsto \sqrt{A}$ is an internal operation on. the set of positive semidefinite elements of $\mathcal{C}$.

2) $\sqrt{A}$ is not constructed using <u>holomorphic calculus</u>. However it is constructed using <u>power series</u>.

<u>Proof</u>.

① $$A \dashrightarrow \tilde{A} = \frac{A}{\|A\|} \dashrightarrow \tilde{A} = 1 - \underbrace{(1-\tilde{A})}_{\|1-\tilde{A}\| \leq 1.} \dashrightarrow$$

$\|\tilde{A}\| = 1$, $\tilde{A} \geq 0$

$$\longrightarrow \tilde{B} = f(\cancel{1-}\tilde{A}). \qquad f(z) = \sqrt{1-z} = 1 - \sum_{n\geq 1} c_n z^n$$

$$\tilde{B} = 1 - \sum_{n\geq 1} c_n (1-\tilde{A})^n$$

$$\longrightarrow B = \sqrt{\|A\|} \cdot \tilde{B} = \sqrt{\|A\|} \cdot \left[1 - \lim_{N\to\infty} \sum_{k=1}^{N} c_n \left(1 - \frac{A}{\|A\|}\right)^n\right].$$

This construct explicitly:

$$B = \underbrace{\lim_{N\to\infty} p_N(A)}_{\text{converge in op. norm}}, \qquad p_N(z) = \sqrt{\|A\|}\cdot\left(1 - \sum_{n\geq 1}^{N} c_n\left(1-\frac{z}{\|A\|}\right)^n\right)$$

i) $B^* = \lim_{N\to\infty}\left[\overline{p}_N(\bar{z})\Big|_{z=A^*}\right] = \lim_{N\to\infty} p_N(A) = B.$

ii). $B^2 = \lim_{N\to\infty}(p_N(A))^2 = \left(\sqrt{\|A\|}\cdot\sqrt{\frac{A}{\|A\|}}\right)^2 = A.$

iii). $\langle Bx, x\rangle = \sqrt{\|A\|}\left\langle f\left(1-\frac{A}{\|A\|}\right)x, x\right\rangle =$

$$R = 1 - \frac{A}{\|A\|} \geq 0, \quad \|R\| \leq 1.$$

$$= \sqrt{\|A\|}\left\langle x - \sum_{n\geq 1} c_n R^n x, x\right\rangle = \sqrt{\|A\|}\left(\|x\|^2 - \sum_{n\geq 1} c_n \underbrace{\langle R^n x, x\rangle}_{\leq \|x\|^2}\right)$$

$$\geq \sqrt{\|A\|}\cdot\left(\|x\|^2 - \underbrace{\left(\sum_n c_n\right)}_{=1}\|x\|^2\right) \geq 0.$$

$\underline{B \geq 0.}$

iv) If $C\cdot A = A\cdot C \longrightarrow C\cdot B = B\cdot C.$ $\xleftarrow{N\to\infty}$

because: $C\cdot p_N(A) = p_N(A)\cdot C.$

② Uniqueness:

Assume $C = C^* \geq 0$, $C^2 = A$. Need to show $C = B$.

$C^2 = A \to A\cdot C = C^3 = C\cdot A \quad \Rightarrow A\cdot C = C\cdot A.$

$\Rightarrow B\cdot C = C\cdot B$ (B constructed earlier).

$$(C+B)\cdot(C-B) = \underset{=A}{C^2} \underbrace{- CB + BC}_{=0} - \underset{=A}{B^2} = 0.$$

$$\Downarrow$$

$(C-B)\cdot C\cdot(C-B) =$

$= (C-B)B(C-$

Take $x \in H$. Let $y = (C-B)x$.

$\langle Cy, y\rangle + \langle By, y\rangle = 0.$

But $\langle Cy, y\rangle \geq 0.$

$\langle By, y\rangle \geq 0$

$\to \langle Cy, y\rangle = 0, \ \langle By, y\rangle = 0.$

Since $|\langle Cy, z\rangle| \leq \underbrace{\sqrt{\langle Cy, y\rangle}}_{=0} \cdot \sqrt{\langle Cz, z\rangle}, \ \forall z \in H.$

$\Rightarrow \langle Cy, z\rangle = 0, \ \forall z \in H \Rightarrow Cy = 0, \ \forall y = (C-B)x.$

Similarly: $By = 0.$

$\forall x \in H:$ $C(C-B)x = 0$, $B(C-B)x = 0$ $\Rightarrow C(C-B) = B(C-B) = 0.$

$(B-C)^2 = B(B-C) - C(B-C) = 0$

$\forall x \in H: \|(B-C)x\|^2 = \langle (B-C)x, (B-C)x\rangle = \langle (B-C)^2 x, x\rangle = 0.$

$\Rightarrow B - C = 0 \Rightarrow \underline{\underline{B = C}}.$ ■

For $A \in B(H)$,

<u>Definition</u>. $|A| = \sqrt{A^* A}$. , where $\sqrt{\cdot}$ is the map introduced in the <u>Square Root lemma</u>.

<u>Note</u>: $|A| \in B(H)$, $|A|^* = |A|$, $|A| \geq 0$.