Polar Decomposition for Self Adjoint Operators.

Theorem. Let $A \in B(H)$, $A = A^*$. Let $A = U \cdot |A|$ be its polar decomposition and let P denote the orthogonal projection onto $\ker(A)^\perp = \overline{\mathrm{Ran}(A)}$. Then:

(a) $U = U^*$

(b) $U^2 = P$

(c) If $B \in B(H)$ s.t. $A \cdot B = B \cdot A$ then $U \cdot B = B \cdot U$.

In particular, $U \cdot |A| = |A| \cdot U$ (d'): $A = |A^*| \cdot U_A$

$$U A = A U$$

(d) Let $H_0 = \ker(A) = \ker(U)$, $H_+ = \ker(U - 1) = \{\varphi \in H : U\varphi = \varphi\}$ and $H_- = \ker(U + 1) = \{\varphi \in H : U\varphi = -\varphi\}$.

Then:

$$H = H_- \oplus H_0 \oplus H_+ \quad \text{(orthogonal decomposition)}$$

and. $H_0 \perp H_+$, $H_0 \perp H_-$, $H_+ \perp H_-$

(e) If P_+, P_0, P_- are the orthogonal projections onto H_+, H_0, H_- then:

If $\varphi \in H_0 + H_+$ then $\langle A\varphi, \varphi \rangle \geq 0$.

If $\varphi \in H_0 + H_-$ then $\langle A\varphi, \varphi \rangle \leq 0$.

(f) Let
$$A_+ = P_+ A = A P_+ = P_+ A P_+$$
$$A_- = P_- A = A P_- = P_- A P_-$$

Then:
$$A = A_+ + A_-$$
$$|A| = A_+ - A_-$$

Proof.

Recall: $\quad U_{A^*} = U_A^* \quad , \quad A = U \cdot |A|.$

Since $A = A^* \longrightarrow U_A = U_A^* \longrightarrow$ (a).

(b): $P = P_{\overline{\mathrm{Ran}(A)}} = U U^* = U^2$

(c): "Tricky".

Base on the following Lemma:

Lemma. Let $C \geq 0$ and P be the orthogonal projection onto $\ker(C)^\perp$

Then for $\varepsilon > 0$:

(a) $\| (C+\varepsilon)^{-1} \| \leq \frac{1}{\varepsilon}$

(b) $\| C \cdot (C+\varepsilon)^{-1} \| \leq 1.$

(c) $\lim_{\varepsilon \downarrow 0} \| C \cdot (C+\varepsilon)^{-1} C - C \| = 0$ $\longrightarrow$ Convergence in C^*-algebra generated by C.

(d) $\forall \varphi \in H, \quad \lim_{\varepsilon \downarrow 0} C (C+\varepsilon)^{-1} \varphi = P \varphi$ $\longrightarrow$ Convergence in W^*-al[gebra] generated by C.

(d)

$H_0 = \ker(A) = \ker(U) = \ker(P)$.

$H_+ := \ker(U-1) = \{\varphi \in H : U\varphi = \varphi\}$.

$H_- := \ker(U+1) = \{\varphi \in H : U\varphi = -\varphi\}$.

Let $x \in \mathrm{Ran}(P)$: $x = Px = U^2 x$.

On the other hand: $(U+1)(U-1) = (U-1)(U+1) = U^2 - 1$

$(U+1)(U-1)x = U^2 x - x = 0$.

If. $x \in H_+ \to Ux = x \dashrightarrow \underset{\overset{\shortparallel}{Px}}{U^2 x} = Ux = x \Rightarrow x \in \mathrm{Ran}(P)$

$x \in H_- \dashrightarrow Ux = -x \dashrightarrow \underset{\overset{\shortparallel}{Px}}{U^2 x} = -Ux = x \Rightarrow x \in \mathrm{Ran}(P)$

i) $H_+, H_- \subset \mathrm{Ran}(P)$.

ii). If $x \in H_+$, $y \in H_-$: $Ux = x$
$Uy = -y$.

$\langle x, y\rangle = \langle Ux, y\rangle = \langle x, Uy\rangle = \langle x, -y\rangle = -\langle x, y\rangle$.

$\Rightarrow \langle x, y\rangle = 0 \Rightarrow H_+ \perp H_-$.

If $z \in H_0$: $Pz = 0 \Leftrightarrow Uz = 0$

$x \in H_+$: $\langle x, z\rangle = \langle Ux, z\rangle = \langle x, Uz\rangle = \langle x, 0\rangle = 0$.

$y \in H_-$: $\langle y, z\rangle = \langle -Uy, z\rangle = \langle -y, Uz\rangle = \langle -y, 0\rangle = 0$.

$\Rightarrow H_0 \perp H_+$, $H_0 \perp H_-$

$\Rightarrow H_0 \oplus H_- \oplus H_+$ is an orthogonal direct sum.

let $x \in \mathrm{Ran}\, P$. let $y = Ux + x$

$$z = x - Ux$$

Note: $x = \frac{1}{2}(y+z)$.

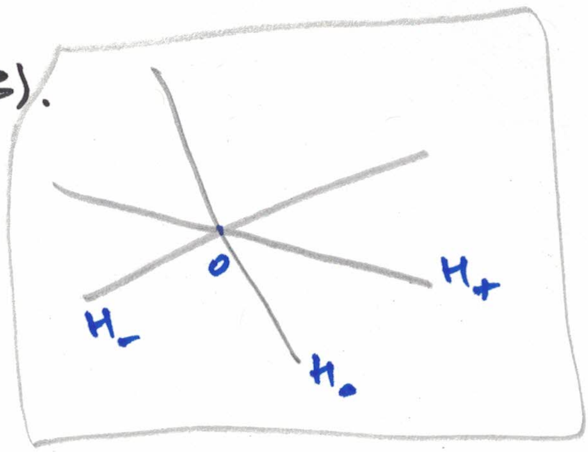
$$Uy = U^2x + Ux = Px + Ux = x + Ux = y. \Rightarrow y \in H_+$$

$$Uz = Ux - U^2x = Ux - Px = Ux - x = -z. \Rightarrow z \in H_-$$

We obtained $\mathrm{Ran}\, P = H_+ \oplus H_-$

$\longrightarrow$ $H = H_- \oplus H_0 \oplus H_+$.

(e).



P_+, P_-, P_0 orthogonal projections.

Fact: $P_+ = \frac{1}{2}(1+U)U^2$

$P_- = \frac{1}{2}(1-U)U^2$

$P_0 = 1 - P = 1 - U^2$

let $\varphi \in H_0 + H_+$: $\varphi = \varphi_+ + \varphi_0$: $U\varphi_+ = \varphi_+$

$U\varphi_0 = 0. \dashrightarrow A\varphi_0 =$

$$\langle A\varphi, \varphi\rangle = \langle A(\varphi_+ + \varphi_0), \varphi_+ + \varphi_0\rangle = \langle A\varphi_+, \varphi_+\rangle =$$

$$= \langle |A| U\varphi_+, \varphi_+\rangle = \langle |A|\varphi_+, \varphi_+\rangle \geq 0.$$

let $\varphi \in H_0 + H_-$: $\varphi = \varphi_- + \varphi_0$: $U\varphi_- = -\varphi_-$, $U\varphi_0 = 0$

$\Rightarrow A\varphi_0 = 0$

$$\langle A\varphi, \varphi\rangle = \langle A\varphi_-, \varphi_-\rangle = \langle |A| \overbrace{U\varphi_-}^{-\varphi_-}, \varphi_-\rangle = -\langle |A|\varphi_-, \varphi_-\rangle \leq 0$$

(f).

Note: $\mathbb{1} = P_+ + P_0 + P_-$

i). $P_+ A = A P_+$: because: $AU = UA$

$$\Rightarrow A \tfrac{1}{2}(1+U)U^2 = \tfrac{1}{2}(1+U)U^2 A$$

$P_+ = \frac{1}{2}(1+U)U^2$

$$\left(\frac{1}{2}(1+U)U^2\right)^2 = \frac{1}{4}(1+2U+U^2)U^4 = \frac{1}{4}(1+2U+P)P =$$

$$= \frac{1}{4}(\underbrace{P+P^2}_{2P} + 2UP) = \frac{2}{4}(1+U)P = \frac{1}{2}(1+U)U^2$$

$P_- A = A P_-$: similarly.

$P_+ A P_+ = P_+ A = A P_+$

$P_- A P_- = P_- A = A P_-$.

$$A = (P_+ + P_0 + P_-)A(P_+ + P_0 + P_-) =$$

$$= P_+ A P_+ + \cancel{P_+ A P_0} + P_+ A P_- + \cancel{P_0 A P_+} + \cancel{P_0 A P_0} + \cancel{P_0 A P_-} +$$

$$+ P_- A P_+ + \cancel{P_- A P_0} + P_- A P_-$$

But: $A P_0 = P_0 A = P_0 A P_0 = 0$.

$P_+ \cdot P_- = P_- P_+ = 0$. $\dashrightarrow$ $P_+ A P_- = P_- A P_+ = 0$.

$\Rightarrow A = P_+ A P_+ + P_- A P_- = A_+ + A_-$

By (e): $A_+ \geq 0$, $A_- \leq 0$.

Set. ~~[illegible]~~ $B = A_+ - A_- \geq 0$

$B = B^* \geq 0$.

$$B^2 = (A_+ - A_-)^2 = A_+^2 - \underbrace{A_+ A_-}_{=0} - \underbrace{A_- A_+}_{=0} + A_-^2 =$$

$$= A_+^2 + A_-^2 = A_+^2 + A_+ A_- + A_- A_+ + A_-^2 = (A_+ + A_-)^2 = A$$

$$= A^* A$$

$\Rightarrow$ By uniqueness of $|\cdot|$ $\Rightarrow$ $\underline{B = |A|}$. ☒

Remark: $U = P_+ - P_-$

Recall "Hahn' Theorem": Any signed measure.

$\mu = \mu_+ - \mu_-$: write as a diff. of two measures

Example.

Let $H = L^2[-1,1]$.

$A : H \to H$, $A(f)(x) = x \cdot f(x)$.

Facts: $A = A^*$, $\sigma(A) = \mathrm{Ran}(x) = [-1,1]$.

$|A|^2 f = A^2 f$: $(|A|^2 f)(x) = x^2 \cdot f(x) = |x|^2 \cdot f(x)$.

$\Rightarrow (|A| f)(x) = |x| \cdot f(x)$

Polar decomp: $A = U \cdot |A|$.

$$(Uf)(x) = \begin{cases} -f(x), & x < 0. \\ 0, & x = 0 \\ +f(x), & x > 0. \end{cases} = \mathrm{sign}(x) \cdot f(x).$$

$\to H_0 = \ker(A) = \{0\}$, $H_+ = \{f : Uf = f\} = \{f \in L^2[-1,1] : \mathrm{supp}(f) \subset [0,1]\}$

Here $H_+ \cong L^2[0,1]$

$$H_- = \{f : \mathcal{U}f = -f\} = \{f \in L^2[-1,1] : \operatorname{supp}(f) \subset [-1,0]\}.$$

$$\operatorname{sign}(x) \cdot f(x) = -f(x).$$

$$H_- \cong L^2[-1,0].$$

$$P_+ : f \longmapsto 1_{[0,1]} \cdot f$$

$$P_- : f \longmapsto 1_{[-1,0]} \cdot f.$$

$$\mathcal{U} = P_+ - P_-$$

$$f \longmapsto \int_{-1}^{1} f(x) \cdot \underbrace{\operatorname{sign}(x)\, dx}_{d\mu \text{ - signed measure}} =$$

$$= \int_0^1 f(x)\,dx - \int_{-1}^0 f(x)\,dx =$$

$$= \int_{-1}^1 f\, d\mu_+ - \int_{-1}^1 f\, d\mu_-$$

$\mu_+ \to$ induced by $1_{[0,1]}dx$

$\mu_- \to \quad 1_{[-1,0]}dx$

Resolution of Identity

Let H be a Hilbert space.

Definition A resolution of identity is a function $E: \mathbb{R} \to B(H)$

$t \longmapsto E_t$ so that:

(1) For each t, E_t is an orthogonal projection on H.

(2) For $t < s$, $E_t \le E_s$ (as quadratic forms) $\Leftrightarrow$ $\operatorname{Ran} E_t \subseteq \operatorname{Ran} E_s$

(3) $\displaystyle s\text{-}\lim_{t \to -\infty} E_t = 0$, $\displaystyle s\text{-}\lim_{t \to +\infty} E_t = 1$ (identity of H).

$$\left(\forall \varphi \in H, \ \lim_{t \to -\infty} \| E_t \varphi \| = 0, \quad \lim_{t \to \infty} \| E_t \varphi - \varphi \| = 0 \right).$$

(4). $\displaystyle s\text{-}\lim_{t \downarrow t_0} E_t = E_{t_0}$ (strongly right-continuous)

In other words, a resolution of identity is a monotone increasing and strongly right-continuous projection-valued function.

Definition: Support of $(E_t)_t$: the $\overbrace{\text{interval}}^{\text{smallest}}$ $[a,b]$ s.t.

(1) $\forall \varepsilon > 0$, $E_{a-\varepsilon} = 0$

(2) $E_b = 1$

Why: Let $g \in C[-R,R]$, with $\underbrace{[a,b]}_{\text{support of } (E_t)_t} \subset [-R,R]$.

Define:

$$\Phi_n(g) = \sum_{j=-2^n\cdot R}^{2^n\cdot R} g\left(\frac{j}{2^n}\right)\cdot\left(E_{\frac{j}{2^n}} - E_{\frac{j-1}{2^n}}\right).$$

Can show: $(\Phi_n(g))_n$ is a Cauchy sequence in $B(H)$.

$\dashrightarrow \exists \underbrace{\lim_{n\to\infty} \Phi_n(g)}_{\text{operator norm.}}$ in $B(H)$.

$\longrightarrow$ [Continuous Functional Calculus on $B(H)$ associated to $(E_t)_t$.

$$\Phi(g) = \lim_{n\to\infty} \Phi_n(g).$$

How to get a resolution of identity from polar factorization:

Take $A = A^* \in B(H)$. Let $\sigma(A) \subset [a,b] \subset \mathbb{R}$.

For each $t \in \mathbb{R}$ $\dashrightarrow$ $A - t\cdot 1 = U_t \cdot |A - t\cdot 1|$. ← partial isometry

$\dashrightarrow$ let $P_-(t), P_0(t), P_+(t)$ denote the orth. projections as in previous theorem.

Set.

$$E_t = P_-(t) + P_0(t).$$

Then: If $t < a$ $\dashrightarrow$ $P_-(t), P_0(t) = 0$.

Because. $A - t \cdot 1 > 0$.

$|A - t \cdot 1| = A - t \cdot 1 \Rightarrow U_t = 1$.

If $t > b$ $\dashrightarrow$ $P_-(t) = 1$. , $P_0(t), P_+(t) = 0$.

$|A - t \cdot 1| = -(A - t \cdot 1)$.

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