L3

Hilbert Spaces

Assume. $(H, +)$ is a vector space.

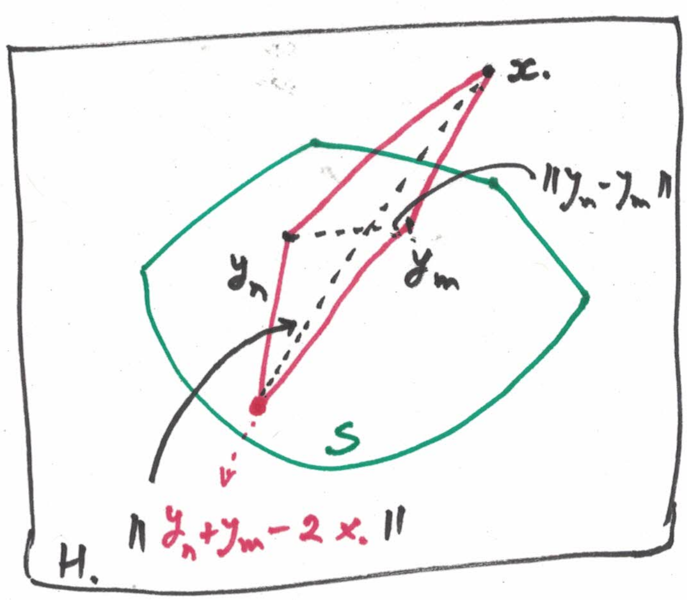
Definition. A set $S \subset H$ is said convex if:
$\forall x, y \in S \quad \forall t \in [0,1], \quad (1-t)x + ty \in S.$

Lemma. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space.
Assume $S \subset H$ is a closed convex subset of H, $S \neq \emptyset$.
Let $x \in H$. Then there exists a unique element $z \in S$ closest to x, i.e., $\|z - x\| \leq \|y - x\|, \ \forall y \in S$.

Proof.

Let $d = \inf_{y \in S} \|y - x\|$. Goals: 1) Find $z \in S$ s.t. $\|z - x\| = d$.
2) If $z' \in S$ s.t. $\|z' - x\| = d$ then $z' = z$.

Step 1.
$\exists (y_n)_{n \geq 1}$ s.t. $\|y_n - x\| \to d$.
$y_n \in S$.



Step 2. Claim: $(y_n)_{n \geq 1}$ is Cauchy.
Use parallelogram identity:

$$2\left(\|x-y_n\|^2+\|x-y_m\|^2\right)=\|y_n-y_m\|^2+\|y_n+y_m-2x\|^2$$

$$\|y_n-y_m\|^2=2\|x-y_n\|^2+2\|x-y_m\|^2-4\left\|x-\frac{y_n+y_m}{2}\right\|^2.$$

$$S \underline{\text{convex}} \Rightarrow \frac{y_n+y_m}{2}\in S$$

$$\left\|x-\frac{y_n+y_m}{2}\right\|\geq d.$$

$$\|y_n-y_m\|^2\leq 2\|x-y_n\|^2+2\|x-y_m\|^2-4d^2$$

$$0\leq \lim_{N\to\infty}\ \sup_{n,m\geq N}\|y_n-y_m\|^2\leq 2d^2+2d^2-4d^2=0$$

$$\lim_{N\to\infty}\ \sup_{n,m\geq N}\|y_n-y_m\|=0.$$

$$\forall \varepsilon>0\ \exists N_\varepsilon\ \forall n,m\geq N_\varepsilon,\quad \|y_n-y_m\|\leq\varepsilon.$$

$\Rightarrow (y_n)_n$ Cauchy.

<u>Step 3</u>. $(y_n)_n$ Cauchy $\xrightarrow[\text{complete.}]{H}$ $(y_n)_n$ is convergent in H.

$S\subset H$ is closed $\Rightarrow$ $(y_n)_n$ is convergent in S.

$$\exists\ z=\lim_{n\to\infty}y_n,\quad z\in S.$$

and $\|z-x\|=\lim_{n\to\infty}\|y_n-x\|=d.$ $\longrightarrow$ Goal 1 (existence).

Uniqueness:

Assume $z, z' \in S$ s.t. $\|z-x\| = \|z'-x\| = d \leq \|y-x\|$ $\forall y \in S$.

Parallelogram:

$$\|z-z'\|^2 + \|z+z'-2x\|^2 = 2\underbrace{\|x-z\|^2}_{d} + 2\underbrace{\|x-z'\|^2}_{d.}$$

$$0 \leq \|z-z'\|^2 = 2d^2 + 2d^2 - 4\|x - \underbrace{\tfrac{z+z'}{2}}_{\in S}\|^2 \leq 4d^2 - 4d^2 = 0.$$

$\longrightarrow$ $\|z-z'\| = 0$ $\to$ $z = z'$.

Bounded linear Operators: $B(V,W)$

Assume $(V, \|\cdot\|)$ and $(W, |||\cdot|||)$ are two normed vectorspaces

Definition A map $T: V \to W$ is called a bounded linear operator if:

(1) T is a linear map, i.e., $T(ax+by) = aT(x) + bT(y)$ $\forall a,b \in \mathbb{C}$, $x,y \in V$.

(2) There exists a constant $K \geq 0$ s.t.

$\forall x \in V$, $|||T(x)|||_W \leq K \cdot \|x\|_V$

~~Def.~~ Notation:

$$\|T\|_{B(V,W)} = \inf\{K \geq 0 \text{ s.t., } \forall x \in V: |||T(x)||| \leq K.\|x\|$$

<u>Theorem</u>. Assume $T : V \to W$ is a linear map, where $(V, \|\cdot\|)$, $(W, |||\cdot|||)$ are normed vector spaces.

Then the following are equivalent:

(1) T is continuous at some point $x_0 \in V$.

(2) T is continuous (on V).

(3) T is bounded.

<u>Pf. - sketch</u>:

(1) $\Leftrightarrow$ (2) follows from linearity:

$$\forall z \in V, \text{ if } z_n \to z \text{, in } V. \text{ then } x_n = z_n - z + x_0,$$

$$x_n \to x_0$$

$$T(z_n) = T(z_n + z - x_0) = T(x_n) + T(z) - T(x_0).$$

Assume T is cont. at x_0:

$$\lim_{n\to\infty} T(z_n) = \underbrace{\lim_{n\to\infty} T(x_n)}_{T \text{ cont. at } x_0} + T(z) - T(x_0) = T(x_0) + T(z) - T(x_0)$$

$$= T(z).$$

(1) $\Leftrightarrow$ (3), Set $x_0 = 0$.

Assume. $(x_n)_{n\geq 1}$ in V s.t. $\lim_{n\to\infty} x_n = 0$. and $T(x_n) \to T(0) = 0$

want to show T is <u>bounded</u>. $\Leftrightarrow$ $\dfrac{|||T(x)|||}{\|x\|} \leq K, \forall x.$

By contradiction: $\exists (y_n)_{n\geq 1}$, s.t. $|||T(y_n)|||/\|y_n\| \geq n$, $y_n \in V$

let $x_n = \frac{1}{\sqrt{n}\|y_n\|} \cdot y_n = \frac{1}{\sqrt{n} \cdot \|y_n\|} y_n \in V.$

$$\|x_n\| = \frac{1}{\sqrt{n}\,\|y_n\|} \cdot \|y_n\| = \frac{1}{\sqrt{n}} \longrightarrow 0, \text{ as } n \to \infty.$$

If T is continuous $\rightarrow \lim\limits_{n\to\infty} |||T(x_n)||| = 0.$

But $T(x_n) = \dfrac{T(y_n)}{\sqrt{n}\,\|y_n\|}$

$$|||T(x_n)||| = \frac{|||T(y_n)|||}{\sqrt{n} \cdot \|y_n\|} \geq \frac{n}{\sqrt{n}} = \sqrt{n}.$$

$\rightarrow \lim\limits_{n\to\infty} |||T(x_n)||| = +\infty \rightarrow$ Contradiction.

If T bounded: $\exists\, K: \ |||T(x)||| \leq K \cdot \|x\|$

let $x_n \longrightarrow 0.$

$$0 \leq |||T(x_n)||| \leq K \cdot \|x_n\|$$

$n \to \infty$

By the squeeze lemma: $\lim\limits_{n\to\infty} |||T(x_n)||| = 0$

$\Rightarrow \lim\limits_{n\to\infty} T(x_n) = 0.$

$\longrightarrow T$ continuous at $0.$

$\blacksquare$

Remark:

$$\inf\left\{K\geq 0:\ \forall x\in V,\ x\neq 0,\ \frac{|||T(x)|||}{\|x\|}\leq K\right\} =$$

$$= \sup\left\{\frac{|||T(x)|||}{\|x\|},\ x\in V,\ x\neq 0\right\}.$$

Therefore:

$$\|T\|_{B(V,W)} = \sup_{\substack{x\neq 0\\ x\in V}} \frac{|||T(x)|||}{\|x\|}.$$

~~Theorem~~.

Definition / Notation: We let $B(V,W)$ denote the set of bounded linear operators from $(V,\|\cdot\|)$ to $(W,|||\cdot|||)$.

Theorem:

(1) $B(V,W)$ is a linear space (complex vector space).

(2) $\|\cdot\|_{B(V,W)}$ is a norm on $B(V,W)$.

(1) ⇒ (2) ⇒ $\left(B(V,W), \|\cdot\|_{B(V,W)}\right)$ is a normed vector space.

(3) If $(W,|||\cdot|||)$ is complete then $\left(B(V,W), \|\cdot\|_{B(V,W)}\right)$ is complete as well, hence a Banach space.

Proof:

(1): linearity $\dashrightarrow$ check.
& boundedness.

(2) Norm:

(i) positivity: $\|T\|_{B(V,W)} \geq 0$. –

$$\|T\|_{B(V,W)} = 0 \dashrightarrow \sup_{x \neq 0} \frac{|||T(x)|||}{\|x\|} = 0$$

$$\Rightarrow |||T(x)||| = 0, \ \forall x.$$

$$\longrightarrow T(x) = 0, \ \forall x \Rightarrow \underline{T = 0}.$$

(2) homogeneity:

$$\|a \cdot T\|_{B(V,W)} = \sup_{x \neq 0} \frac{|||a \cdot T(x)|||}{\|x\|} = \sup_{x \neq 0} \frac{|a| \cdot |||T(x)|||}{\|x\|}$$

$$= |a| \cdot \sup_{x \neq 0} \frac{|||T(x)|||}{\|x\|} = |a| \cdot \|T\|_{B(V,W)}.$$

(3) Δ inequality:

$$T, S \in B(V,W): \ \|T + S\|_{B(V,W)} = \sup_{x \neq 0} \frac{|||(T+S)(x)|||}{\|x\|} \leq$$

$$\leq \sup_{x \neq 0} \left(\frac{|||T(x)|||}{\|x\|} + \frac{|||S(x)|||}{\|x\|} \right) \leq \sup_{x \neq 0} \frac{|||T(x)|||}{\|x\|} + \sup_{x \neq 0} \frac{|||S(x)|||}{\|x\|}$$

$$= \|T\|_{B(V,W)} + \|S\|_{B(V,W)}.$$

(3) completness.

Assume $(T_n)_{n\geq 1}$ is a Cauchy sequence in $(B(V,W), \|\cdot\|_{B(V,W)})$ and W is a complete normed vector space.

Want: Construct $T \in B(V,W)$ s.t. $\lim_{n\to\infty} T_n = T$.

Step 1 Construction of T.

~~For any~~ Fix $x \in V$.

Claim: $(T_n(x))_{n\geq 1}$ is a Cauchy sequence in W.

by sup-form of $\|\cdot\|_{B(V,W)}$

$$|||T_n(x) - T_m(x)||| = |||(T_n - T_m)(x)||| \leq \|T_n - T_m\|_{B(V,W)} \cdot \|x\|$$

$$0 \leq \lim_{N\to\infty} \sup_{n,m\geq N} |||T_n(x) - T_m(x)||| \leq \underbrace{\left(\lim_{N\to\infty} \sup_{n,m\geq N} \|T_n - T_m\|_{B(V,W)}\right)}_{=0} \cdot \|x\|$$

$$\Rightarrow \lim_{N\to\infty} \sup_{n,m\geq N} \|T_n(x) - T_m(x)\| = 0 \rightarrow (T_n(x))_n \text{ Cauchy.}$$

Therefore $(T_n(x))_{n\geq 1}$ is convergent in W, because W is complete.

let $T(x) = \lim_{n\to\infty} T_n(x)$.

Step 2 claim: T is linear: $T(ax+by) = \cdots = aT(x) + bT(y)$

Step3. T is bounded.

$(T_n)_{n\geq 1}$ is Cauchy in $\left(B(V,W), \|\cdot\|_{B(V,W)}\right)$.

$\longrightarrow \left(\|T_n\|_{B(V,W)}\right)_{n\geq 1}$ is also Cauchy.

(By Δ ineq.
$$\left| \|T_n\|_{B(V,W)} - \|T_m\|_{B(V,W)} \right| \leq \|T_n - T_m\|_{B(V,W)} \Big).$$

$\longrightarrow \left(\|T_n\|_{B(V,W)}\right)_n$ is bounded:

$$\exists\ K \geq 0 \quad \text{s.t.} \quad \|T_n\|_{B(V,W)} \leq K, \ \forall n.$$

For every $x \neq 0$:

$$\frac{|||T(x)|||}{\|x\|} = \lim_{n\to\infty} \frac{|||T_n(x)|||}{\|x\|} \leq \frac{K \cdot \|x\|}{\|x\|} = K.$$

$$\Rightarrow \sup_{x\neq 0} \frac{|||T(x)|||}{\|x\|} \leq K \longrightarrow T \text{ is } \underline{\text{bounded}}.$$

Step 4.

Need to show $T_n \to T$ in $\|\cdot\|_{B(V,W)}$ topology, i.e.

$$\lim_{n\to\infty} \| T_n - T \|_{B(V,W)} = 0.$$

For this, recall Cauchy assumption:

$$\lim_{N\to\infty} \sup_{n,m\geq N} \| T_n - T_m \|_{B(V,W)} = 0.$$

Fix $\varepsilon > 0$. Let N be s.t. $\| T_n - T_m \|_{B(V,W)} \leq \frac{\varepsilon}{2}$, $\forall n,m \geq N$.

Therefore, for every $x \in V$, $\|x\| = 1$ and $n > N$:

$$|||(T_n - T)x|||_W = \lim_{m\to\infty} |||(T_n - T_m)x|||_W =$$

$$= \limsup_{m\to\infty} |||(T_n - T_m)x|||_W \leq \limsup_{m\to\infty} \Big[\underbrace{\| T_n - T_m \|_{B(V,W)}}_{\leq \frac{\varepsilon}{2}} \cdot \underbrace{\|x\|_V}_{1} \Big] \leq$$

$$\leq \frac{\varepsilon}{2} < \varepsilon.$$

Thus $\| T_n - T \|_{B(V,W)} < \varepsilon$, $\forall n > N$.

This shows $\lim_{n\to\infty} \| T_n - T \|_{B(V,W)} = 0$.

This proves $(B(V,W), \|\cdot\|_{B(V,W)})$ is a complete normed vector space, hence Banach.

Remark: Even in $(V, \|\cdot\|)$ may not be complete,

$(B(V,W), \|\cdot\|_{B(V,W)})$ is always complete when $(W, |||\cdot|||)$ is

Special Case : $W = \mathbb{C}$ (or $\mathbb{R}$).

Definition. Let $(V, \|\cdot\|)$ be a normed vector space. The space $\left(B(V,\mathbb{C}), \|\cdot\|_{B(V,\mathbb{C})}\right)$ is called the **dual space** to $(V, \|\cdot\|)$, denoted $(V^*, \|\cdot\|_{V^*})$

Remark:

$(V^*, \|\cdot\|_{V^*})$ is always complete, hence a Banach space