L4

Hilbert Spaces (2)

Let $(V, \langle\cdot,\cdot\rangle)$ be a vector space with inner product.

Definition Let $E \subset V$ be a linear subspace. The set $E^{\perp}$ is called the orthogonal complement of E in V, where

$$E^{\perp} = \{ x \in V : \forall w \in E, \langle x, w\rangle = 0 \}.$$

Lemma. Assume. $(H, \langle\cdot,\cdot\rangle)$ is a Hilbert space and $E \subset H$ is a linear subspace. Then $E^{\perp}$ is a closed linear space.

Pf.

- If. $x_1, x_2 \in E^{\perp}$ and $a, b \in \mathbb{C}$ then $\forall w \in E$,

$$\langle a x_1 + b x_2, w\rangle = a \langle x_1, w\rangle + b\langle x_2, w\rangle = 0$$

$$\Rightarrow a x_1 + b x_2 \in E^{\perp}$$

- Assume. $(x_n)_n \in E^{\perp}$ and $(x_n)_n$ converges in $(H, \langle\cdot,\cdot\rangle)$: $\exists\, x = \lim_{n\to\infty} x_n \in H$.

$$\underline{\forall w \in E} : \langle x_n, w\rangle = 0 \rightarrow \langle \lim_{n\to\infty} x_n, w\rangle = 0 \Rightarrow \langle x, w\rangle = 0$$

$$\Rightarrow x \in E^{\perp}$$

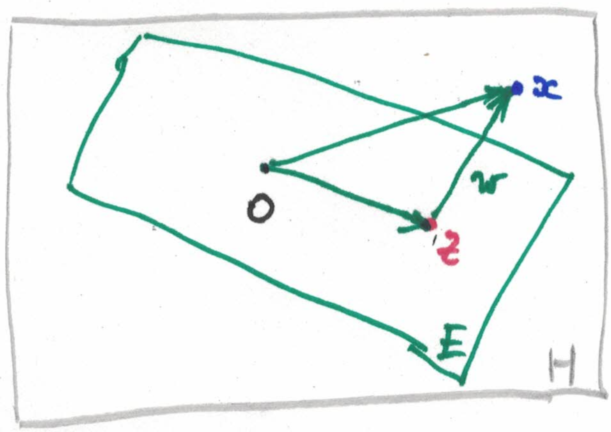
Theorem. Assume $(H, \langle \cdot,\cdot \rangle)$ is a Hilbert Space and $E \subset H$ is a closed linear subspace.

(1) For every $x \in H$ there are unique. $z \in E$ and $w \in E^{\perp}$ such that $x = z + w$.

(2) If. $E \neq H$ then $E^{\perp} \neq \{0\}$.

Proof.

(1). If $E = H$ then $E^{\perp} = \{0\}$.



and: $\underset{\substack{\| \\ z \in E}}{x} = x + \underset{\substack{\in \\ E^{\perp}}}{0}.$

If $E = \{0\}$ then $E^{\perp} = H$

$$x = \underset{\substack{\| \\ z}}{0} + \underset{\substack{\| \\ w}}{x}.$$

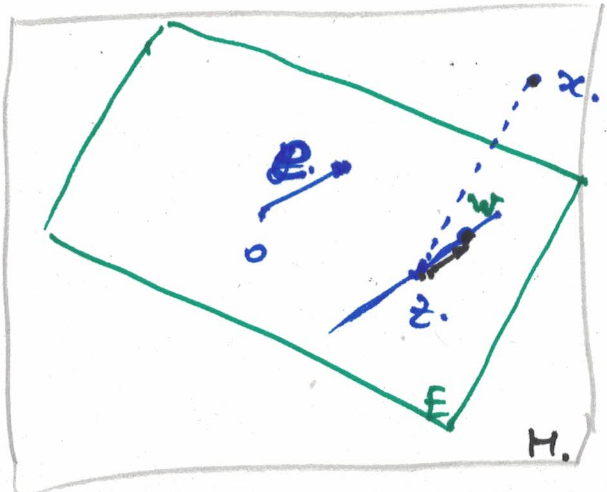
If. $\{0\} \subsetneq E \subsetneq H$.

$x \in H$: Recall we showed last time: If $S \overset{\neq \phi}{\subset} H$ is a closed convex set then. $\exists!\ z \in S$ s.t. $\|x - z\| = d_S(x) := \inf \|y - x\|$ yes.

Need to show: set $w = x - z$ then $w \in E^{\perp}$.

$x \in H$,

$$z = \underset{y \in E}{\operatorname{argmin}} \|x - y\|$$



Take $\underset{e \neq 0}{e} \in E$, $t \in \mathbb{R}$:

$$\|x - (z + te)\|^2 \geq \|x - z\|^2$$

$w = x - z$: $$\underbrace{\|w - te\|^2}_{\cancel{\|w\|^2} - \langle te, w\rangle - \langle w, te\rangle + \|te\|^2} \geq \|w\|^2$$

$$\cancel{\|w\|^2} - \langle te, w\rangle - \langle w, te\rangle + \|te\|^2 \geq \cancel{\|w\|^2}$$

$$\underbrace{t^2 \cdot \|e\|^2 - 2t \operatorname{Re}\langle e, w\rangle}_{\text{quadratic. } \|e\|^2 \neq 0.} \geq 0, \ \forall t \in \mathbb{R}$$

$$\Downarrow$$

$$\boxed{\operatorname{Re}\langle e, w\rangle = 0}.$$

Repeat the argument: $\|x - (z + t\,ie)\|^2 \geq \|x - z\|^2$, $\forall t \in \mathbb{R}$

$$\cdots \Rightarrow \operatorname{Re}\langle ie, w\rangle = 0$$

$$\Leftrightarrow \operatorname{Im} \boxed{\langle e, w\rangle = 0}$$

$$\Rightarrow \langle e, w\rangle = 0 : \forall e \in E \Rightarrow w \in E^\perp$$

We obtained: $x = z + w$, with $z \in E$, $w \in E^\perp$

Since $\langle z, w\rangle = 0$. $\Rightarrow$ $\boxed{\|x\|^2 = \|z\|^2 + \|w\|^2}$ (Pythagora relation).

Uniqueness: $E \cap E^\perp = \{0\}$. $\dashrightarrow$ $E \oplus E^\perp$ is an algebraic direct sum

(2) Assume $E \subsetneq H$: closed linear subspace, $E \neq H$.

$\dashrightarrow$ Take $x \in H \setminus E \longrightarrow$ by part (1): $x = z + w$

with $\underline{w \in E^{\perp}}$ and. $\underline{w \neq 0}$ (otherwise : $z = x \in E$).

$\Rightarrow E^{\perp} \neq \{0\}$.

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<u>Definition</u>. Let $E \subset H$ be a closed subspace of the Hilbert space $(H, \langle \cdot, \cdot \rangle)$.
We let $P_E : H \to E$ denote,

$$P_E(x) = \operatorname*{argmin}_{y \in E} \|x - y\|$$

i.e., the unique $z \in E$ s.t. $\|x - z\| \leq \|x - y\|$, $\forall y \in E$.

— .

Since the decomposition $x = z + w$ corresponds to the direct sum $H = E \oplus E^{\perp}$
it follows that P_E is a <u>linear map</u>.

Since. $\|x\|^2 = \|z\|^2 + \|x - z\|^2 \geq \|z\|^2$

$$\Rightarrow \|P_E\|_{B(H,H)} = \sup_{x \neq 0} \frac{\|P_E(x)\|}{\|x\|} \leq 1.$$

But, if $x \in E (\neq \{0\})$ then $P_E(x) = x \Rightarrow \underline{\|P_E\|_{B(H,H)} = 1}$

Notation : $B(V) = B(V, V)$.

[We obtained for any $E \subset H$ a closed subspace, $P_E \in B(H)$ and $\|P_E\|_{B(H)} = 1$.].

<u>Theorem</u> [Riesz Representation Theorem of the dual of a Hilbert space]

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space.

(a) For each $z \in H$, let $\ell_z : H \to \mathbb{C}$, $\ell_z(x) = \langle x, z\rangle$ denote a linear map. Then ℓ_z is bounded (hence continuous) linear map on H and $\|\ell_z\|_{H^*} = \|z\|_H$. (recall: $\|\ell_z\|_{H^*} = \|\ell_z\|_{B(H,\mathbb{C})}$

(b) For each $\ell \in H^*$, i.e., $\ell : H \to \mathbb{C}$ a bounded linear functional there exists a unique $z \in H$ s.t. $\ell = \ell_z$, i.e. $\forall x \in H$,

$$\ell(x) = \langle x, z\rangle.$$

Furthermore, $\|\ell\|_{H^*} = \|z\|_H$.

<u>Proof</u>.

(a). Take $z \in H$, set $\ell_z : H \to \mathbb{C}$, $\ell_z(x) = \langle x, z\rangle$.

Clearly ℓ_z is a linear map:

Take $x \in H$:

$$|\ell_z(x)| = |\langle x, z\rangle| \overset{\text{by C-S}}{\leq} \|x\| \cdot \|z\|$$

$\Longrightarrow$ ℓ_z is a bounded linear map hence $\ell_z \in H^*$.

$\Rightarrow \|\ell_z\| \leq \|z\|$.

But: if $z \neq 0$: $\dfrac{|\ell_z(z)|}{\|z\|} = \dfrac{|\langle z, z\rangle|}{\|z\|} = \|z\|$

if $z = 0 \to \ell_z = 0$

$\Big\} \to \|\ell_z\| = \|z\|$.

(b) let $\ell \in H^*$, i.e. $\ell : H \to \mathbb{C}$, ℓ continuous linear map.

Then $E = \ker(\ell) = \{y \in H : \ell(y) = 0\}$ is a closed linear subspace of H.

$= \ell^{-1}(\{0\})$.

(i) Case 1: If $E = H$ $\dashrightarrow$ $\ell(y) = 0, \forall y \in H$ $\dashrightarrow$ $\ell = 0$

$\dashrightarrow$ $\ell = \ell_z$, for $z = 0$ and $\|\ell\| = 0 = \|z\|$.

(ii) Case 2: If $E \subsetneq H$. We know: $E^{\perp} \neq \{0\}$.

Let $z_1 \in E^{\perp}$, $z_1 \neq 0$. and $\ell(z_1)$ real.

~~let $z \in H \setminus E$, $\ell(z) \neq 0$~~.

Set $z = \ell(z_1) \cdot \dfrac{z_1}{\|z_1\|^2}$. Note: $\ell(z_1) \neq 0$, $\Rightarrow z \neq 0$.

~~assume $\ell(z_1) \in \mathbb{R}$~~.

Claim: $\ell = \ell_z$, i.e., $\forall x \in H$, $\ell(x) = \langle x, z \rangle$.

Take $x \in H$. Let $w = x - \dfrac{\ell(x)}{\ell(z)} \cdot z \in H$.

But: $\ell(w) = \ell(x) - \ell\left(\dfrac{\ell(x)}{\ell(z)} \cdot z\right) = \ell(x) - \dfrac{\ell(x)}{\cancel{\ell(z)}} \cdot \cancel{\ell(z)} = 0$

$\longrightarrow w \in E = \ker(\ell)$.

and: $x = \dfrac{\ell(x)}{\ell(z)} \cdot z + w$.

Now:

$$\langle x, z \rangle = \left\langle \frac{\ell(x)}{\ell(z)} z + w, z \right\rangle = \frac{\ell(x)}{\ell(z)} \cdot \|z\|^2 + \underbrace{\langle \overset{E}{\overset{\uparrow}{w}}, \underset{E^\perp}{\underset{\downarrow}{z}} \rangle}_{=0} =$$

$$= \ell(x) \cdot \frac{\|z\|^2}{\ell(z)}$$

$$\ell(z) = \ell\left(\ell(z_1) \cdot \frac{z_1}{\|z_1\|^2}\right) = \frac{(\ell(z_1))^2}{\|z_1\|^2}$$

$$\|z\|^2 = \left\| \ell(z_1) \cdot \frac{z_1}{\|z_1\|^2} \right\|^2 = \frac{(\ell(z_1))^2}{\|z_1\|^4} \cdot \|z_1\|^2 = \frac{(\ell(z_1))^2}{\|z_1\|^2}$$

$\ell(z) = \|z\|^2$. Hence: $\langle x, z \rangle = \ell(x)$. $\Rightarrow \ell = \ell_z$

By part (a): $\|\ell\|_{H^*} = \|z\|_H$.

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Remark.

The proof of part ② identified one z s.t. $\ell = \ell_z$.

Is it unique?

YES: Why: Assume $z, z' \in H$ s.t. $\ell = \ell_z = \ell_{z'}$,

Then $\langle x, z \rangle = \langle x, z' \rangle, \ \forall x \in H$

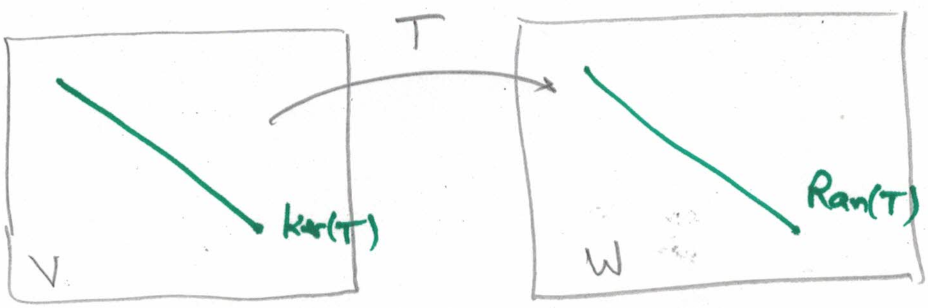
$$\Rightarrow \langle x, z - z' \rangle = 0 \xrightarrow{x = z - z'} \|z - z'\|^2 = 0$$

$$\longrightarrow \underline{\underline{z = z'}}.$$

Interpretation:

Recall the (Grassman/Frobenius) isomorphisms:

Let $T : V \longrightarrow W$ be a linear map between two finite dim. vector spaces.



Then: $V = \ker(T) \oplus \cancel{\mathrm{Ran}} \left(\ker(T)\right)^{\perp}$

and: $\boxed{\ker(T)^{\perp} \simeq \mathrm{Ran}(T)}$.

In our case: $\ell : H \to \mathbb{C} \dashrightarrow (\ker(\ell))^{\perp} \simeq \mathrm{Ran}(\ell)$ — either $\{0\}$ or $\mathbb{C}$.

E.

Either way: $\dim \mathrm{Ran}(\ell) \in \{0, 1\}$

$\ell = 0$

$\ell \neq 0$, but $E^{\perp}$ has dimension 1.