Hilbert Spaces (3)

Def. Let $(H, \langle\cdot,\cdot\rangle)$ and $(V, \langle\langle\cdot,\cdot\rangle\rangle)$ be two Hilbert spaces. Then they are called isomorphic if there exists $U: H \to V$ a linear invertible map such that $\|Ux\|_V = \|x\|_H$, $\forall x \in H$.

Such U is called a unitary transformation.

Corollary of Riesz Rep. Theorem:

Theorem. Assume $(H, \langle\cdot,\cdot\rangle)$ is a Hilbert space.

Then. $H^* \simeq H$, i.e., they are isomorphic.

Specifically, if $l \in H^*$ then $U(l) = z \in H$ s.t. $l(x) = \langle x, z\rangle$, $\forall$

$U: H^* \to H$ defines a unitary map.

Remark. What if $(H, \langle\cdot,\cdot\rangle)$ is a vector space with scalar product, but possibly not complete. Then what is $H^* = ?$

Claim: H^* is ~~the~~ completion of $(H, \langle\cdot,\cdot\rangle)$.

a completion

Step 1: There exists a completion of $(H, \langle \cdot,\cdot \rangle)$ as a metric space. Let $\overline{H}$ denote this completion.

Step 2. $\langle \cdot,\cdot \rangle$ on H extends uniquely to a scalar product on $\overline{H}$,

$$\left(\overline{H}, \langle \cdot,\cdot \rangle_{\overline{H}}\right).$$

$H \overset{i}{\hookrightarrow} \overline{H}$, isometric dense embedding.

$i : H \to \overline{H}$, i linear. $i(H)$ dense in $\overline{H}$

$\| i(x) \|_{\overline{H}} = \|x\|_H \longrightarrow \langle i(x), i(y) \rangle_{\overline{H}} = \langle x, y \rangle_H$

$\longrightarrow \left(\overline{H}, \langle \cdot,\cdot \rangle_{\overline{H}}\right)$ is a Hilbert space.

Step 3.

$\longrightarrow \overline{H} \cong \overline{H}^*$

But:

Claim: $\overline{H}^* = H^*$

If. $\ell \in \overline{H}^*$: $\ell : \overline{H} \to \mathbb{C}$ bounded linear functional.

$\xrightarrow{j} \ell|_H \in H^*$

If. $\ell \in H^*$: $\ell : H \to \mathbb{C}$: linear map and: $|\ell(x)| \leq C \cdot \|x\|_H$.

$\xrightarrow{j^{-1}} \exists!$ extension, $\overline{\ell} : \overline{H} \to \mathbb{C}$ s.t. $\overline{\ell}|_H = \ell$.

$\ell \in \overline{H}^*$, $\|\ell\|_{\overline{H}^*} = \sup\limits_{x \in \overline{H}} |\ell(x)| = \sup\limits_{x \in H} |\ell(x)| = \|\ell|_H\|_{H^*} \rightarrow$ $\underline{j \text{ is unitary}}$.

$H \subset \overline{H}$ is dense.

Conclusion:

$$\overline{H} \simeq \overline{H}^* \simeq H^*$$

Thus ~~the~~ the completion of $(H, \langle \cdot, \cdot \rangle)$ is given by H^*.

Another Corollary of Riesz Rep. Th. for Hilbert spaces:

Theorem. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. Let $T : H \times H \to \mathbb{C}$ be a function that satisfies:

1). $T(\cdot, z)$ is linear for every $z \in H$, i.e. $\forall x, y \in H$, $\forall a, b \in \mathbb{C}$,

$$T(ax+by, z) = a\, T(x,z) + b\, T(y,z).$$

2) $T(z, \cdot)$ is antilinear for every $z \in H$, i.e. $\forall x, y \in H$, $\forall a, b \in \mathbb{C}$,

$$T(z, ax+by) = \overline{a} \cdot T(z,x) + \overline{b}\, T(z,y).$$

3) There exists $C > 0$ s.t.

$$\forall x, y \in H, \quad |T(x,y)| \le C \cdot \|x\| \cdot \|y\|.$$

Then:

There exists a unique bounded linear map $A : H \to H$ such that, $\forall x, y \in H$, $\quad T(x,y) = \langle A(x), y \rangle$.

Furthermore, $\|A\| \le C$, and:

$$\sup_{\substack{\|x\|=1 \\ \|y\|=1}} |T(x,y)| = \sup_{\|x\|=1} \|A(x)\| = \|A\|_{B(H)}.$$

Proof.

Construct A: For every $x \in H$ need to find $Ax = ?$

Fix $x \in H$.

Consider the linear map, $y \mapsto \overline{T(x,y)}$

let $\ell : H \to \mathbb{C}$, $\ell(y) = \overline{T(x,y)}$.

$|\ell(y)| = |\overline{T(x,y)}| = |T(x,y)| \leq C \cdot \|x\| \cdot \|y\|$.

$\to$ ℓ is a bounded linear functional:

By Riesz Rep. Th., $\exists !\, z \in H$ s.t. $\ell(y) = \langle y, z\rangle = \overline{\langle z, y\rangle}$

Thus: $\overline{T(x,y)} = \ell(y) = \overline{\langle z, y\rangle} \Rightarrow T(x,y) = \langle z, y\rangle, \forall y$.

let $A(x) = z$. $\dashrightarrow$ Easy to check that A is linear.

and. $\|A(x)\| = \|z\| = \sup\limits_{\|y\|=1} |\langle z, y\rangle| = \sup\limits_{\|y\|=1} |T(x,y)| \leq C \|x\|$

$\Rightarrow$ A is a bounded linear transformation: $A \in B(H)$

Since $\langle A(x), y\rangle = T(x,y) \Rightarrow$

$\Rightarrow \|A\|_{B(H)} = \sup\limits_{x \in H} \|Ax\| = \sup\limits_{\substack{\|x\|=1 \\ \|y\|=1}} |T(x,y)|$ $\boxed{}$

<u>Corollary</u>. Assume. $(H, \langle\cdot,\cdot\rangle)$ is a Hilbert space and $A: H \to H$ a bounded linear map. Then there exist a unique bounded linear map $B: H \to H$ such that $\forall x, y \in H$, $\langle Ax, y\rangle = \langle x, By\rangle$.

<u>Furthermore</u> $\|A\|_{B(H)} = \|B\|_{B(H)}$.

<u>Definition/Notation</u>. The map B in this corollary is called <u>the adjoint of A</u> and denoted A^*. $\longrightarrow$ $\|A\|_{B(H)} = \|A^*\|_{B(H)}$.

<u>Proof</u>.

Given $A: H \to H$ $\dashrightarrow$ constructs, $T: H \times H \to \mathbb{C}$,

$$T(x,y) = \langle Ax, y\rangle. \dashrightarrow \overline{T}: H \times H \to \mathbb{C},$$

$\overline{T}(y,x) = \overline{T(x,y)}$: $\overline{T}$ is linear in the ~~second~~ first term & antilinear in the ~~first~~ second term

<u>and</u>. $|\overline{T}(y,x)| \leq C \cdot \|x\| \cdot \|y\|$, where $C: |T(x,y)| \leq C \cdot \|x\| \|y\|$

$\longrightarrow$ by previous theorem: $\exists\, B: H \to H$ bounded s.t.

$$\overline{T}(y,x) = \langle B(y), x\rangle.$$

$$\longrightarrow \langle Ax, y\rangle = T(x,y) = \overline{\overline{T}(y,x)} = \overline{\langle B(y), x\rangle} = \langle x, B(y)\rangle, \ \forall x, y.$$

$$\dashrightarrow \|A\|_{B(H)} = \|B\|_{B(H)} = \|A^*\|_{B(H)}.$$

$\blacksquare$

<u>Formal</u>:

$$(H \otimes \overline{H})^* \simeq B(H)$$

<u>Orthonormal Bases</u>

<u>Assume</u>. $(H, \langle \cdot, \cdot \rangle)$ is a <u>Hilbert space</u>.

$\boxed{\text{Assume}}$: H is <u>separable</u>. ($\exists$ a countable dense subset) and $\dim H = \infty$.

<u>Theorem</u>. There exists a set $\{e_n\}_{n \geq 1}$ such that:

1) (ORTHONORMALITY): $\langle e_n, e_m \rangle = \begin{cases} 1, & n = m \\ 0, & n \neq m. \end{cases}$

2). For every $x \in H$, $\forall \varepsilon > 0$ $\exists N, c_1 \ldots c_N \in \mathbb{C}$ s.t.

$$\| x - (c_1 e_1 + \ldots + c_N e_N) \| < \varepsilon.$$

<u>Proof</u>.

<u>Step 1</u>. Construct inductively: $\{e_1, e_2, \ldots \}$:

i) Take $e_1 \neq 0$, normalize $\|e_1\| = 1$.

ii) Assume $\{e_1 \ldots e_{n-1}\}$ have been constructed.

Pick $e_n \in \{e_1 \ldots e_{n-1}\}^\perp$: $\|e_n\| = 1$. <u>Zorn's lemma</u>.

$\dashrightarrow$ The set $\{e_1, e_2, \ldots\}$ is <u>maximal</u> in the sense that there is <u>no</u> $f \in H$, $f \neq 0$ s.t. $\langle f, e_n \rangle = 0$, $\forall n$.

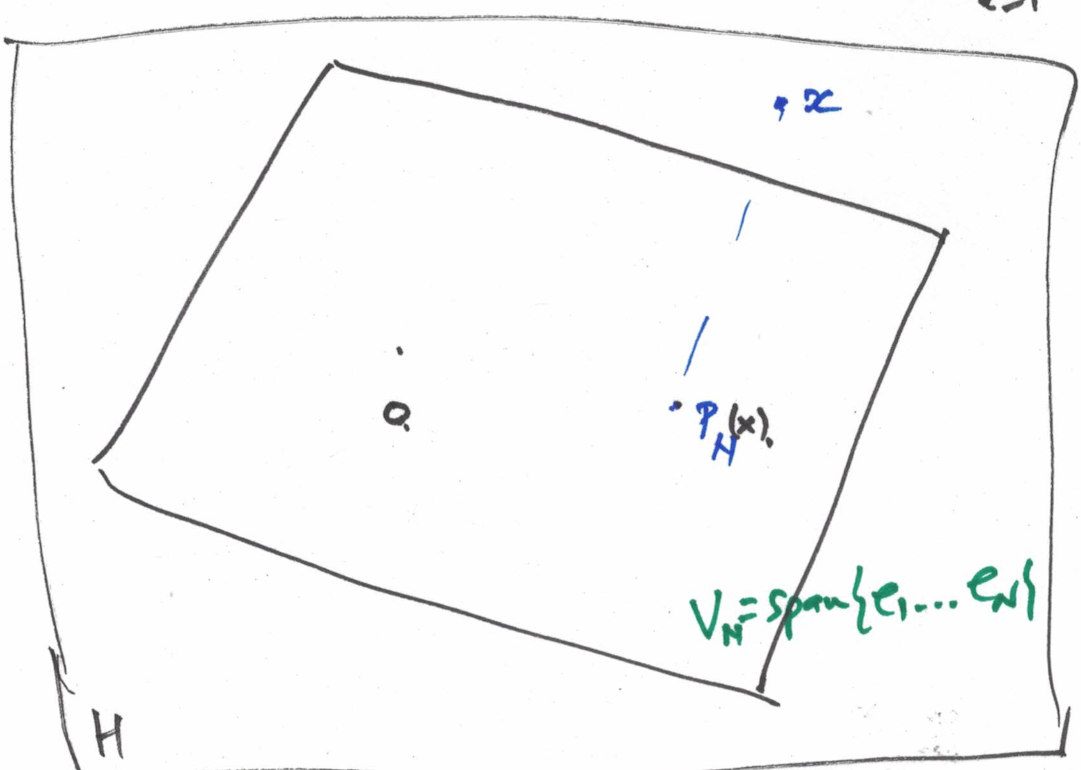
Step 2.

Let $x = \sum_{k=1}^{n} c_k e_k$

$$\|x\|^2 = \left\langle \sum_{k=1}^{n} c_k e_k, \sum_{j=1}^{n} c_j e_j \right\rangle = \cdots = \sum_{k=1}^{n} |c_k|^2$$

Also note: $c_k = \langle x, e_k \rangle$

Therefore: $\forall x \in H \dashrightarrow$ Consider $P_N x = \sum_{k=1}^{N} \langle x, e_k \rangle e_k$



$P_N : H \to V_N \subset H$ is the "orthonormal" proj. onto V_N

$$P_N(x) = \underset{y \in V_N}{\arg\min} \|x - y\|. \qquad \|x\|^2 = \|P_N(x)\|^2 + \|x - P_N(x)\|^2$$

$$\|P_N(x)\|^2 \le \|x\|^2 \longrightarrow \boxed{\sum_{k=1}^{N} |\langle x, e_k \rangle|^2 \le \|x\|^2}, \ \forall N.$$

Take $N \to \infty$: $$\boxed{\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \leq \|x\|^2}$$ Bessel Inequality.

Let's consider. $(P_n(x))_{n\geq 1} \in H$.

<u>Claim</u>: This is a Cauchy sequence:

Take $n<m$:

$$\|P_n(x) - P_m(x)\|^2 = \left\| \sum_{k=1}^{n} \langle x, e_k\rangle e_k - \sum_{k=1}^{m} \langle x, e_k\rangle e_k \right\|^2 =$$

$$= \left\| \sum_{k=n+1}^{m} \langle x, e_k\rangle e_k \right\|^2 = \sum_{k=n+1}^{m} |\langle x, e_k\rangle|^2$$

Since: $\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 < \infty \Rightarrow \lim_{N\to\infty} \sup_{n,m\geq N} \|P_n(x) - P_m(x)\| = 0$

H complete $\to \exists \lim_{n\to\infty} P_n(x) = z \in H$.

If $z = x \to$ end of proof.: $\lim_{n\to\infty} \|x - P_n(x)\| = 0$.

If $z \neq x$.

<u>Claim</u>: $w := x - z \perp P_n(x)$, $\forall n$.

<u>Why</u>: Take. $N \geq 1$. $\langle w, e_N\rangle = \langle x, e_N\rangle - \langle z, e_N\rangle = \langle x, e_N\rangle - \lim_{n\to\infty} \langle P_n(x), e_N\rangle$

$= \langle x, e_N\rangle - \langle x, e_N\rangle = 0$

We Obtained: $\langle x-z, e_N\rangle = 0, \ \forall N \geq 1.$

By maximality of $\{e_1, e_2, \dots\}$ $\longrightarrow$ $x-z=0$.

Hence: $z = x$. $\longrightarrow \bigcup_{n\geq 1} \operatorname{Span}\{e_1 \dots e_n\}$ is dense in H.

Furthermore:

We obtained: The series $\sum_{k=1}^{\infty} \langle x, e_k\rangle e_k$ converges to x in $\|\cdot\|$-topo[logy]

$$\lim_{n\to\infty} \|x - \sum_{k=1}^{n} \langle x, e_k\rangle e_k\| = 0. \quad \blacksquare$$

Also:

1) $\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 = \|x\|^2$, $\forall x \in H$.

by polarization.

2) $\langle x, y\rangle = \sum_{k=1}^{\infty} \langle x, e_k\rangle \langle e_k, y\rangle$, $\forall x, y \in H$.