Convexity and Duality

Theorem [Hahn–Banach Theorem – The Real Case].

Let X be a real vector space and $p: X \to \mathbb{R}$ a convex function. Let $Y \subset X$ be a linear subspace and $\ell: Y \to \mathbb{R}$ a linear map obeying:

$$\ell(y) \leq p(y), \quad \forall y \in Y$$

Then there exists a linear map $L: X \to \mathbb{R}$ such that:

(1) $L(x) \leq p(x), \quad \forall x \in X$

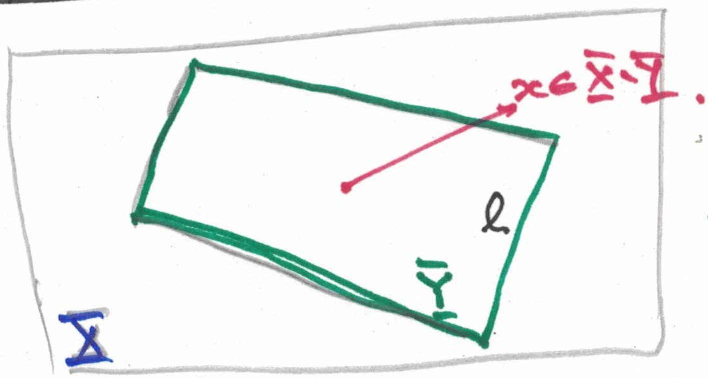
(2) $L\big|_Y = \ell.$

Recall: p is convex: $p((1-t)x + ty) \leq (1-t)p(x) + t\,p(y)$

$$\forall x, y \in X, \quad \forall t \in [0,1].$$

(Typically: $p(x) = a \cdot \|x\| \; \ldots$).

Idea of Proof:



Step 1 (Main): Extension by 1 di…

$\ell \to \tilde{L} : \operatorname{span}\{Y \oplus x\} \to \mathbb{R}$.

$\vdots$

Step 2:

* [If X is normed v.s. & separable …
* ---> extension in a countable # of steps

* [Otherwise] → Use Zorn's lemma. to obtain an extension to X

Relationship to ONB:

Assume $\mathcal{E} = \{e_\alpha : \alpha \in A\}$ an orthonormal set in H ↳ Hilbert space.

1) $Y = \overline{\operatorname{span}\{e_\alpha\}}$

either $Y = H \to$ done.

or $Y \subsetneq H \dashrightarrow \exists\, e' \in H \setminus Y$, $\|e'\| = 1$ & $e' \perp \{e_\alpha\}_{\alpha \in A}$

$\longrightarrow \mathcal{E}' = \{e_\alpha, \alpha \in A\} \cup \{e'\}$

2) Repeat

Proof of Hahn–Banach (Real Case)

On Y: $\quad \ell(y) \le p(y), \ \forall y \in Y \xrightarrow{y \to -y} -\ell(y) \le p(-y).$

$$\Rightarrow \quad 0 \le p(y) + p(-y). \quad \text{and.} \quad 0 \le p(0).$$

Step 1: $\boxed{\text{Extension.}}$

Lemma Let $Y \subset X$ be a proper linear subspace of the real vector space X. Let $p: X \to \mathbb{R}$ be convex and $\ell: Y \to \mathbb{R}$ be a linear map such that $\ell(y) \le p(y), \ \forall y \in Y$. Let $x \in X \setminus Y$, $x \ne 0$. and set $\tilde{Y} = Y \oplus \operatorname{span}\{x\} = \{y + \alpha \cdot x;\ y \in Y, \alpha \in \mathbb{R}\} \subset X$.

Then there exists $\tilde{L}: \tilde{Y} \to \mathbb{R}$ a linear map such that:

(1) $\tilde{L}(z) \le p(z), \ \forall z \in \tilde{Y}$

(2) $\tilde{L}|_Y = \ell.$

Proof of Lemma.

Since $\forall z \in \tilde{Y}$, $\quad z = y + \alpha \cdot x \quad , y \in Y, \alpha \in \mathbb{R}$

$\to$ any linear map $\tilde{L}$ on $\tilde{Y}$ should satisfy:

$$\tilde{L}(z) = \tilde{L}(y + \alpha \cdot x) = \underbrace{\tilde{L}(y)} + \alpha \cdot \tilde{L}(x)$$

We want $\tilde{L}|_Y = \ell. \quad \Rightarrow \quad \ell(y)$

$$\to \quad \tilde{L}(z) = \ell(y) + \alpha \cdot \tilde{L}(x).$$

We need: Find a $\boxed{\lambda \in \mathbb{R}}$ $\quad (\lambda \to \tilde{L}(x_0))$.

such that $\tilde{L}(y + \alpha \cdot x_0) = \ell(y) + \alpha \cdot \lambda$

satisfies: $\tilde{L}(y+\alpha \cdot x) \leq p(y+\alpha\cdot x)$, $\forall y \in \overline{Y}$, $\forall \alpha \in \mathbb{R}$.

$$\Longleftrightarrow \quad \boxed{\ell(y) + \alpha\cdot\lambda \leq p(y+\alpha\cdot x), \quad \forall y \in \overline{Y}, \ \forall \alpha \in \mathbb{R}.}$$

If $\alpha = 0 \dashrightarrow \ell(y) \leq p(y), \ \forall y \in \overline{Y}$ ok by hypothesis.

If. $\alpha > 0$: $\quad \ell(y) + \alpha\cdot\lambda \leq p(y+\alpha\cdot x)$.

$$\lambda \leq \frac{p(y+\alpha\cdot x) - \ell(y)}{\alpha} \quad : \text{need.}$$

If. $\alpha < 0 \dashrightarrow$ let $\beta = -\alpha > 0$:

$$\ell(y) - \beta\cdot\lambda \leq p(y - \beta\cdot x).$$

$$\lambda \geq \frac{\ell(y) - p(y-\beta x)}{\beta}$$

Given (α, β): $\alpha, \beta > 0 \dashrightarrow$

$$\lambda \in \underbrace{\left[\frac{\ell(y) - p(y-\beta x)}{\beta}, \frac{p(y+\alpha\cdot x) - \ell(y)}{\alpha}\right]}_{I_{y;\alpha,\beta}}.$$

$\boxed{\text{Claim}}$: $\left[\text{For any finite } \{(\alpha_1,\beta_1),\dots,(\alpha_N,\beta_N)\} : \bigcap_{k=1}^{N} I_{y;\alpha_k,\beta_k} \neq \phi. \right]$

$$\iff \forall\, \alpha,\beta>0 \quad \frac{\ell(y)-p(y-\beta x)}{\beta} \leq \frac{p(y+\alpha x)-\ell(y)}{\alpha} \quad (*)$$

Need to check $(*)$:

$(*) \iff \quad \alpha\, \ell(y) - \alpha\, p(y-\beta x) \leq \beta\, p(y+\alpha x) - \beta\, \ell(y).$

$\iff \quad (\alpha+\beta)\, \ell(y) \leq \alpha \cdot p(y-\beta x) + \beta \cdot p(y+\alpha x)$

$\iff \quad \ell(y) \leq \frac{\alpha}{\alpha+\beta}\, p(y-\beta x) + \frac{\beta}{\alpha+\beta}\, p(y+\alpha x).$

But : hypothesis on ℓ.

$$\ell(y) \leq p(y) = p\left(\frac{\alpha}{\alpha+\beta}(y-\beta x) + \frac{\beta}{\alpha+\beta}(y+\alpha x)\right) \leq$$

convexity of p

$$\leq \frac{\alpha}{\alpha+\beta}\, p(y-\beta x) + \frac{\beta}{\alpha+\beta}\, p(y+\alpha x)$$

Thus: $\boxed{I_{y;\alpha,\beta} \neq \phi}$

$\longrightarrow$ Any Finite intersection $\bigcap_{k=1}^{N} I_{y;\alpha_k,\beta_k} \neq \phi.$

By finite intersection property of compact sets $\Rightarrow \bigcap_{y\in Y} \bigcap_{\alpha,\beta>0} I_{y;\alpha,\beta} \neq \phi$

To conclude:

We showed:

$$\exists \lambda \in \bigcap_{y \in \underline{Y}} \bigcap_{\alpha, \beta > 0} \left[\frac{\ell(y) - p(y - \beta x)}{\beta}, \frac{p(y + \alpha x) - \ell(y)}{\alpha} \right].$$

$\dashrightarrow$ For such λ: $\tilde{L}(y + \alpha . x) \leq p(y + \alpha . x), \forall y \in Y, \alpha \in$

where $\tilde{L}(y + \alpha . x) = \ell(y) + \alpha . \lambda$

End of $\boxed{\text{LEMMA}}$

Step 2: $\boxed{\text{MAXIMALITY}}$ By Zorn's lemma.

Consider pairs $\mathcal{P} = \{ (\tilde{Y}, \tilde{L})$ of linear subspaces $\tilde{Y} \subset \underline{X}$ and linear maps $\tilde{L} : \tilde{Y} \to \mathbb{R}$ such that: (1) $Y \subset \tilde{Y}$

(2) $\tilde{L}(z) \leq p(z), \forall z \in \tilde{Y}$, (3) $\tilde{L}|_Y = \ell$

and: (4) If $(\tilde{Y}_1, \tilde{L}_1), (\tilde{Y}_2, \tilde{L}_2) \in \mathcal{P}$:

$$\tilde{L}_1|_{\tilde{Y}} = \tilde{L}_2|_{\tilde{Y}} \text{ where } \tilde{Y} = \tilde{Y}_1 \cap \tilde{Y}_2 \}.$$

On $\mathcal{P}$ there is a partial order:

$$(\tilde{Y}_1, \tilde{L}_1) \triangleleft (\tilde{Y}_2, \tilde{L}_2) \text{ if } \tilde{Y}_1 \subset \tilde{Y}_2.$$

$\dashrightarrow$ Extract an ascending chain: $\{ (\tilde{Y}_\alpha, \tilde{L}_\alpha), \alpha \in A \}$.

such that $\forall \alpha, \beta \in A$ either $(\tilde{Y}_\alpha, \tilde{L}_\alpha) \triangleleft (\tilde{Y}_\beta, \tilde{L}_\beta)$ or $(\tilde{Y}_\beta, \tilde{L}_\beta) \triangleleft (\tilde{Y}_\alpha, \tilde{L}_\alpha)$

Such ascending chains are not empty, because $(\underline{Y}, \ell)$ is in each.

~~By Zorn's lemma: $\mathcal{P}$ has a maximal element (or a maximal upper bound): $\exists (Y_0, L_0) \in \mathcal{P}$ such that.~~

~~If. $(Y_1, L_1) \in \mathcal{P}$ with. $(Y_0, L_0) \lhd (Y_1, L_1)$ then. $(Y_1, L_1) = (Y_0, L_0)$~~

Let. $\mathcal{C}$ be an <u>ascending chain</u>: $\mathcal{C} \subset \mathcal{P}$

Then define: $\tilde{\tilde{Y}} = \bigcup \tilde{Y}_\alpha$. $\longrightarrow$ <u>Note</u>: $\tilde{\tilde{Y}}$ is a <u>linear space</u> $\tilde{\tilde{Y}} \subset \underline{X}$.

where: $(\tilde{Y}_\alpha, \tilde{L}_\alpha) \in \mathcal{C}$

and: $\tilde{\tilde{L}} : \tilde{\tilde{Y}} \longrightarrow \mathbb{R}$: if $x \in \tilde{\tilde{Y}}$, then $\exists \alpha$ s.t. $x \in \tilde{Y}_\alpha$

$\downarrow$ linear map. $\qquad \perp \longrightarrow \tilde{\tilde{L}}(x) = \tilde{L}_\alpha(x)$.

<u>Claim</u>: $(\tilde{\tilde{Y}}, \tilde{\tilde{L}})$ is a maximal element:

<u>Why</u>: $\forall (\tilde{Y}_\alpha, \tilde{L}_\alpha) \in \mathcal{C}$: $\tilde{Y}_\alpha \subset \tilde{\tilde{Y}}$ and. $\tilde{\tilde{L}}\big|_{\tilde{Y}_\alpha} = \tilde{L}_\alpha$

By Zorn's lemma : There exists a maximal upper bound: $(Y_0, L_0) \in \mathcal{P}$

<u>such</u>. $\forall (Y_1, L_1) \in \mathcal{P}$ with. $(Y_0, L_0) \lhd (Y_1, L_1)$ then. $Y_1 = Y_0, L_1 = L_0$

Therefore: $Y_0 = \underline{X}$. $\dashrightarrow$ L_0 satisfies the Hahn-Banach theorem

<u>Use Case</u> : $(\underline{X}, \|\cdot\|)$ is a normed vector space.

let $\ell : Y \subset \underline{X} \longrightarrow \mathbb{R}$ a bounded (continuous) linear functional on Y : $\|\ell(y)\| \leq$ $\leq c \cdot \|y\|$

$\longrightarrow \exists\, L : \underline{X} \longrightarrow \mathbb{R}$: $|L(x)| \leq C \cdot \|x\|, \forall x \in \underline{X}$

$L|_Y = \ell.$

$\longrightarrow \underline{\|L\| = \|\ell\|}$ & $L|_Y = \ell.$

$\left[\; \cdots\!\!\longrightarrow \underline{X} \hookrightarrow \underline{X}^{**} \right.$ <u>isometric</u> embedding. $\left.\right]$.