Complex Hahn-Banach Theorem & Consequences

Theorem [Hahn-Banach – complex case]. Let $\underline{X}$ be a complex vector space and $p : \underline{X} \to \mathbb{R}$ a convex function that is also symmetric, in the sense that $p(z \cdot x) = p(x)$, $\forall z \in \mathbb{C}, |z| = 1$, $\forall x \in \underline{X}$.

Let $\underline{Y} \subset \underline{X}$ be a complex linear subspace and $\ell : \underline{Y} \to \mathbb{C}$ be a linear map obeying:

$$|\ell(y)| \leq p(y), \quad \forall y \in \underline{Y}$$

Then there exists a linear map $L : \underline{X} \to \mathbb{C}$ such that:

(1) $|L(x)| \leq p(x), \quad \forall x \in \underline{X}$

(2). $L\big|_{\underline{Y}} = \ell.$

Proof.

Idea: $\underline{X} \dashrightarrow$ realification (realization) of $\underline{X}$ as a real vector space.

In this case. $\{x, i \cdot x\}$ is linearly independent $(x \neq 0)$.

$\underline{Y} \longrightarrow$ real linear subspace of $\underline{X}$

$\ell : \underline{Y} \to \mathbb{C} \dashrightarrow \lambda : \underline{Y} \longrightarrow \mathbb{R}, \quad \lambda(y) = \mathrm{Real}[\ell(y)].$

Note:

$$\ell(y) = \mathrm{Real}(\ell(y)) + i \underbrace{\mathrm{Imag}(\ell(y))}_{= -\mathrm{Re}(\ell(iy))} = \lambda(y) - i\lambda(iy).$$

Because: $\ell(iy) = i\,\ell(y) = i\,(\mathrm{Re}(\ell(y)) + i\,\mathrm{Im}(\ell(y))) = -\mathrm{Imag}(\ell(y)) + i \cdot \mathrm{Real}(\ell(y$

$$\lambda(y) = \text{Real}\,(\ell(y)) \leq |\ell(y)| \leq p(y), \ \forall y \in \underline{Y}.$$

$\longrightarrow$ By real Hahn-Banach, there exists $\Lambda : \underline{X} \longrightarrow \mathbb{R}$ s.t.

(i) Λ is $\mathbb{R}$-linear.

(ii) $\Lambda|_{Y} = \lambda$

(iii) $\Lambda(x) \leq p(x), \ \forall x \in \underline{X}$.

Consider: $L : \underline{X} \longrightarrow \mathbb{C}, \quad L(x) = \Lambda(x) - i\,\Lambda(ix)$.

Claim: This L satisfies the condition of the theorem.

(i) $L|_{Y} = \ell$: $L|_{Y}(y) = \Lambda(y) - i\,\Lambda(iy) = \lambda(y) - i\,\lambda(iy) =$

$$= \ell(y), \ \forall y \in \underline{Y}.$$

(ii). L is $\mathbb{C}$-linear.

$$L(ix) = \Lambda(ix) - i\,\Lambda(-x) = i\Lambda(x) + i^2 \cdot \Lambda(-ix) =$$

$$= i\left[\Lambda(x) - i\,\Lambda(ix)\right] = i\,L(x).$$

$\longrightarrow$ $L(\alpha \cdot x) = L(ax + ibx) = L(ax) + L(ibx) = a\,L(x) + i\,b\,L(x)$

$\alpha = a + ib \in \mathbb{C}$ $\qquad = (a+ib)\,L(x) = \alpha \cdot L(x)$.

$L(x_1 + x_2) = L(x_1) + L(x_2)$ $\quad \checkmark$.

(iii) $x \in \underline{X}$. ~~[illegible]~~

$$L(x) = e^{i\theta} \cdot |L(x)| \longrightarrow |L(x)| = e^{-i\theta} L(x) = L(e^{-i\theta} \cdot x) =$$

$$= \underbrace{\Lambda(e^{-i\theta} \cdot x)} - \underbrace{i\Lambda(e^{-i\theta}\, ix)}$$

$\longrightarrow \Lambda(e^{-i\theta} i x) = 0.$

$$\longrightarrow |L(x)| = \Lambda(e^{-i\theta}\cdot x) \underset{\uparrow \\ (iii)}{\leq} p(e^{-i\theta}\cdot x) \underset{\uparrow \\ p \text{ is symmetric.}}{=} p(x).$$

$\blacksquare$

<u>Remark</u>. Other extension results.

"Lipschitz extension."

<u>Theorem</u> [Kirszbraun's Extension Result]. Assume $(V, \langle \cdot, \cdot \rangle)$ and $(W, \langle\langle \cdot, \cdot \rangle\rangle)$ are two Hilbert spaces and. $S \subset V$ is a subset of V. Given $f : S \to W$ a Lipschitz function with Lipschitz constant L, i.e., $\forall x, y \in S : \|f(x) - f(y)\|_W \leq L \cdot \|x - y\|$ there exists $F : V \to W$ such that:

(i) $F|_S = f$.

(ii) F is Lipschitz with same Lipschitz constant L.

(F: isometric extension).

Assume $(\underline{X}, \|\cdot\|)$ is a Normed Linear Space. ($\mathbb{C}$-vector space).

Recall the dual space:

$$\underline{X}^* = \Big\{ \ell : \underline{X} \to \mathbb{C}, \quad \ell \text{ linear and } \underline{\text{bounded}} \Big\}.$$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Leftrightarrow$ continuous

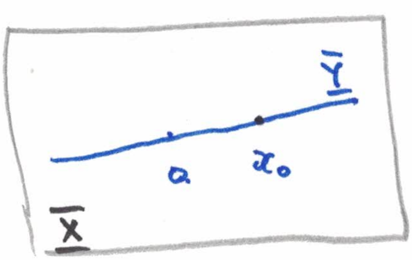
$$\|\ell\|_{\underline{X}^*} = \sup_{\substack{x \in \underline{X} \\ \|x\| = 1}} |\ell(x)|.$$

We showed: $(\underline{X}^*, \|\cdot\|_*)$ is a Banach space.

<u>Theorem</u>. Assume $(\underline{X}, \|\cdot\|)$ is a normed linear space with dual $\underline{X}^*$. For every $x_0 \in \underline{X}$ there exist $\ell \in \underline{X}^*$ such that:

$$\ell(x_0) = \|x_0\|, \qquad \|\ell\| = 1.$$

<u>Proof</u>:



let $Y = \operatorname{span}\{x_0\}$.

If $x_0 = 0 \to$
$\to$ immediate

If $x_0 \neq 0$:

$$Y = \operatorname{span}\{x_0\} = \{\alpha \cdot x_0, \ \alpha \in \mathbb{C}\}.$$

let $\lambda : Y \to \mathbb{C}$, $\begin{cases} \lambda(x_0) = \|x_0\|. \\ \lambda(\alpha \cdot x_0) = \alpha \cdot \|x_0\|. \end{cases}$

be this linear functional

By H-B there exits $\ell : \underline{X} \to \mathbb{C}$: $\ell|_Y = \lambda \to \ell(x_0) = \|x_0\|$

$\alpha \in \mathbb{C}$: $\quad |\lambda(\alpha \cdot x_0)| = |\alpha| \cdot \|x_0\| = \|\alpha \cdot x_0\|. \dashrightarrow$ For $p : \underline{X} \to \mathbb{R}$.

$p(x) =$ [redacted] $\cdot \|x\|$.

$$|\lambda(y)| \leq p(y), \ \forall y \in \underline{Y}$$

The extension

$$|\ell(x)| \leq p(x) = \|x\|$$

$$\left.\begin{array}{l} \longrightarrow \ \|\ell\| \leq 1. \\ \text{But } |\ell(x_0)| = \|x_0\| \end{array}\right\} \longrightarrow \|\ell\| = 1.$$

∎

__Corollary.__

(1) If $x_0 \neq 0$ there exists $\ell \in X^*$ s.t. $\ell(x_0) \neq 0$.

(2) If $x, y \in X$, $x \neq y$ there exists $\ell \in X^*$ s.t. $\ell(x) \neq \ell(y)$.

(3)
$$\|x\|_X = \sup_{\substack{\ell \in X^* \\ \|\ell\|_{X^*} \leq 1}} |\ell(x)| = \max_{\substack{\ell \in X^* \\ \|\ell\|_{X^*} \leq 1}} |\ell(x)| = \ell_0(x)$$

for some $\ell_0 \in X^*$, $\|\ell_0\|_{X^*} = 1$

__Proof.__

(1). – immediate.

(2). $\longrightarrow$ Consider $z = x - y \neq 0 \xrightarrow{\text{By part 1}} \exists\, \ell \in X^* : \ell(z) \neq 0$

$\longrightarrow \ell(x) \neq \ell(y)$.

(3). Take. $\ell \in X^*$, $\|\ell\|_{X^*} \leq 1$

$$\longrightarrow |\ell(x)| \leq \|\ell\|_{X^*} \cdot \|x\| \leq \|x\| \Rightarrow \sup_{\substack{\ell \in X^* \\ \|\ell\|_{X^*} \leq 1}} |\ell(x)| \leq \|x\|.$$

But, by previous theorem: $\exists\, l_0 \in X^*$, $\|l_0\|_{X^*} = 1$.

s.t. $l_0(x) = \|x\|$

$\longrightarrow$ conclusion of ~~theorem~~ corollary. $\boxtimes$

<u>Note</u>:

$$\|l\|_{X^*} = \sup_{\substack{x \in X \\ \|x\| \leq 1}} |l(x)|$$

$$\|x\|_{X} = \sup_{\substack{l \in X^* \\ \|l\|_{X^*} \leq 1}} |l(x)|.$$

<u>Embedding in Double Dual</u>.

Assume $(X, \|\cdot\|)$ is a normed linear space.

We construct $(X^*, \|\cdot\|_{X^*})$ the dual $\longrightarrow$ Banach space.

Construct: $X^{**} = (X^*)^*$, the double dual space.

$(X^{**}, \|\cdot\|_{X^{**}})$ Banach space.

$i : X \longrightarrow X^{**}$, $\quad i(x)(l) = l(x)$, $\forall x \in X$
$\forall l \in X^*$.

<u>Theorem</u>. i is an isometry. In particular,

i is injective (one-one), $\|i(x)\|_{X^{**}} = \|x\|_X$.

If X is complete then $i(X)$ is a <u>closed</u> subspace of X^{**}.

<u>Proof</u>

Take $x \in X$:

$$\|i(x)\|_{X^{**}} = \sup_{\substack{\ell \in X^* \\ \|\ell\|_{X^*} = 1}} |i(x)(\ell)| = \sup_{\substack{\ell \in X^* \\ \|\ell\|_{X^*} = 1}} |\ell(x)| = \|x\|_X$$

i linear $\longrightarrow$ If $i(x) = i(y) \Rightarrow i(x-y) = 0 \longrightarrow \|x-y\| = 0$
$\longrightarrow x = y$.
$\longrightarrow$ i is injective.

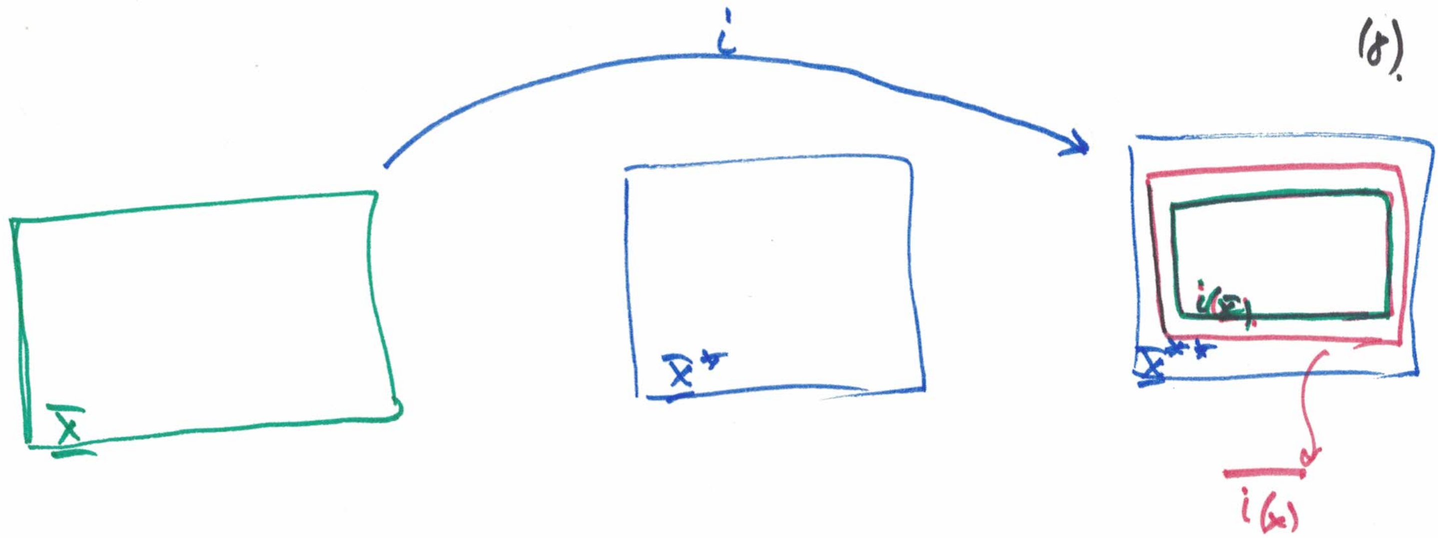
i isometry $\longrightarrow$ $\mathrm{Ran}(i) = i(X)$ is closed.

Take $(y_n)_n \in \mathrm{Ran}(i)$, $(y_n)_n$ Cauchy.

$\rightarrow \exists! \, (x_n)_n \in X$: $\|x_n - x_m\| = \|i(x_n) - i(x_m)\|$
$= \|y_n - y_m\|$.

$\longrightarrow (x_n)_n$ Cauchy in X.

X complete $\rightarrow \exists\, z = \lim_n x_n \in X \Rightarrow \lim_n y_n = i(z) \in i(X)$
$\Rightarrow i(X)$ is closed. ∎



If. $\overline{i(X)} = X^{**}$ → Then X = completion of (X) is called <u>reflexive</u>.

If $\overline{i(x)} \neq X^{**}$ then completion of X is not reflexive.

$\overline{i(X)}$ can be interpreted as a completion of X.

<u>Fact</u>. [Milman- Pettis Theorem]. Uniform convexity → Reflexivity.

<u>Definition</u>

A Banach space $(X, \|\cdot\|)$ is called reflexive if

$$i(X) = X^{**}$$

<u>Examples</u> : ℓ^p, $L^p(X)$: $1 < p < \infty$.

ℓ^p ↓ infinite dimensional.

However if $\dim X < \infty$ then $(X, \|\cdot\|)$ is always reflexive. Hence ℓ^1_n, ℓ^∞_n are reflexive but not uniformly co