Homework #1
Due: Thursday, September 13

1. (3pt) Prove that a set C is a closed set if and only if for every convergent sequence \((x_n)\) in C, its limit is also in C.

2. Let B be a subset of \(\mathbb{R}^m\). Define the closure of \(R\) by:
   \[
   \overline{R} = \{x \in \mathbb{R}^m : \forall \varepsilon > 0, \exists y \in B \text{ such that } d(x, y) < \varepsilon\}
   \]
   Prove the following:
   a) (2pt) The set \(\overline{R}\) is a closed set containing \(R\).
   b) (2pt) If \(C\) is a closed set such that \(R \subset C\), then \(\overline{R} \subset C\).
   c) (2pt) The set \(R\) is closed if and only if \(\overline{R} = R\).
   d) (2pt) \(\overline{R} = \bigcap \{C : R \subset C = \overline{C}\}\).
   e) (2pt) The closure of an open Euclidian ball is a closed Euclidian ball.

3. Let \(D\) be an open set of \(\mathbb{R}^d\) and let \(C\) be a compact set contained in \(D\). Set
   \[
   \rho = \inf \{d(x, y) : x \in C, y \notin D\}.
   \]
   (a) (4pt) Show that \(\rho > 0\). Is this true if \(C\) is just closed?
   (b) (2pt) Show that \(\{y : d(x, y) \leq \rho / 2 \text{ for some } x \in C\}\) is a compact set contained in \(D\).
   (c) (2pt) Show that \(\{y : d(x, y) < \rho / 2 \text{ for some } x \in C\}\) is an open set.

4. (4pt) Construct a sequence of rational numbers \((r_n)\) that includes every positive rational number. Modify the construction to include every rational number. Given an interval \(I\), construct a sequence of rational numbers \((r_n)\) that includes every rational number in the interval \(I\).

5. (2pt) A set \(B\) is countable provided there exists a sequence \((x_k)\) such that \(x \in B\) if and only if \(x = x_k\) for some \(k\). Show that \(\mathbb{Q}^m\) is countable.

6. (4pt) Prove that every open set is a union of Euclidian balls \(B_r(x)\) where \(r\) is rational, and the center \(x\) has rational coordinates.

7. (2pt) Let \(f : \Omega \to \mathbb{R}^n\) be a function, where \(\Omega\) is an open subset of \(\mathbb{R}^m\). Prove that \(f\) is continuous at every point of \(\Omega\) if and only if \(f^{-1}(U) = \{x \in \Omega : f(x) \in U\}\) is an open set for every open set \(U\) of \(\mathbb{R}^n\).

8. (2pt) Let \(f : \Omega \to \mathbb{R}^n\) be a function, where \(\Omega\) is an open subset of \(\mathbb{R}^m\). Prove that \(f\) is continuous at every point of \(\Omega\) if and only if \(f^{-1}(V)\) is a closed set for every closed set \(V\) of \(\mathbb{R}^n\).
9. (2pt) Let \( f : \Omega \to \mathbb{R}^n \) be a function, where \( \Omega \) is an open subset of \( \mathbb{R}^n \). Prove that \( f \) is continuous at \( x \in \Omega \) if and only if for every sequence \((x_k)_k\) converging to \( x \), the sequence \((f(x_k))_k\) converges to \( f(x) \).

10. (5pt) Give an example of a bijective and continuous function \( f : \mathbb{R} \to \mathbb{R} \) so that \( f^{-1}([0,1]) \) is not compact. (Here \( \mathbb{R} \) is the set of real numbers, but no other assumptions are made).

11. (2pt) Find \( A,B \) such that \( A|x| \leq \|x\|_\infty \leq B|x| \). For \( d=2 \) graph the sets \( \{x : |x| \leq 1\} \) and \( \{x : \|x\|_\infty \leq 1\} \).

12. Show by example that Arzela-Ascoli’s Theorem is false when each of the following hypotheses are individually deleted from the statement of the result:
   (a) (2pt) The interval \( I \) is bounded.
   (b) (2pt) The sequence \((f_m)_m\) of functions is equicontinuous.
   (c) (2pt) The sequence \( \{f_m(t)\} \) is bounded for every \( t \in I \).

Total: 50 pts