Homework #1
Due: Thursday, September 13

1. (3pt) Prove that a set C is a closed set if and only if for every convergent sequence $(x_n)_n$ in C, its limit is also in C.

2. Let B be a subset of R^m. Define the *closure* of R by:
$$\overline{R} = \{x\,; \mathrm{R} \cap \mathrm{B}_{\varepsilon}(y) \neq \varnothing, \forall \varepsilon > 0\}$$
Prove the following:
   a) (2pt) The set $\overline{R}$ is a closed set containing R.
   b) (2pt) If C is a closed set such that $R \subset C$, then $\overline{R} \subset C$.
   c) (2pt) The set R is closed if and only if $R = \overline{R}$
   d) (2pt) $\overline{R} = \bigcap \{C : R \subset C = \overline{C}\}$
   e) (2pt) The closure of an open Euclidian ball is a closed Euclidian ball.

3. Let D be an open set of R^d and let C be a compact set contained in D. Set $\rho = \inf\{|x-y| : x \in C, y \notin D\}$.
   (a) (4pt) Show that $\rho > 0$. Is this true if C is just closed?
   (b) (2pt) Show that $\{y : |x-y| \leq \rho/2 \text{ for some } x \in C\}$ is a compact set contained in D
   (c) (2pt) Show that $\{y : |x-y| < \rho/2 \text{ for some } x \in C\}$ is an open set.

4. (4pt) Construct a sequence of rational numbers $(r_n)_n$ that includes every positive rational number. Modify the construction to include every rational number. Given an interval I, construct a sequence of rational numbers $(r_n)_n$ that includes every rational number in the interval I.

5. (2pt) A set B is *countable* provided there exists a sequence $(x_k)_k$ such that $x \in B$ if any only if $x = x_k$ for some k. Show that $\mathbf{Q}^m$ is countable.

6. (4pt) Prove that every open set is a union of Euclidian balls $B_r(x)$ where r is rational, and the center x has rational coordinates.

7. (2pt) Let $f : \Omega \to \mathrm{R}^n$ be a function, where Ω is an open subset of $\mathbf{R}^m$. Prove that f is continuous at every point of Ω if and only if $f^{-1}(U) := \{x \in \Omega \mid f(x) \in U\}$ is an open set for every open set U of $\mathbf{R}^n$.

8. (2pt) Let $f : \Omega \to \mathrm{R}^n$ be a function, where Ω is an open subset of $\mathbf{R}^m$. Prove that f is continuous at every point of Ω if and only if $f^{-1}(V)$ is a closed set for every closed set V of $\mathbf{R}^n$.

9. (2pt) Let $f:\Omega \to \mathrm{R}^n$ be a function, where Ω is an open subset of $\mathbf{R}^m$. Prove that f is continuous at $x \in \Omega$ if and only if for every sequence $(x_k)_k$ converging to x, the sequence $(f(x_k))_k$ converges to f(x).

10. (5pt) Give an example of a bijective and continuous function f:R->R so that $f^{-1}([0,1])$ is not compact. (Here R is the set of real numbers, but no other assumptions are made).

11. (2pt) Find A,B such that $A|x| \le \|x\|_\infty \le B|x|$. For d=2 graph the sets $\{x : |x| \le 1\}$ and $\{x : \|x\|_\infty \le 1\}$.

12. Show by example that Arzela-Ascoli's Theorem is false when each of the following hypotheses are individually deleted from the statement of the result:
    (a) (2pt) The interval I is bounded.
    (b) (2pt) The sequence $(f_m)_m$ of functions is equicontinuous.
    (c) (2pt) The sequence $\{f_m(t)\}$ is bounded for every $t \in I$.

Total: 50 pts