THE HRT CONJECTURE FOR TWO CLASSES OF SPECIAL CONFIGURATIONS

KASSO A. OKOUDJOU AND VIGNON OUSSA

Abstract. The HRT (Heil-Ramanathan-Topiwala) conjecture stipulates that the set of any finitely many time-frequency shifts of a non-zero square Lebesgue integrable function is linearly independent. The present work settles two special cases of this conjecture, namely, the cases where the set of time-frequency shifts has cardinality $N+1$ such that either N of the points lie on some integer lattice and the last point is arbitrary, or N of the points are on a line, while the last point does not belong this line. In both cases, we prove that the HRT conjecture holds appealing mainly to various forms of the ergodic theorem. We note that, in recent years, the latter case has been the subject of many investigations–notably, the subcase where $N=3$– and our work completely resolves it.

1. Introduction

For a nonzero function $f \in L^2(\mathbb{R})$, a time-frequency shift of f is a function of the type

$$M_y T_x f(t) = e^{-2\pi i y t} f(t-x) \quad (x,y) \in \mathbb{R}^2.$$

The operator M_y is called a modulation or frequency-shift operator, and T_x is an operator acting by shifting f in time domain. Generally, modulation and translation operators do not commute, and in fact, these two families of unitary operators generate a non-commutative group called the Heisenberg group. The ubiquitous nature of the Heisenberg group and its relevance across numerous subjects within harmonic analysis is a fact that has been extensively documented in the literature [17]. For instance, the Heisenberg group is fundamental to the foundation of time-frequency or Gabor analysis and is a source of important examples in frame theory [12, 15]. For a historical development, as well as an in-depth treatment of Gabor analysis and connection with topics such as frame theory, we refer the reader to [7, 8, 12, 14].

We recall that given a countable subset Λ of $\mathbb{R}^2$, the collection of functions

$$\mathcal{G}(f,\Lambda) = \{M_y T_x f : (x,y) \in \Lambda\}$$

is called a Gabor system. For comparison, the system obtained by substituting modulation for dilation operators is often called a wavelet system. The underlying groups for these

Date: October 11, 2021.
2000 *Mathematics Subject Classification.* Primary 42C15 Secondary 42C40.
Key words and phrases. HRT conjecture, Ergodic maps, Zak transform.

systems: the Heisenberg group and the affine group, share many striking similarities, as documented in [23, 22]. However, there are also several instances in which these systems behave in drastically different ways. For example, it is well-known that there exist finite systems produced by time-dilation operators which are linearly dependent. This property guarantees the existence of a scaling function as a solution to a suitable functional equation–a central ingredient in the construction of multiresolution orthonormal wavelets [15]. However, to this date, we do not know the extent to which an analogous statement holds for finite systems of time-frequency shifts. In fact, all progress made in this direction suggests that any finite system of time-frequency shifted copies of a nonzero function is linearly independent. This conjecture, first formulated by Heil, Ramanathan, and Topiwala, is now known as the HRT Conjecture [16] and may be stated as follows.

Conjecture 1. [16] *(HRT Conjecture) Given a fixed finite set $\Lambda \subset \mathbb{R}^2$, and a nonzero function $f \in L^2(\mathbb{R})$, the associated finite Gabor system, $\mathcal{G}(f, \Lambda)$ is linearly independent.*

For the specific case where all the points lie on the same line (which is equivalent to stating that all modulation parameters are identical or all the translation parameters are identical), a straightforward application of the Fourier transform verifies the conjecture. Generally, however, the problem is much more challenging than one might initially anticipate. To this date, the most general result is attributed to Linnell [18], who was able to corroborate the conjecture for the case where Λ is a shift of a finite subset of a discrete subgroup of $\mathbb{R}^{2d}, d \in \mathbb{N}$.

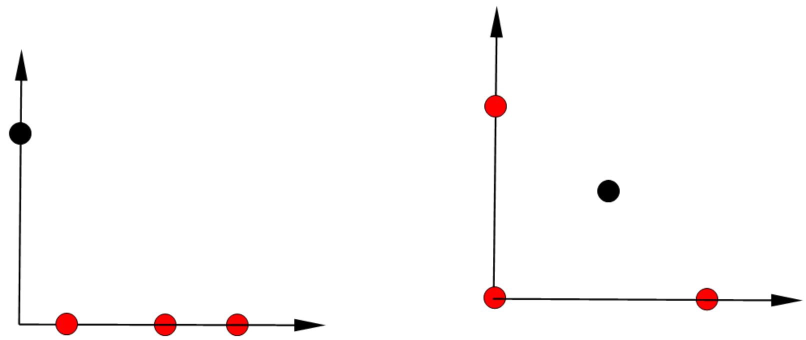
In terms of the number of points, the smallest case not handled by Linnell's result can only occur for specific configurations of a set Λ containing four points. To date, the status of the HRT conjecture for sets of four points can be summarized as follows.

Proposition 1. *Let $\Lambda \subset \mathbb{R}^2$ be such that $\#\Lambda = 4$, and let $0 \neq f \in L^2(\mathbb{R})$. The finite Gabor system $\mathcal{G}(f, \Lambda)$ is linearly independent in each of the following cases:*

(1) *Λ is $(2,2)-$configuration, that is two of the points are on a line and the other two on a parallel line, and $0 \neq f \in L^2(\mathbb{R})$ is arbitrary,* [4, 5]*;*
(2) *Λ is $13-$configuration, that is three of the points are on a line and the fourth point off that line, and $0 \neq f \in \mathcal{S}(\mathbb{R})$* [4]*;*
(3) *Λ is an arbitrary set of four points, but extra restrictions are imposed on f, e.g.,* [2, 3, 21].

Still, the general problem of the HRT conjecture for four points remains unsettled, and the main aim of this paper is to present new results, including the proof of the HRT conjecture for all $(1,3)$ configurations and all L^2 functions.

Our work is motivated by the two special cases of 4-point HRT, namely the $(1,3)$ configurations, as well as well the case in which three of the points are on the $\mathbb{Z}^2$-lattice and the fourth is arbitrary.



(A) $(1,3)$-configuration

(B) $(1,\mathbb{Z}^2)$-configuration.

FIGURE 1. Examples of four-point configurations considered in the sequel.

For example, the configuration $\{T_{x_1}f, T_{x_2}f, T_{x_3}f, Mf\}$, for arbitrary $x_1, x_2, x_3 \in \mathbb{R}$ falls into the first category, while the configuration $\{f, Mf, Tf, M_\alpha T_\beta f\}$ falls into the second, see Figure 1.

Our results will deal with generalization of these two cases to the setting of $N+1$ time-frequency shifts with $N \geq 3$. In particular, our results are summarized as follows.

Theorem 2. *Let $\{(x_k, y_k)\}_{k=1}^N \cup \{(\alpha, \beta)\} \subset \mathbb{R}^2$ be such that, either*

(i) $\{(x_k, y_k)\}_{k=1}^N \subset \mathbb{Z}^2$, and $(\alpha, \beta) \in \mathbb{R}^2 \setminus \mathbb{Z}^2$, or

(ii) $\{(x_k, y_k)\}_{k=1}^N \subset \mathbb{R}^2$ lie on a line L and $(\alpha, \beta) \in \mathbb{R}^2 \setminus L$.

If $f \in L^2(\mathbb{R})$ is such that for some nonzero complex coefficients, $c_1, \cdots, c_N$ the following linear dependency relation holds:

$$\sum_{k=1}^N c_k M_{y_k} T_{x_k} f = M_\alpha T_\beta f,$$

then f must be identically zero.

As will be seen in most of our proofs, Ergodic theory plays a critical role. This is primarily due to the structure of the configurations of interest in this work. For any time-frequency equations of the type $\sum_{k=1}^N c_k M_{y_k} T_{x_k} f = M_\alpha T_\beta f$, or $\sum_{k=1}^N c_k T_{x_k} f = Mf$, we prove that there exist unitary operators that diagonalize the left hand side of each equation. This allows us to reformulate each time-frequency equation in the form $|p(z)F(z)| = |F(z+\gamma)|$ with a varying parameter z, some polynomial p in the variable z, and some fixed parameter γ. The overarching strategy consists of step-wise iterations of $|p(z)F(z)| = |F(z+\gamma)|$ combined with suitable applications of various versions of the Ergodic Theorem and its relatives (such as the Poincaré Recurrence Theorem.) For each of these applications, we refer the interested reader to the following relevant textbooks [6, 24] for detailed treatment.

The rest of the paper is organized as follows. In Section 2, we consider the case of $(1,\mathbb{Z}^2)$ configurations, and state and prove our first main result, namely Theorem 3. In Section 3, we prove our second main result, Theorem 6, which settles the HRT conjecture for all $(1,N)$ configurations.

2. The HRT conjecture for $(1,\mathbb{Z}^2)$-configurations

A collection of $N+1$ points $\Lambda = \{(x_k,y_k)\}_{k=1}^N \cup \{(\alpha,\beta)\} \subset \mathbb{R}^2$ is a $(1,\mathbb{Z}^2)$-configuration if $\{(x_k,y_k)\}_{k=1}^N \subset \mathbb{Z}^2$ and (α,β) is arbitrary. We refer to Figure 1 for an illustration in the case $N=3$. Our first result establishes the HRT conjecture for all such configurations.

Theorem 3. *Fix $\{(x_k,y_k) : 1 \le k \le N\} \subset \mathbb{Z}^2$, and let $(\alpha,\beta) \in \mathbb{R}^2 \setminus \mathbb{Z}^2$. If there exists a function $f \in L^2(\mathbb{R})$ such that*

$$\sum_{k=1}^{N} c_k M_{y_k} T_{x_k} f = M_\alpha T_\beta f \tag{1}$$

for some nonzero complex coefficients $c_1,\cdots,c_N$, then f must be the zero function.

An immediate consequence of Theorem 3, is the follwing result which answers a sub-conjecture of C. Heil, [15, Conjecture 11.39].

Corollary 4. *For any function $0 \neq f \in L^2(\mathbb{R})$, the finite collection of time-frequency shifts $\{f, M_1 f, T_1 f, M_\alpha T_\beta f\}$ is linearly independent in $L^2(\mathbb{R})$ for any point $(\alpha,\beta) \in \mathbb{R}^2$.*

Theorem 3 can also be extended to $(1, A\mathbb{Z}^2)$-configurations where A is any 2×2 symplectic matrix, that is an invertible matrix whose determinant is 1.

Corollary 5. *Suppose that A is a 2×2 matrix with $\det A = 1$. The HRT conjecture is true for any $0 \neq f \in L^2(\mathbb{R})$ and any set of $N+1$ point of the form $\{(x_k,y_k) : 1 \le k \le N\} \cup \{(\alpha,\beta)\}$ where $\{(x_k,y_k) : 1 \le k \le N\} \subset A\mathbb{Z}^2$, and $(\alpha,\beta) \in \mathbb{R}^2$.*

Proof. When $(\alpha,\beta) \in A\mathbb{Z}^2$, we are back in the lattice case, which was already settled by Linell. Therefore, we may assume that $(\alpha,\beta) \in \mathbb{R}^2 \setminus A\mathbb{Z}^2$, and suppose that there, HRT was false for a function f and one such set of $N+1$ points. We may then use the fact that A is a symplectic matrix to find a function g (image of f under a metaplectic transformation) such that g and the set of points $\{(x_k', y_k') : 1 \le k \le N\} \cup \{(\alpha',\beta')\}$ where $\{(x_k', y_k') : 1 \le k \le N\} \subset \mathbb{Z}^2$. But this will contradict Theorem 3. $\square$

2.1. **Discussion of the proof of Theorem 3.** As we will see, the concept of convergence and divergence of infinite products will also be of central importance in our proofs. As such, we would like to review here some of the important notions needed in the sequel, and we refer to [1] for details. Let $(a_j)_{j\geq 1}$ be a sequence of nonzero real or complex numbers. If the sequence of partial products

$$\left(\prod_{j=1}^{n} a_j\right)_{n\in\mathbb{N}}$$

converges to a nonzero limit then we say that the infinite product $\prod_{j=1}^{\infty} a_j$ is convergent. Suppose a finite number of factors a_j are equal to zero, and the infinite product obtained by removing these factors converges. In that case, we say that the infinite product converges to zero. Otherwise, we define the infinite product as divergent to zero. We note that if $\prod_{j=1}^{\infty} a_j$ is convergent, then $\lim_{j\to\infty} a_j = 1$, see [1].

To prove Theorem 3, we note that any finite set of time-frequency operators parametrized by a collection of finite points on an integer lattice in $\mathbb{R}^2$, forms a collection of pairwise commuting operators. As such, the Spectral Theorem [13] guarantees the existence of a unitary operator which diagonalizes the operator

$$J = \sum_{k=1}^{N} c_k M_{y_k} T_{x_k}.$$

In our case, this unitary operator is the Zak transform $Z : L^2(\mathbb{R}) \to L^2([0,1]^2)$ defined formally, by

$$Zf\,(t,\omega) = \sum_{k\in\mathbb{Z}} f\,(t+k)\, e^{-2\pi k\omega i}. \tag{2}$$

We summarize below the main properties of the Zak transform and refer to [12] for a complete introduction to this transform.

Lemma 1. *The Zak transform Z satisfies the following properties.*

(1) Z is a unitary operator $L^2(\mathbb{R})$ onto $L^2\left([0,1)^2\right)$.

(2) For $\varphi \in L^2(\mathbb{R})$ and $(t,\omega) \in [0,1)^2$, we have:

- *(a) $[ZT\varphi](t,\omega) = e^{-2\pi\omega i} \cdot [Z\varphi](t,\omega)$.*
- *(b) $[ZM\varphi](t,\omega) = e^{-2\pi i t} \cdot [Z\varphi](t,\omega)$.*
- *(c) For each $\alpha, \beta > 0$, $[ZM_\alpha T_\beta \varphi](t,\omega) = e^{-2\pi i\alpha t} \cdot [Z\varphi](t-\beta, \omega+\alpha)$.*
- *(d) For any integer j, $[ZT\varphi](t+j,\omega) = e^{2\pi\omega j i}\,[ZT\varphi](t,\omega)$.*
- *(e) For any integer j, $[ZT\varphi](t,\omega+j) = [ZT\varphi](t,\omega)$.*

Applying Zak transform to the linear dependence equation $Jf = M_\alpha T_\beta f$, and letting $F = Zf$, for almost every $(t,w) \in [0,1]^2$, we have

$$\sum_{k=1}^{N} c_k e^{-2\pi i y_k t} e^{-2\pi i x_k \omega} \cdot F\,(t,\omega) = e^{-2\pi i\alpha t} \cdot F\,(t-\beta, \omega+\alpha)\,, \tag{3}$$

or equivalently,

$$p\left(t,\omega\right)\cdot F\left(t,\omega\right)=e^{-2\pi i\alpha t}\cdot F\left(t-\beta,\omega+\alpha\right),\tag{4}$$

where $p:\mathbb{R}^2\to\mathbb{C}$ is the non-zero trigonometric polynomial given by

$$p\left(t,\omega\right)=\sum_{k=1}^{N}c_k e^{-2\pi i y_k t}e^{-2\pi i x_k\omega}=\sum_{k=1}^{N}c_k e^{-2\pi i\langle(t,\omega),(y_k,x_k)\rangle}.\tag{5}$$

It follows that for almost every $(t,w)\in[0,1]^2$,

$$\left|p\left(t,\omega\right)\right|\cdot\left|F\left(t,\omega\right)\right|=\left|F\left(t-\beta,\omega+\alpha\right)\right|\iff\left|F(z+\gamma)\right|=|p(z)|\cdot|F(z)|,$$

where, for simplicity in the notations, we set $\gamma=(-\beta,\alpha)$ and $z=(t,\omega)$. Iterating, the last equation, we see that for almost every $z\in[0,1]^2$ and for any positive integer $n\geq 1$,

$$\left|F\left(z+n\gamma\right)\right|=\left|\prod_{j=0}^{n-1}p\left(z+j\gamma\right)\right|\cdot\left|F\left(z\right)\right|.\tag{6}$$

We let S_F be the essential support of F in $[0,1]^2$ and let $K_F=[0,1]^2\setminus S_F$ be its complement. By Lusin's theorem, there exists a set of positive measure $\Lambda\subset S_F$, such that the restriction $|F|_{|\Lambda}$ of $|F|$ onto Λ is continuous. Finally, owning to the analyticity of $|p(z)|^2$, we may assume that Λ contains no zeroes of $p(z)$.

The final tool needed in our proof is the measure-preserving map $\sigma:[0,1)^2\to[0,1)^2$ such that for any $z\in[0,1)^2$, $\sigma(z)$ is the unique element in $[0,1)^2$ which is a solution to the equation $z+\gamma\mathbb{Z}^2=\sigma(z)+\mathbb{Z}^2$. In other words, from here on, we will regard the range of σ as a connected transversal for the $\mathbb{Z}^2$-cosets in $\mathbb{R}^2$.

Observe that, (6) suggests that, at least for continuous functions, the behavior of $|F|$ at $z+n\gamma$ is fully determined by its value at a single point such as z. Thus, if for instance, F is assumed to be continuous, since p is predetermined; for any unknown function f for which the HRT conjecture is true, the restriction of $|F|$ to the set $\{z+n\gamma:n\in\mathbb{N}\}$ is completely determined by its value at a single element such as z. Even if, F is not assumed to be continuous, since the Zak transform converges almost everywhere and since (6) imposes a rather stringent condition on F, and, in principle, $Jf=M_\alpha T_\beta f$ should be incompatible with the square-integrability of nonzero f. Our main concern in this section is to fully substantiate these claims.

2.2. **Proof of Theorem 3.**

Proof of Theorem 3. The proof is twofold. First, we assume that $\{1, \alpha, \beta\}$ is linearly independent over the rationals. Secondly, we deal with the situation where this set is dependent over the rationals.

Case 1: Let us assume that $\{1, \alpha, \beta\}$ is linearly independent over the rationals. It follows that for any $z \in [0,1]^2$, the set $\{\sigma(z + n\gamma) : n \in \mathbb{N}\}$ is dense in $[0,1]^2$, and that σ defines an ergodic map on the unit square (viewed as a fundamental domain of an integer lattice). Furthermore, according to Poincaré Recurrence Theorem, for almost every $\lambda \in \Lambda$, $\sigma^n(\lambda) \in \Lambda$ for infinitely many n.

Next, assume that that both S_F (the support of F) and its complement K_F in S, have positive Lebesgue measure as subsets of the unit square. It follows then, that there exists $n \in \mathbb{N}$, such that $\sigma^n(S_F) \cap K_F$ has positive Lebesgue measure in $[0,1]^2$, which is a contradiction (see Equation (6)). Since $F \neq 0$, we conclude that $|S_F| = 1$ (or that $|K_F| = 0$), where $|E|$ denotes the Lebesgue measure of $E \subset \mathbb{R}^2$.

We now see that $n \geq 1$ and any $z \in S_F$, (6) is equivalent to

$$\left|\frac{F(\sigma^n(z))}{F(z)}\right| = \left|\prod_{j=0}^{n-1} p\left(\sigma^j(z)\right)\right|. \tag{7}$$

Let $z_0 \in S_F$ and fix $\lambda_0 \in \Lambda$. Choose a subsequence $\{\sigma^{n_k}(\lambda_0)\}_{k\geq 1} \subset \Lambda$ such that $\lim_{k\to\infty} \sigma^{n_k}(\lambda_0) = z_0$. Appealing to (7), we derive the following:

$$\left|\frac{F(z_0)}{F(\lambda_0)}\right| = \lim_{k\to\infty}\left|\frac{F(\sigma^{n_k}(\lambda_0))}{F(\lambda_0)}\right| = \lim_{k\to\infty}\left|\prod_{j=0}^{n_k-1} p\left(\sigma^j(\lambda_0)\right)\right|.$$

Moreover, applying the natural log to each side of the previous equation, the following is immediate:

$$\lim_{k\to\infty}\sum_{j=0}^{n_k-1} \ln\left|p\left(\sigma^j(\lambda_0)\right)\right| = \ln\left|\frac{F(z_0)}{F(\lambda_0)}\right| < \infty.$$

Set $s_k = \sum_{j=0}^{n_k-1} \ln\left|p\left(\sigma^j(\lambda_0)\right)\right|$ and note that $\lim_{k\to\infty} s_{k+1} - s_k = 0$, which implies that

$$\sum_{k=1}^{\infty}(s_{k+1} - s_k) = \lim_{K\to\infty}\sum_{k=1}^{K}(s_{k+1} - s_k) = \ln\left|\frac{F(z_0)}{F(\lambda_0)}\right| - s_1 < \infty.$$

Now the sequence $\{s_{k+1} - s_k\}_{k\geq 1}$ can be written as $\{\ln|p(\sigma^{m_k}(\lambda_0))|\}_{k\geq 1}$ where the increasing sequence $\{m_k\}$ is the enumeration of $\{n_k + \ell : \ell = 0, 1, 2, \ldots, n_{k+1} - 1 - n_k,\ k = 1, 2, \ldots\}$. We observe that the original sequence $\{\ln|p(\sigma^{n_k}(\lambda_0))|\}_{k\geq 1}$ is a subsequence of the new sequence

$\{\ln|p(\sigma^{m_k}(\lambda_0))|\}_{k\geq 1}$. We can then write that

$$\sum_{k=1}^{\infty} \ln|p(\sigma^{m_k}(\lambda_0))| < \infty.$$

This implies that

$$\lim_{k\to\infty} \ln|p(\sigma^{m_k}(\lambda_0))| = 0 \implies \lim_{k\to\infty} \ln|p(\sigma^{n_k}(\lambda_0))| = 0$$

and as such, $\lim_{k\to\infty}|p(\sigma^{n_k}(\lambda_0))| = |p(z_0)| = 1$.

We conclude that $|p(z)| = 1$ on $[0,1]^2$. Note however, that $|p|$ is not a constant function, since $|p|^2$ is real-analytic (a trigonometric polynomial, more precisely.) This produces the desired contradiction.

Case 2: Assume now that the set $\{1, \alpha, \beta\}$ is rationally dependent. Since the case where the parameters α, β are rationals (see [19, Proposition 6.3] and [18]) has already been settled, to complete the proof, it remains to address the subcase where either α or β is irrational. Without loss of generality, assume that $0 \neq \beta \notin \mathbb{Q}$ and write $\alpha = r + s\beta$ for some rational numbers r, s. For each $\lambda \in [0,1)^2$, let

$$L(\lambda) = \{\lambda + t\gamma \mod 1 : t \in \mathbb{R}\}$$

be the toral line passing through λ and γ. Then $L(\lambda)$ can be identified with a finite union of parallel line segments inside the two-dimensional unit square. Note that these toral lines partition the unit square $[0,1)^2$.

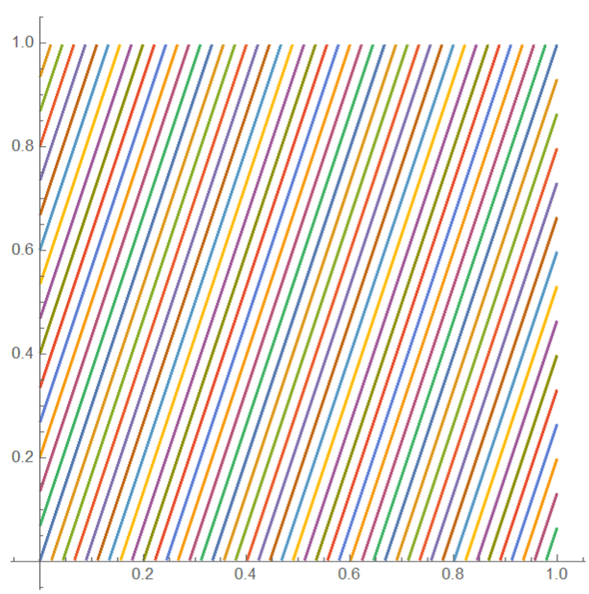


FIGURE 2. An illustration of several distinct toral lines in $[0,1)^2$

In this case, σ fails to act ergodically on the (entire) unit square. Next, we aim to show that the trigonometric polynomial p in (5) is invariant under the action of σ. Indeed, for $t \in \mathbb{R}$, let $z_t \in [0,1)^2$ be the unique element such that $\lambda + t\gamma - z_t = q \in \mathbb{Z}^2$. It follows that $\sigma(\lambda + t\gamma) = z_t$, and that

$$\begin{aligned} p(\sigma(\lambda+t\gamma)) &= \sum_{k=1}^{N} c_k e^{-2\pi i\langle \sigma(\lambda+t\gamma),(y_k,x_k)\rangle} = \sum_{k=1}^{N} c_k e^{-2\pi i\langle z_t,(y_k,x_k)\rangle} \\ &= \sum_{k=1}^{N} c_k e^{-2\pi i\langle z_t+q,(y_k,x_k)\rangle} = p(\lambda+t\gamma). \end{aligned}$$

By the continuity of p, we conclude that $p(\sigma(z)) = p(z)$ for each $z \in L(\lambda)$. In addition, since the action of σ is ergodic (with respect to a normalized arclength measure) on the toral lines $L(\lambda)$, it follows that p must be constant on each of these lines.

We proceed as in the first case and choose a set of positive measure $\Lambda \subset S_F$ such that the restriction $|F|_{|\Lambda}$ of $|F|$ onto Λ is continuous. Let $\lambda \in \Lambda$. Using the fact that σ is measure-preserving, once again, we may apply Poincaré Recurrence Theorem to assert the existence of a subsequence $(\sigma^{n_k}(\lambda))_{k\in\mathbb{N}} \subset \Lambda$ such that $\lim_{k\to\infty} \sigma^{n_k}(\lambda) = \lambda_0 \in \Lambda$. Using the same arguments as in Case 1, we conclude that

$$\left|\frac{F(\lambda_0)}{F(\lambda)}\right| = \lim_{k\to\infty}\left|\frac{F(\sigma^{n_k}(\lambda))}{F(\lambda)}\right| = \lim_{k\to\infty}\left|\prod_{j=0}^{n_k-1} p\left(\sigma^j(\lambda)\right)\right|,$$

which implies that $\lim_{k\to\infty} |p(\sigma^{n_k}(\lambda))| = |p(\lambda_0)| = 1$. We now use the fact that $(\sigma^{n_k}(\lambda))_{k\in\mathbb{N}} \subset L(\lambda)$ to conclude that $\lambda_0 \in L(\lambda)$. It follows that $|p(\lambda)| = |p(\lambda_0)| = 1$.

Since the collection of toral lines $L(\lambda)$ is a partition of the unit square (up to a set of Lebesgue measure zero) and since Λ is assumed to have positive Lebesgue measure, then $|p|^2 = 1$ on a set of positive Lebesgue measure. Hence $|p|^2$ (being real analytic), and $|p|$ must be constant everywhere on the unit square. This would imply that all but one of the coefficients $c_k s$ is nonzero. This produces a desired contradiction. □

3. The HRT for $(1, n)$-configurations

In this section, we prove the HRT conjecture for all $(1,n)-$configurations. In particular, the result settles the $(1,3)-$configurations and should be compared to the results in [4, 5, 20, 21].

Theorem 6. *Suppose that $\{x_k\}_{k=1}^{N} \subset \mathbb{R}$ is a set of N distinct points. Assume that there exists a function $f \in L^2(\mathbb{R})$ such that*

$$\sum_{k=1}^{N} c_k T_{x_k} f = Mf \tag{8}$$

for some nonzero complex coefficients $c_1, \cdots, c_N$. Then f must be the zero function.

3.1. **Discussion of the proof of Theorem 6.** Since the case where all translation parameters x_k are rational is already established [19, 18, Proposition 6.3], without loss of generality, we may assume that

$$\dim_{\mathbb{Q}}\left(\mathbb{Q}+\mathbb{Q}x_1+\cdots+\mathbb{Q}x_N\right)>1.$$

Although the rationality condition above implies that at least, one of the translation parameters must be irrational, it is perhaps worth pointing out that it is not necessarily the case that $\{1, x_1, \cdots, x_N\}$ is linearly independent over the rationals. Applying Fourier transform to each side of Equation (8) and letting F be the Fourier transform of f, it follows that

$$\left(\sum_{k=1}^{N} c_k e^{-2\pi i x_k \xi}\right)\cdot F(\xi)=F(\xi+1)\ \text{ a.e.} \tag{9}$$

Next, we consider the trigonometric polynomial

$$p(\xi)=\sum_{k=1}^{N} c_k e^{-2\pi i x_k \xi}.$$

The following will be quite convenient for the proof of our main result.

Lemma 2. *p is a matrix coefficient of an N-dimensional unitary representation $\mathbb{R}$.*

Proof. Indeed, it is not hard to verify that $p(\xi)=\left\langle e^{\xi D}c, v\right\rangle$ where

$$D=\begin{bmatrix}-2\pi i x_1 & & \\ & \ddots & \\ & & -2\pi i x_N\end{bmatrix}\ e^{\xi D}=\begin{bmatrix}e^{-2\pi i \xi x_1} & & \\ & \ddots & \\ & & e^{-2\pi i \xi x_N}\end{bmatrix}:=A.$$

Additionally

$$v=\begin{bmatrix}1\\ \vdots \\ 1\end{bmatrix}, c=\begin{bmatrix}c_1\\ \vdots \\ c_N\end{bmatrix}.$$

□

Each quantity $p(\xi)=\left\langle e^{\xi D}c, v\right\rangle$ is obtained by letting $\mathbb{R}$ acts on $\mathbb{C}^N$ via the action $(\xi, c)\mapsto e^{\xi D}c$ and this produces a family of scalars $\left(\left\langle e^{\xi D}c, v\right\rangle\right)_{\xi\in\mathbb{R}}$ obtained by projecting $e^{\xi D}c$ continuously on the function v described above. As ξ runs through the set of real numbers, $e^{\xi D}c$ traces out a curve wrapping itself densely around a torus in $\mathbb{C}^N$ passing through, c.

Observe, that generally, the countable group generated by matrix $A=e^D$ is not closed in the subspace topology of the General Linear Group which A belongs to. In what follows, let

G be the topological closure of the countable group

$$\Gamma = \{A^j = e^{jD} : j \in \mathbb{Z}\}$$

generated by A. Additionally, let H be the topological closure (as a subset of the General Linear Group) of the 1-parameter group $\left\{e^{\xi D} : \xi \in \mathbb{R}\right\}$. We refer to [11] for more background.

Lemma 3. *G is a compact abelian group which is either homeomorphic to a torus or to a finite union of translates of tori of the same positive dimension.*

Proof. By assumption, since $\dim_{\mathbb{Q}}(\mathbb{Q} + \mathbb{Q}x_1 + \cdots + \mathbb{Q}x_N) > 1$, the set $\{x_1, \cdots, x_N\}$ must contain at least one irrational number. Without loss of generality, we may also assume that $x_1 \notin \mathbb{Q}$, and additionally, $\{1, x_1, \cdots, x_m\}$ is a basis for the rational space $\mathbb{Q}x_1 + \cdots + \mathbb{Q}x_m + \mathbb{Q}$. In other words, we may rewrite the remaining parameters $x_{m+1}, \cdots, x_N$ as follows:

$$x_{m+1} = u_{m+1} + \sum_{l=1}^{m} d_{m+1,l} x_l \text{ for } (u_{m+1}, d_{m+1,1,\dots} d_{m+1,m}) \in \mathbb{Q}^{m+1}$$

$$\vdots$$

$$x_N = u_N + \sum_{l=1}^{m} d_{N,l} x_l \text{ for } (u_N, d_{N,1,\dots} d_{N,m}) \in \mathbb{Q}^{m+1}.$$

Furthermore, we also assume that every member of the set $\{1, x_1, \cdots, x_N\}$ has a rationalized denominator. Next, let L be the least common multiple of all of these denominators. For every integer j, let $j = qL + r$, that is $r = j \bmod L$. It follows that the diagonal entries of A^j where $A = e^D$ are

$$\left(e^{-2\pi i x_1(qL+r)}, \cdots, e^{-2\pi i x_m(qL+r)}, e^{-2\pi i\left(u_{m+1}+\sum_{l=1}^{m} d_{m+1,l}x_l\right)(qL+r)}, \cdots, e^{-2\pi i\left(u_N+\sum_{l=1}^{m} d_{N,l}x_l\right)(qL+r)}\right).$$

Letting $z_1 = e^{-2\pi i x_1}, \cdots, z_m = e^{-2\pi i x_m}$, we obtain $A^j = A^{qL+r} = A^r D_{(u,q,L)}$ where $D_{(u,q,L)}$ is a diagonal matrix with diagonal entries

$$z_1^{qL}, \cdots, z_m^{qL}, e^{-2\pi i(u_{m+1})(qL)} \left(\prod_{l=1}^{m} z_l^{d_{m+1,l}qL}\right), \cdots, e^{-2\pi i(u_N)(qL)} \left(\prod_{l=1}^{m} z_l^{d_{N,l}qL}\right).$$

In summary, the closure of the countable group generated by A is a topological closed subgroup of the General Linear Group which is also of the form

$$G = \bigcup_{r=0}^{L-1} A^r K,$$

where K is a connected component of the identity of the group. □

Remark 7. *In addition to Lemma 3, we also need the following facts.*

(1) Since G is a locally compact group, up to normalization, G admits a unique measure which is invariant under left translations by elements of G. Moreover, this invariant

measure on G is the sum of measures $\frac{1}{L}\sum_{k=1}^{L}\mu_k$ where each μ_k is the push-forward of the invariant measure μ_0 on the connected component K to A^kK via the translation map $g \mapsto A^k g$.

(2) Generally H and G do not coincide and since G is a closed subgroup of H, there exists a subset Z of H such that $\{zG : z \in Z\}$ is a transversal for H/G [9]. Additionally, the transversal Z admits up to normalization, a unique H-invariant measure. We endow each set zG with the following measure. For a suitable subset zE of zG, $\mu_{zG}(zE) = \mu_G(E)$ where μ_G is a fixed invariant measure on G. Additionally, we shall assume that μ_G is normalized to satisfy $\mu_G(G) = 1$.

(3) Note that the group generated by e^D is generally, not a dense subset of H. Therefore, rotation by e^D is generally not an ergodic map on H. However, e^D does act ergodically on every coset of G in H. Precisely, for each fixed z in a transversal Z, of H/G, G acts on zG as follows. Given $g \in G$ and $zh \in zG$, $gzh = zgh = z(gh) \in zG$. Since the countable subgroup $\left\{e^{jD} : j \in \mathbb{Z}\right\}$ is dense in G, for every fixed $z \in Z$ where $Z \subseteq H$ is a measurable transversal for H/G, the map $\sigma : zG \to zG$ defined by $\sigma(zh) = e^D zh = ze^D h$ is an ergodic map with respect to the measure μ_{zG}.

We are now ready to prove Theorem 6.

3.2. **Proof of Theorem 6.**

Proof. Let S be the support of F and assume that S does not contain any zero of the polynomial p (this is possible since the set of these zeroes is at most countable.) For each $\xi \in S$, we can iterate (9) so that for any positive integer n, the following holds true:

$$|F(\xi+n)| = \left(\prod_{j=0}^{n-1}|p(\xi+j)|\right)|F(\xi)| \text{ a.e.} \tag{10}$$

and additionally

$$|F(\xi-n)| = \left(\prod_{j=1}^{n}|p(\xi-j)|\right)^{-1}|F(\xi)| \text{ a.e.} \tag{11}$$

Since F is square-integrable,

$$\infty > \int_S |F(\xi)|^2\, d\xi = \int_{S\cap[0,1]}\sum_{k=-\infty}^{\infty}|F(\xi+k)|^2\, d\xi.$$

This implies that for a.e. $x \in S\cap[0,1]$, we have

$$\sum_{k=-\infty}^{\infty}|F(\xi+k)|^2 = |F(\xi)|^2 + \sum_{k=1}^{\infty}\left(|F(\xi+k)|^2 + |F(\xi-k)|^2\right) < \infty.$$

Appealing to (10) and (11), we derive the following

$$\sum_{k=-\infty}^{\infty} |F(\xi+k)|^2 = |F(\xi)|^2 \cdot \left(1 + \sum_{k=1}^{\infty} \left(\prod_{j=0}^{k-1} |p(\xi+j)|^2 + \prod_{j=1}^{k} |p(\xi-j)|^{-2} \right) \right) < \infty. \tag{12}$$

By Egorov's theorem, there exists a set $\Lambda \subset S \cap [0,1]$ of positive Lebesgue measure such that

$$\lim_{k\to\infty} \prod_{j=0}^{k-1} |p(\xi+j)|^2 = \prod_{j=0}^{\infty} |p(\xi+j)|^2 = \lim_{k\to\infty} \prod_{j=1}^{k} |p(\xi-j)|^2 = \prod_{j=1}^{\infty} |p(\xi-j)|^{-2} = 0$$

uniformly on Λ.

Moving forward, formally define functions P, Q on set the $e^{\Lambda D}$ such that

$$P\left(e^{\xi D}\right) = \prod_{j=0}^{\infty} |p(\xi+j)| \text{ and } Q\left(e^{\xi D}\right) = \prod_{j=1}^{\infty} |p(\xi-j)|^{-1}.$$

Since $X = \overline{e^{\Lambda D}}$ has positive measure in H, we may extend the definition of P and Q to X such that $Q = P = 0$ on X. Let Z be a measurable transversal for H/G in G. Since X has positive μ_H-measure, there exists a unique H-invariant measure μ_Z [11, Theorem 2.51] on Z such that

$$\mu_H(X) = \int_H 1_X(h)\, d\mu_H(h) = \int_Z \int_G 1_X(zg)\, d\mu_G(g)\, d\mu_Z(z) > 0. \tag{13}$$

Equation (13) gives a disintegration of the Haar measure on H into a family of measures, parameterized by the transversal Z and defined over the cosets of G. Additionally, by virtue of (13), there exists at least one coset zG of G such that $zG \cap X$ has positive measure in zG. Fixing such a coset, the matrix $h \in zG \cap X$, viewed as an element of H may be factored uniquely such that $h = zg$, $z \in Z$ and additionally,

$$P(h) = P(zg) = \prod_{j=0}^{\infty} \left|\left\langle zge^{jD}c, v\right\rangle\right| = Q(h) = Q(zg) = \prod_{j=1}^{\infty} \left|\left\langle zge^{-jD}c, v\right\rangle\right|^{-1} = 0$$

for all g in a set Ω of positive measure in G.

Fix $\epsilon \in (0,1)$. Choose a natural number L, sufficiently large to ensure that for all $n > L$, and $zg \in z\Omega$,

$$\begin{cases} \prod_{j=0}^{n-1} \left|\left\langle zge^{jD}c, v\right\rangle\right| < \epsilon \\ \prod_{j=1}^{n} \left|\left\langle zge^{-jD}c, v\right\rangle\right|^{-1} < \epsilon. \end{cases}$$

Since for $z \in Z$, the map $\sigma : zG \to zG$ defined by $\sigma(z_1 g_1) = e^D z_1 g_1$ is ergodic, and by the Ergodic theorem, there exists $n > L$ such that $\mu_{zG}(\sigma^n(z\Omega) \cap z\Omega) > 0$. For $g \in \sigma^n(z\Omega) \cap z\Omega$, we may write $g = z\omega$ and

$$\prod_{j=1}^{n} \left|\left\langle z\omega e^{-jD}c, v\right\rangle\right|^{-1} = \prod_{j=1}^{n} \left|\left\langle g e^{-jD}c, v\right\rangle\right|^{-1} < \epsilon$$

for $n > L$. Similarly, since $\sigma^{-n}(g) \in z\Omega$, the following inequality is satisfied:

$$\prod_{j=0}^{n-1} \left|\left\langle \sigma^{-n}(g) e^{jD}c, v\right\rangle\right| = \prod_{j=0}^{n-1} \left|\left\langle g e^{(j-n)D}c, v\right\rangle\right| < \epsilon.$$

The change of variables $\ell = j - n$ yields

$$\prod_{\ell=-n}^{-1} \left|\left\langle g e^{\ell D}c, v\right\rangle\right| = \prod_{\ell=1}^{n} \left|\left\langle g e^{-\ell D}c, v\right\rangle\right| < \epsilon.$$

In summary, we have found $n > L$ such that

$$\begin{cases} \prod_{j=1}^{n} \left|\left\langle g e^{-jD}c, v\right\rangle\right|^{-1} < \epsilon \\ \prod_{j=1}^{n} \left|\left\langle g e^{-jD}c, v\right\rangle\right| < \epsilon, \end{cases}$$

which is a contradiction, since $0 < \epsilon < 1$. □

Remark 8.

(a) *We say that two configurations Λ_1, Λ_2 of n distinct points in $\mathbb{R}^2$ are equivalent if and only if there exists a symplectic matrix $A \in SL(2,\mathbb{R})$ such that $\Lambda_2 = A\Lambda_1$. This relation defines an equivalence relation on the set of all n-point configurations. As a result, we can partition the collection of all sets of n distinct points in $\mathbb{R}^2$ under this equivalence relation. Therefore, when we choose an n-point configuration, we may assume without any loss of generality that it is a representative for a corresponding equivalence class. Furthermore, when $n \geq 3$ we may assume that the chosen (representative) n-point configuration includes the set $\{(0,0),(0,1),(a,0)\}$ where $a \neq 0$,* [10, 12, 16].

(b) *We now consider the case where $n = 4$. By Theorem 6, the assumption $g \in \mathcal{S}(\mathbb{R})$ can be removed from part (2) of Proposition 1. Consequently, to settle the conjecture for sets of 4 points, one needs to consider configurations that are equivalent to $\Lambda_g = \{(0,0),(0,1),(a,0),(\alpha,\beta)\}$ that are neither $(1,3)-$configuration, $(2,2)-$configuration, nor $(1,\mathbb{Z}^2)-$configurations. Let $0 \neq g \in L^2(\mathbb{R})$ and suppose that Λ_g is the collection of all 4-point configurations for which the HRT conjecture fails for this function g. The ultimate goal is to prove that such collection is empty. Thus far, we can only say that it does not contain any $(1,3)-$configuration, $(2,2)-$configuration, nor a $(1,\mathbb{Z}^2)-$configuration. A question that has not received much attention is to understand the* *structures of Λ_g. For example, is it true that if $g_1 \neq g_2 \in L^2(\mathbb{R})$, then Λ_{g_1} and Λ_{g_2} are disjoint? In particular, suppose that $\mathcal{G}(g, \Lambda_g)$ is linearly dependent, and let Λ' be equivalent to Λ_g, then is it the case that $\mathcal{G}(g, \Lambda')$ is also linearly dependent?*

Acknowledgment

K.A.O. was partially supported by a grant from the National Science Foundation grant DMS 1814253.

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Department of Mathematics, Tufts University, Medford MA 02131, USA
Email address: `Kasso.Okoudjou@tufts.edu`

Department of Mathematics, Bridgewater State University, Bridgewater, MA 0235, USA
Email address: `vignon.oussa@bridgew.edu`