PROOF OF THE HRT CONJECTURE FOR (2,2) CONFIGURATIONS

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ABSTRACT. We prove that for any 4 points in a (2-2) configuration, there is no linear dependence between the associated time-frequency translates of any $L^2(\mathbb{R})$ function.

1. INTRODUCTION

The following conjecture, known as the HRT conjecture appears in [4]. See also [5] for an ample discussion on the subject.

Conjecture 1.1. *Let $(t_j,\xi_j)_{j=1}^n$ be $n \geq 2$ distinct points in the plane. Then there is no nontrivial L^2 function $f:\mathbb{R}\to\mathbb{C}$ satisfying a nontrivial linear dependence*

$$\sum_{j=1}^n d_i f(x+t_j)e^{2\pi i \xi_j x}=0,$$

for a.e. $x\in\mathbb{R}$.

The conjecture follows trivially when the points $(t_j,\xi_j)_{j=1}^n$ are collinear. The conjecture was proved when $(t_i,\xi_i)_{i=1}^n$ sit on a lattice, [6], using von Neumann algebras techniques. See also [1], [3], for more elementary alternative arguments. In particular, this is the case with any 3 points. But the question whether the conjecture holds for *arbitrary* 4 points is open. Progress on that has been made by the first author in [2] using a number theoretical approach, and we briefly discuss it below.

We will call an $(2,2)$ configuration, any collection of 4 distinct points in the plane, such that there exist 2 distinct parallel lines each of which containing 2 of the points. One of the results in [2] is

Theorem 1.2. *Conjecture 1.1 holds for special $(2,2)$ configurations $(0,0),(1,0),(0,\alpha),(1,\beta)$*

(a) if

$$\liminf_{n\to\infty} n\log n \min\{\|n\frac{\beta}{\alpha}\|,\|n\frac{\alpha}{\beta}\|\}<\infty$$

(b) if at least one of α,β is rational

In either case, no nontrivial solution f can exist satisfying minimal decay

$$\lim_{\substack{|n|\to\infty\\ n\in\mathbb{Z}}} |f(x+n)|=0, \quad a.e.\ x$$

In this paper we prove the strongest possible statement about (2,2) configurations, namely

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Theorem 1.3. *Conjecture 1.1 holds for all* $(2,2)$ *configurations. Moreover, when the points sit in a special (2,2) configuration* $(0,0),(1,0),(0,\alpha),(1,\beta)$*, no nontrivial solution* f *can exist satisfying minimal decay*

$$\lim_{\substack{|n|\to\infty\\ n\in\mathbb{Z}}} |f(x+n)| = 0, \quad a.e.\ x$$

The general approach for proving this theorem is the one developed in [2]. We first reduce to the case of special configurations, by applying metaplectic transformations. Then we turn the hypothetical linear dependence into a recurrence. The contribution from β is estimated by using the *conjugates trick*. The novelty of our approach here is in the way we treat the contribution coming from the terms containing α. In particular, we exploit the Diophantine behavior of α at more than one scale.

2. Proof of the main theorem

Define $[x]$, $\{x\}$, $\|x\|$ to be the integer part, the fractional part and the distance to the nearest integer of x. For two quantities A, B that vary, we will denote by $A \lesssim B$ or $A = O(B)$ the fact that $A \le CB$ for some universal constant C, independent of A and B. In general, $A \lesssim_p B$ means that the implicit constant is allowed to depend on the parameter p. The notation $A \sim_p B$ means that $A \lesssim_p B$ and $B \lesssim_p A$. If no parameter is specified, the implicit constants are implicitly understood to depend on the (harmless) fundamental parameters introduced in the proof of Theorem 1.3. For a set $A \subset \mathbb{R}$, we will denote by $|A|$ its Lebesgue measure, and if the set is finite, $|A|$ will represent its cardinality. Finally, we define $e(x) := e^{2\pi i x}$.

Let $0 < \alpha < 1$ be irrational. Let $\frac{p_k}{N_k}$ be the k^{th} convergent of α, so that

$$|\alpha - \frac{p_k}{N_k}| \le \frac{1}{N_k N_{k+1}}, \tag{1}$$

and

$$p_k N_{k-1} - p_{k-1} N_k = (-1)^{k-1} \tag{2}$$

Since

$$N_k \le N_{k+1},$$

there exists an infinite set $E \subset \mathbb{N}$ and a constant $D = D(\alpha)$ such that for each $k \in E$ we have

$$\frac{N_k}{N_{k+1}} \le D \min_{j\le k} \frac{N_j}{N_{j+1}}. \tag{3}$$

Define $\frac{1}{M_k} := N_k^2 |\alpha - \frac{p_k}{N_k}|$. Of course, $M_k \ge 1$ for each k.

The following proposition is the main new ingredient in this paper.

Proposition 2.1. *Let* $k \in E$ *be odd, and* $0 < \delta < \frac{1}{100}$*. Define* $N := N_k$*,* $p := p_k$*,* $M := M_k$*. Then, for each* $x \in [0,1]$ *such that*

$$\min\{\frac{\|x\|}{N}, \|x - n\alpha\|, \|x - \frac{n}{N}\| : 1 \le n \le N\} \ge \frac{\delta}{N} \tag{4}$$

we have

$$\prod_{n=1}^{N} |e(x) - e(\alpha n)| \sim_\delta 1. \tag{5}$$

Remark 2.2. The key thing in (5) is that the similarity constant does not depend on N.

Proof Fix x satisfying (4). We will compare $\prod_{n=1}^{N} |e(x) - e(\alpha n)|$ to

$$\prod_{n=1}^{N} |e(x) - e(\frac{np}{N})| = \prod_{n=1}^{N} |e(x) - e(\frac{n}{N})| = |e(Nx) - 1| \sim_\delta 1,$$

and prove that their ratio is $\sim_\delta 1$. This is reasonable to expect, since, due to (1), we have for each $1 \le n \le N$

$$|n\alpha - \frac{np}{N}| \le \frac{1}{N} \tag{6}$$

First, let $1 \le n_1, n_2, \dots, n_{200} \le N$ be such that

$$\|x - \frac{n_i p}{N}\| \le \frac{100}{N}$$

Due to (4) and (6), we get that

$$\delta^3 \lesssim \prod_{i=1}^{200} \frac{|e(x) - e(\alpha n_i)|}{|e(x) - e(\frac{n_i p}{N})|} \lesssim \delta^{-1}. \tag{7}$$

Next, we analyze

$$\prod_{\substack{n=1\\ n\neq n_i}}^{N} \frac{|e(x) - e(\alpha n)|}{|e(x) - e(\frac{np}{N})|}.$$

Note that

$$\frac{|e(x) - e(\alpha n)|}{|e(x) - e(\frac{np}{N})|} = \left|1 + \frac{1 - e(\alpha n - \frac{np}{N})}{e(x - \frac{np}{N}) - 1}\right|,$$

and that

$$\left|\frac{1 - e(\alpha n - \frac{np}{N})}{e(x - \frac{np}{N}) - 1}\right| \le \frac{10}{N\|x - \frac{np}{N}\|} < \frac{1}{2}.$$

Thus,

$$\sum_{\substack{n=1\\ n\neq n_i\\ \|x-\frac{np}{N}\|\ge\delta}}^{N} \frac{1}{N\|x - \frac{np}{N}\|} \lesssim \sum_{N\delta \le i \le N} \frac{1}{i} \lesssim \log(\delta^{-1}).$$

Using this and the fact that

$$1 + x \le e^x, \quad 0 < x < 1$$
$$e^{-10x} \le 1 - x, \quad 0 < x < 1/2,$$

we get

$$\prod_{\substack{n=1\\ n\neq n_i\\ \|x-\frac{np}{N}\|\ge\delta}}^{N} \left|1 + \frac{1 - e(\alpha n - \frac{np}{N})}{e(x - \frac{np}{N}) - 1}\right| \sim_\delta 1 \tag{8}$$

Denote by

$$A := \{1 \le n \le N : n \ne n_i, \ \ \|x - \frac{np}{N}\| < \delta\}$$

Using the fact that for $z \in \mathbb{R}$ with $|z| < \frac{1}{10}$

$$1/2 \le \frac{|e(z)-1|}{2\pi|z|} < 2,$$

we get for each $n \in A$

$$\left|\frac{1-e(\alpha n - \frac{np}{N})}{e(x-\frac{np}{N})-1}\right| < \frac{\frac{10n}{N^2M}}{\frac{100}{N}} < \frac{1}{10}.$$

It is easy to check that for each $z \in \mathbb{C}$ with $|z| < \frac{1}{10}$ we have

$$e^{-O(|z|^2)} \le \left|\frac{1+z}{e^z}\right| \le e^{O(|z|^2)}.$$

Apply this inequality to each $z_n := \frac{1-e(\alpha n - \frac{np}{N})}{e(x-\frac{np}{N})-1}$. We have seen that $|z_n| \lesssim \frac{1}{N\|x-\frac{np}{N}\|}$, and hence

$$\sum_{n\in A}|z_n|^2 \lesssim 1.$$

It follows that

$$\prod_{n\in A}\left|1+\frac{1-e(\alpha n - \frac{np}{N})}{e(x-\frac{np}{N})-1}\right| \sim \left|e^{\sum_{n\in A}\frac{1-e(\alpha n - \frac{np}{N})}{e(x-\frac{np}{N})-1}}\right|.$$

Let $\alpha - \frac{p}{N} := \frac{t}{N^2}$, so $M|t| = 1$. Note that since $\|x\| \ge \delta$, it follows that

$$|x - \{\frac{np}{N}\}| < \frac{1}{2} \tag{9}$$

for each $n \in A$. By invoking Taylor expansions, (9), and using that

$$|\frac{1}{e(y)-1} - \frac{1}{2\pi i y}| \lesssim 1$$

for $|y| < \frac{1}{2}$, we get that

$$\sum_{n\in A}\frac{1-e(\alpha n - \frac{np}{N})}{e(x-\frac{np}{N})-1} = -\sum_{n\in A}\frac{tn}{N^2(x-\{\frac{np}{N}\})} + O(1).$$

We rewrite

$$\sum_{n\in A}\frac{tn}{N^2(x-\{\frac{np}{N}\})} = t\sum_{\substack{n=1\\ \delta\ge|x-\frac{n}{N}|\ge\frac{100}{N}}}^{N}\frac{\frac{n^*}{N}}{(Nx-n)},$$

where $n^* := p^{-1}n \mod N$, and p^{-1} is the inverse of p mod N. Our next goal is to prove that

$$\frac{1}{M}\left|\sum_{\substack{n=1\\ \delta\ge|x-\frac{n}{N}|\ge\frac{100}{N}}}^{N}\frac{\frac{n^*}{N}}{(Nx-n)}\right| = O(1). \tag{10}$$

Since k is odd, it follows from (2) that $p^{-1} = N_{k-1}$. Let

$$\alpha = \langle a_0, a_1, \dots \rangle := a_0 + \frac{1}{a_1 + \dots}$$

be the continued fraction expansion of α. We have for each $i \geq 2$

$$p_i = a_i p_{i-1} + p_{i-2},$$
$$N_i = a_i N_{i-1} + N_{i-2},\ N_0 = 1,\ N_1 = a_1.$$

Due to (3) we have $a_i \leq DM$ for each $i \leq k+1$.

Note that $\rho_i := N_i/N_{i-1}$ satisfies

$$\rho_i = a_i + \frac{1}{\rho_{i-1}},\ \rho_1 = a_1.$$

Thus,

$$N/p^{-1} = N_k/N_{k-1} = \langle a_k, a_{k-1}, \dots, a_1 \rangle.$$

The thing that matters is that all a_i are $O(M)$. Thus, from the recurrence above, the convergents of N/p^{-1}, denote them by M_l/c_l, have the property that

$$M_{l+1} \lesssim M M_l \tag{11}$$

for each $l \leq k$ (and similarly for c_l, but this will be irrelevant).

It is known that the l^{th} convergent of p^{-1}/N will equal $\frac{c_{l-1}}{M_{l-1}}$, and that the last convergent will equal p^{-1}/N. Choose l_0 such that $\frac{N\delta}{M^{3/2}} \lesssim M_{l_0}^{3/2} < N\delta$. This is possible due to (11). Reasoning as before, we get

$$\frac{1}{M} \Big| \sum_{\substack{n=1 \\ |x - \frac{n}{N}| \geq \frac{M_{l_0}^{3/2}}{N}}}^{N} \frac{\frac{n^*}{N}}{(Nx-n)} \Big| \lesssim \frac{1}{M} \sum_{N \geq i \gtrsim M_{l_0}^{3/2}} \frac{1}{i} \lesssim \frac{\log M + \log(\delta^{-1})}{M} \lesssim_\delta 1.$$

Next, we observe that the remaining part of the sum can be written as

$$\frac{1}{M} \sum_{|j| < M_{l_0}^{3/2}} \frac{\{u + \frac{N_{k-1} j}{N}\}}{j} + O(1),$$

where u is a number whose value is completely irrelevant.

Note that if, say, $M^5 > N$ then the sum above is trivially bounded by $\frac{1}{M} \sum_{|j| < M^5} \frac{1}{|j|} = O(1)$, and we are fine. Otherwise, we can choose $l_1 < l_0$ such that $M^4 \lesssim M_{l_1} < M^5$. The sum above restricted to $|j| \leq M_{l_1}^{3/2}$ is trivially $O(1)$.

For $l_1 \leq l \leq l_0 - 1$ and $M_l^{3/2} \leq |j| \leq M_{l+1}^{3/2}$, we use that

$$\Big| \frac{N_{k-1}}{N} - \frac{c_l}{M_l} \Big| \leq \frac{1}{M_l M_{l+1}},$$

and thus by (11)

$$\Big| \frac{N_{k-1} j}{N} - \frac{c_l j}{M_l} \Big| \leq \frac{M_{l+1}^{3/2}}{M_l M_{l+1}} \lesssim M^{1/2} M_l^{-1/2}.$$

Define

$$C_l := \{ M_l^{3/2} \leq |j| \leq M_{l+1}^{3/2} : \|u + \frac{c_l j}{M_l}\| \gtrsim M^{1/2} M_l^{-1/2} \}.$$

It follows that

$$|\{M_l^{3/2} \le |j| \le M_{l+1}^{3/2}\} \setminus C_l| \le |\{|j| \le M_{l+1}^{3/2} : \|u + \frac{c_l j}{M_l}\| \lesssim M^{1/2} M_l^{-1/2}\}|$$

$$\lesssim M_{l+1}^{3/2} M^{1/2} M_l^{-1/2},$$

and that for each $j \in C_l$

$$|\{u + \frac{N_{k-1} j}{N}\} - \{u + \frac{c_l j}{M_l}\}| \lesssim M^{1/2} M_l^{-1/2}.$$

So we have the following estimate for the error term corresponding to some l

$$\left| \sum_{M_l^{3/2} < |j| < M_{l+1}^{3/2}} \frac{\{u + \frac{N_{k-1} j}{N}\}}{j} - \sum_{M_l^{3/2} < |j| < M_{l+1}^{3/2}} \frac{\{u + \frac{c_l j}{M_l}\}}{j} \right|$$

$$\lesssim M^{1/2} M_l^{-1/2} \sum_{j \in C_l} \frac{1}{|j|} + \sum_{\substack{M_l^{3/2} \le |j| \le M_{l+1}^{3/2} \\ j \notin C_l}} \frac{1}{|j|} \lesssim M^2 M_l^{-1/2}.$$

Since for each i

$$M_i \ge M_{i-1} + M_{i-2} \ge 2M_{i-2}, \tag{12}$$

and since $M_{l_1} \gtrsim M^4$ it follows that the sum of all error terms is bounded by

$$\sum_{l_1 \le l} \frac{M^2}{M_l^{1/2}} \lesssim 1$$

as desired. But

$$\sum_{M_l^{3/2} < |j| < M_{l+1}^{3/2}} \frac{\{u + \frac{c_l j}{M_l}\}}{j} = \sum_{r=1}^{M_l} \{u + \frac{c_l r}{M_l}\} \sum_{\substack{M_l^{3/2} < |j| < M_{l+1}^{3/2} \\ j = r \bmod M_l}} \frac{1}{j},$$

and this is $O(\frac{1}{M_l^{1/2}})$, since actually

$$\sup_{P > M_l^{3/2}} |\sum_{\substack{M_l^{3/2} < |j| \le P \\ j = r \bmod M_l}} \frac{1}{j}| = O(\frac{1}{M_l^{3/2}})$$

for each r. Summing over $l \ge l_1$ we get using (12)

$$\sum_{l_0 - 1 \ge l \ge l_1} \left| \sum_{M_l^{3/2} < |k| < M_{l+1}^{3/2}} \frac{\{u + \frac{c_l k}{M_l}\}}{k} \right| \lesssim 1.$$

By putting everything together we conclude that (10) holds. ■

An immediate consequence which only requires trivial modifications is the following.

Corollary 2.3. *Let $A, B \in \mathbb{C}$ with $|A| = |B| = 1$. Let also α and N be as in Proposition 2.1. Define*

$$P(x) = A + Be(\alpha x).$$

Then for each $0 < \epsilon < 1$ there exist $c_1(\epsilon, A, B, \alpha), c_2(\epsilon, A, B, \alpha) > 0$ and a set $P(A, B, \epsilon, \alpha, N) \subset [0, 1]$ with measure at least $1 - \epsilon$ such that for each $y \in P(A, B, \epsilon, \alpha, N)$

$$c_2(\epsilon, A, B, \alpha) \geq \prod_{n=-N}^{-1} |P(y + n)| \geq c_1(\epsilon, A, B, \alpha)$$

$$c_2(\epsilon, A, B, \alpha) \geq \prod_{n=0}^{N-1} |P(y + n)| \geq c_1(\epsilon, A, B, \alpha).$$

The relevance of this result for later applications is that while the sets P are allowed to depend on N, the constants c_1, c_2 do not depend on N.

We can now begin the proof of Theorem 1.3. By applying the area preserving affine transformations -also called *metaplectic transforms*- of the plane (such as translations, rotations, shears, and area one rescalings), it suffices to rule out minimal decay (14) for special configurations. See Section 2 in [4] for a discussion on this.

Assume for contradiction that there exists a measurable function $f : \mathbb{R} \to \mathbb{C}$, some $d \in (0, \infty)$ and some $S \subset [0, 1]$ with positive measure such that

$$d < |f(x)| < \infty \text{ for each } x \in S, \tag{13}$$

$$\lim_{\substack{|n|\to\infty \\ n\in\mathbb{Z}}} f(x + n) = 0, \tag{14}$$

and

$$f(x + 1)(A + Be(\alpha x)) = f(x)(E + Fe(\beta x)),$$

for a.e. x, for some fixed $A, B, E, F \in \mathbb{C}$, $\alpha, \beta \in \mathbb{R}$, none of them zero. We can also assume α and β to be irrational, since the rational case was treated in [2]. The same metaplectic transforms allow us to assume $0 < \alpha < 1$. By re-normalizing, we can trivially assume $E = 1$. Let

$$P(x) = A + Be(\alpha x), \quad Q(x) = 1 + Fe(\beta x).$$

Also, the argument from [2] shows that the worst case scenario (and the only one that needs to be considered here) is when $|B| = |A|$. Equivalently, P will have zeros. We comment on this in the end of the argument.

By making S a bit smaller, we can also assume that $S + \mathbb{Z}$ contains no zeros of P and Q.

Note that by Egoroff's Theorem, (14) will allow us to assume (by making S a bit smaller if necessary) that

$$\lim_{\substack{|n|\to\infty \\ n\in\mathbb{Z}}} f(x + n) = 0, \tag{15}$$

uniformly on S.

The parameters $D, \alpha, \beta, A, B, F, \epsilon_1, \epsilon_2, \epsilon_3, c_1, c_2, d, m, \gamma$ (some of which are introduced below) will be referred to as *fundamental parameters*. They will stay fixed throughout the argument, and in particular will not vary with N.

Let us first see how to deal with the contribution coming from the polynomials Q. This is done via the *conjugates trick* introduced in [2]. More precisely, let $F = e(\theta)$. Since S has positive measure, it follows that $1_S * 1_S$ is continuous and that there exists an interval $I \subset [0, 2]$ and $\epsilon_1 > 0$ such that

$$1_S * 1_S(w) > \epsilon_1 \tag{16}$$

for each $w \in I$. We can assume without any loss of generality that $I \subset [0, 1]$. There exists $n' \in \mathbb{N}$ large enough such that $m := [-\frac{2\theta}{\beta} + n'\beta^{-1}] > 0$ and $\gamma := \{-\frac{2\theta}{\beta} + n'\beta^{-1}\} \in I$. It follows from (16) that the set $S' := \{x \in S : \gamma - x \in S\}$ has measure at least ϵ_1. The point of this selection is that for each $n \in \mathbb{Z}$, and each $y := -x - \frac{2\theta}{\beta} + n'\beta^{-1}$, the numbers $1 + Fe(\beta y - n\beta)$ and $1 + Fe(\beta x + n\beta)$ are complex conjugates and thus, for each $L \geq 1$, and each $x \in \mathbb{R}$

$$\prod_{n=-L}^{-1} |Q(\gamma - x + n)| = \prod_{n=m+1}^{L+m+1} |Q(x + n)|. \tag{17}$$

Let S'' be a subset of S' of measure at least $\epsilon_1/2$, and let $\epsilon_2 > 0$ depending only on the fundamental parameters β, F and m such that

$$\prod_{n=0}^{m} |Q(x + n)| \geq \epsilon_2 \tag{18}$$

for each $x \in S''$. Let N be as in Corollary 2.3. Let $\epsilon_3 > 0$ be small enough (depending only on ϵ_1, in particular not depending on N) such that the set

$$S(N) := S'' \cap \{x \in P(A, B, \epsilon_3, \alpha, N)\} \cap \{x : \gamma - x \in P(A, B, \epsilon_3, \alpha, N)\},$$

has positive measure, and thus is non-empty. For each N as above, choose a point $x_N \in S(N)$. Let $z_N := \gamma - x_N$. The recurrence along the orbits of x_N and z_N implies that

$$|f(x_N + N + m + 2)| = |f(x_N)| \frac{\prod_{n=0}^{N+m+1} |Q(x_N + n)|}{\prod_{n=0}^{N+m+1} |P(x_N + n)|}$$

$$|f(z_N - N)| = |f(z_N)| \frac{\prod_{-N}^{n=-1} |P(z_N + n)|}{\prod_{-N}^{n=-1} |Q(z_N + n)|}.$$

Multiply these equalities. Using the fact that x_N, z_N are in S, (13), (17) with $x := x_N$ and $L := N$, (18) with $x := x_N$, Corollary 2.3 and the fact that

$$\prod_{n=N}^{N+m+1} |P(x_N + n)| \leq (2|A|)^{m+2},$$

it follows that

$$|f(x_N + N + m + 2)||f(z_N - N)| \geq \frac{d^2 \epsilon_2 c_1(\epsilon_3, A, B, \alpha)}{(2|A|)^{m+2} c_2(\epsilon_3, A, B, \alpha)}.$$

The important thing is that the constant on the right depends only on the fundamental parameters, and not on N. By letting $N \to \infty$, this will contradict the uniformity assumption (15). This ends the proof of Theorem 1.3, under the assumption that $|A| = |B|$.

If $|A| \neq |B|$, then things are much easier, and have already been addressed in [2]. We briefly recap the argument. By invoking Riemann sums and the fact that the derivative of $\phi(x) := \ln|A + Be(x)|$ satisfies

$$\inf_{x\in[0,1]} |\phi'(x)| \gtrsim_{A,B} 1,$$

we get that

$$|\sum_{n=0}^{N-1} \ln|P(x+n)| - N\int_0^1 \phi| \lesssim_{A,B} 1$$
$$|\sum_{n=-N}^{-1} \ln|P(x+n)| - N\int_0^1 \phi| \lesssim_{A,B} 1,$$

for each $x \in [0,1]$ and each N such that

$$N\|N\alpha\| \leq 1.$$

In particular,

$$|\sum_{n=0}^{N-1} \ln|P(x_N+n)| - \sum_{n=-N}^{-1} \ln|P(z_N+n)|| \lesssim_{A,B} 1$$

and thus

$$\frac{\prod_{-N}^{n=-1}|P(z_N+n)|}{\prod_{n=0}^{N-1}|P(x_N+n)|} \sim_{A,B} 1, .$$

This will replace Corollary 2.3 in the argument above. Everything else will be the same.

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