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VON NEUMANN ALGEBRAS AND LINEAR INDEPENDENCE OF TRANSLATES

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Abstract. For $x, y \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$, define $(x, y)f(t) = e^{2\pi i y t} f(t+x)$ and if $\Lambda \subseteq \mathbb{R}^2$, define $S(f, \Lambda) = \{(x, y)f \mid (x, y) \in \Lambda\}$. It has been conjectured that if $f \neq 0$, then $S(f, \Lambda)$ is linearly independent over $\mathbb{C}$; one motivation for this problem comes from Gabor analysis. We shall prove that $S(f, \Lambda)$ is linearly independent if $f \neq 0$ and Λ is contained in a discrete subgroup of $\mathbb{R}^2$, and as a byproduct we shall obtain some results on the group von Neumann algebra generated by the operators $\{(x, y) \mid (x, y) \in \Lambda\}$. Also, we shall prove these results for the obvious generalization to $\mathbb{R}^n$.

1. Introduction

Let n be a positive integer, let $\mathcal{G}_n$ be the abelian group $\{(x, y) \mid x, y \in \mathbb{R}^n\}$ with the operation addition (so $\mathcal{G}_n \cong \mathbb{R}^{2n}$), and for $x, y \in \mathbb{R}^n$, let $x \cdot y$ denote the dot product $x_1 y_1 + \cdots + x_n y_n$. Let $\mathbb{C} * \mathcal{G}_n$ denote the twisted group ring (a twisted group ring is a particular kind of crossed product) which has $\mathbb{C}$-basis $\{\bar{g} \mid g \in \mathcal{G}_n\}$, and multiplication satisfying $\overline{(a, b)}\,\overline{(x, y)} = e^{2\pi i a \cdot y}\overline{(a + x, b + y)}$. For $g \in \mathcal{G}_n$, we shall often write g instead of $\bar{g}$ if there is no danger of confusion, and then g^{-1} will mean $\bar{g}^{-1}$ rather than $\overline{g^{-1}}$. Let $L^2(\mathbb{R}^n)$ denote the Hilbert space of square integrable functions $\{f \colon \mathbb{R}^n \to \mathbb{C} \mid \int_{\mathbb{R}^n} |f(t)|^2\, dt < \infty\}$ with two functions $f_1, f_2 \in L^2(\mathbb{R}^n)$ being equal if and only if $f_1(t) = f_2(t)$ almost everywhere, and let $\mathcal{B}(L^2(\mathbb{R}^n))$ denote the set of bounded linear operators on $L^2(\mathbb{R}^n)$. Then $\mathbb{C} * \mathcal{G}_n$ acts on the left of $L^2(\mathbb{R}^n)$ according to the rule $(x, y)\, f(t) = e^{2\pi i y \cdot t} f(t + x)$ and extending to the whole of $\mathbb{C} * \mathcal{G}_n$ by $\mathbb{C}$-linearity. To check that this indeed defines an action, we need only verify that $(a, b)\big((x, y)f(t)\big) = \big((a, b)(x, y)\big)f(t)$, which is indeed true because both sides equal $e^{2\pi i(a\cdot y + b\cdot t + y\cdot t)} f(t + a + x)$. Thus we obtain a homomorphism from $\mathbb{C} * \mathcal{G}_n$ into $\mathcal{B}(L^2(\mathbb{R}^n))$. Since $\mathbb{C} * \mathcal{G}_n$ is a simple ring by Lemma 2.1, this homomorphism must be a monomorphism and so we may view $\mathbb{C} * \mathcal{G}_n$ as a $\mathbb{C}$-subalgebra of $\mathcal{B}(L^2(\mathbb{R}^n))$. We shall consider the following conjecture.

Conjecture 1.1. *Let $0 \neq \theta \in \mathbb{C} * \mathcal{G}_n$ and $0 \neq f \in L^2(\mathbb{R}^n)$. Then $\theta f \neq 0$.*

Motivation for studying this problem comes from Gabor analysis and in particular the conjecture on page 2790 of [4]. If $G \leqslant \mathcal{G}_n$, then $\mathbb{C} * G$ will denote the $\mathbb{C}$-subalgebra of $\mathbb{C} * \mathcal{G}_n$ which has $\mathbb{C}$-basis $\{\bar{g} \mid g \in G\}$. Of course when talking

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about discrete subsets of $\mathcal{G}_n$, we are giving $\mathcal{G}_n$ the usual topology from $\mathbb{R}^{2n}$. We shall prove

Theorem 1.2. *Let G be a discrete subgroup of $\mathcal{G}_n$. If $0 \neq \theta \in \mathbb{C} * G$ and $0 \neq f \in L^2(\mathbb{R}^n)$, then $\theta f \neq 0$.*

Of course it follows immediately that if G is a discrete subgroup of $\mathcal{G}_n$, $g \in \mathcal{G}_n$, $0 \neq \theta \in g\mathbb{C} * G$ and $0 \neq f \in L^2(\mathbb{R}^n)$, then $\theta f \neq 0$. This means we can rephrase the above result in terminology closer to that of [4] as follows. For $x, y \in \mathbb{R}^n$ and $f \in L^2(\mathbb{R}^n)$, define $(x,y)f(t) = e^{2\pi i y \cdot t} f(t+x)$ and if $\Lambda \subseteq \mathbb{R}^{2n}$, define $S(f,\Lambda) = \{(x,y)f \mid (x,y) \in \Lambda\}$. Then Theorem 1.2 yields

Proposition 1.3. *Let n be a positive integer, let Λ be a subset of $\mathbb{R}^{2n}$ of the form $g+G$ where G is a discrete subgroup of $\mathbb{R}^{2n}$, and let $0 \neq f \in L^2(\mathbb{R}^n)$. Then $S(f,\Lambda)$ is linearly independent.*

As a byproduct, we shall obtain results on the von Neumann algebra generated by $\mathbb{C} * G$, which we shall denote by $W * G$. Thus $W * G$ is the weak closure of $\mathbb{C} * G$ in $\mathcal{B}(L^2(\mathbb{R}^n))$ and is rather similar to the group von Neumann algebra of G. For $f, g \in L^2(\mathbb{R}^n)$, let $\langle f, g\rangle$ denote the inner product $\int_{\mathbb{R}^n} f(t)\bar{g}(t)\,dt$, where $^-$ denotes complex conjugation, and let $\mathcal{U}(L^2(\mathbb{R}^n))$ denote the set of closed densely defined linear operators [5, §2.7] acting on $L^2(\mathbb{R}^n)$. Then the adjoint α^* of $\alpha \in \mathcal{U}(L^2(\mathbb{R}^n))$ satisfies $\langle \alpha f, g\rangle = \langle f, \alpha^* g\rangle$ whenever $f, g \in L^2(\mathbb{R}^n)$ and αf, $\alpha^* g$ are defined. Of course * restricts to an involution on both $\mathcal{B}(L^2(\mathbb{R}^n))$ and $W * G$. If G is a discrete subgroup of $\mathcal{G}_n$, then $W * G$ is a *finite* von Neumann algebra by Lemma 3.2; also in many cases this can be deduced from Rieffel's paper [7]. In this situation, we let $U * G$ indicate the operators of $\mathcal{U}(L^2(\mathbb{R}^n))$ which are affiliated to $W * G$ [3, p. 150]. The results of [3] (especially theorem 1 and the proof of theorem 10) now show that $(U * G)^* = U * G$, $U * G$ is a $*$-regular ring containing $W * G$, and every element of $U * G$ can be written in the form $\gamma\delta^{-1}$ where $\gamma, \delta \in W * G$. In particular every nonzero divisor in $W * G$ is invertible in $U * G$. We shall prove

Theorem 1.4. *Let G be a discrete subgroup of $\mathcal{G}_n$. Then $W * G$ is a finite von Neumann algebra, every nonzero element of $\mathbb{C} * G$ is invertible in $U * G$, and the set $\{\gamma\delta^{-1} \mid \gamma \in \mathbb{C} * G, 0 \neq \delta \in \mathbb{C} * G\}$ is a division subring of $U * G$.*

Let L be a locally compact group, let G be a torsion free subgroup of L, and let $L^2(L)$ denote the Hilbert space of square integrable functions on L with respect to the left Haar measure on L. Then G acts on the left of $L^2(L)$ according to the rule $gf(l) = f(g^{-1}l)$ for $g \in G, f \in L^2(L), l \in L$. For $f \in L^2(L) \setminus 0$, a closely related problem to Conjecture 1.1 is to determine whether the set $\{gf \mid g \in G\}$ is linearly independent over $\mathbb{C}$. If the von Neumann algebra $W * G$ generated by G is a finite von Neumann algebra, then by using the techniques of this paper, it is possible in many cases to show that the set $\{gf \mid g \in G\}$ is linearly independent. On the other hand if $W * G$ is not a finite von Neumann algebra, then the techniques of this paper cannot be applied. It will usually be the case that $W * G$ is not finite if G is not discrete and has no abelian subgroup of finite index. A specific example would be to let L be the Heisenberg group consisting of upper unitriangular 3 by 3 matrices with entries in $\mathbb{R}$, in other words matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

where $a, b, c \in \mathbb{R}$, and to let $G = L$. Then it is not known in this case whether for $f \in L^2(L) \setminus 0$, the set $\{gf \mid g \in G\}$ is linearly independent.

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2. Notation, terminology and assumed results

The identity of a group will be denoted by either 0 or 1. If n is a positive integer and R is a ring, then $\mathrm{M}_n(R)$ will denote the n by n matrices over R, and we shall let δ_{ij} indicate the Kronecker delta, so $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$. The identity matrix of $\mathrm{M}_n(R)$ will be denoted by I_n, and the zero matrix of $\mathrm{M}_n(R)$ will be denoted by 0_n. We shall view vectors in $\mathbb{R}^n$ as column vectors rather than row vectors. A lattice in $\mathbb{R}^n$ will mean a discrete subgroup of $\mathbb{Z}$-rank n; in other words a discrete subgroup of finite covolume (note that this is a different definition of lattice from that of [4, p. 2791]). If $\alpha = \sum_{g \in \mathcal{G}_n} \lambda_g g \in \mathbb{C} * \mathcal{G}_n$ where $\lambda_g \in \mathbb{C}$ for all $g \in \mathcal{G}_n$, then the support of α, denoted $\operatorname{supp} \alpha$, is the set $\{g \in \mathcal{G}_n \mid \lambda_g \neq 0\}$. We shall use the notation $\|f\|_2$ for the norm $\sqrt{\langle f, f \rangle}$ of an element $f \in L^2(\mathbb{R}^n)$, and $\overline{X}$ for the closure of a subset X in $L^2(\mathbb{R}^n)$. The commutant of a subset A of $\mathcal{B}(L^2(\mathbb{R}^n))$ is $A' = \{x \in \mathcal{B}(L^2(\mathbb{R}^n)) \mid ax = xa \text{ for all } a \in A\}$. If $A = A^*$, then A' is a von Neumann algebra and by von Neumann's double commutant theorem [1, theorem 1.2.1], A is dense in A'' in the weak operator topology. Thus another description of $W * G$ is the double commutant of $\mathbb{C} * G$ in $\mathcal{B}(L^2(\mathbb{R}^n))$. In the case $W * G$ is a finite von Neumann algebra, we can now describe $U * G$ as those unbounded operators in $\mathcal{U}(L^2(\mathbb{R}^n))$ which commute with every element of $(W * G)'$.

Lemma 2.1. *$\mathbb{C} * \mathcal{G}_n$ is a simple ring.*

Proof. Suppose $0 \neq I \lhd \mathbb{C} * \mathcal{G}_n$ with $I \neq \mathbb{C} * \mathcal{G}_n$, and choose $0 \neq \alpha \in I$ with minimal support. If $g \in \operatorname{supp} \alpha$, then $1 \in \operatorname{supp} \bar{g}^{-1}\alpha$ and $\bar{g}^{-1}\alpha \in I$, so we may assume that $1 \in \operatorname{supp} \alpha$. Since $I \neq \mathbb{C} * \mathcal{G}_n$, we may choose $a \in \mathcal{G}_n$ such that $1 \neq a \in \operatorname{supp} \alpha$. Then there exists $g \in \mathcal{G}_n$ such that $\bar{g}\bar{a}\bar{g}^{-1} \neq \bar{a}$, and now we have $0 \neq \bar{g}\alpha\bar{g}^{-1} - \alpha \in I$. This contradicts the minimality of $\operatorname{supp} \alpha$ because $|\operatorname{supp}(\bar{g}\alpha\bar{g}^{-1} - \alpha)| < |\operatorname{supp} \alpha|$, and the result follows. □

If R is a ring and σ is an automorphism of R, then $R_\sigma[X]$ will denote the twisted polynomial ring over R in the indeterminate X, so multiplication is defined by $\sum a_i X^i \sum b_j X^j = \sum_n (\sum_{i+j=n} a_i \sigma^i b_j) X^n$. We say that R is an Ore domain if it is contained in a division ring D, called the division ring of fractions of R, such that every element of D can be written in the form rs^{-1} and also in the form $s^{-1}r$, with $r, s \in R$ and $s \neq 0$. Of course the division ring D containing R is unique up to R-isomorphism. Also if R is contained in a ring D' such that every nonzero element of R is invertible, then the set $\{rs^{-1} \mid r, s \in R \text{ and } s \neq 0\}$ is the division ring of fractions containing R. The following two elementary results are well known.

Lemma 2.2. *Let R be an Ore domain with division ring of fractions D, and let σ be an automorphism of R. Then σ extends uniquely to an automorphism of D, which we shall also call σ, and if $\alpha, \beta \in D_\sigma[X]$, then there exists $r \in R \setminus 0$ such that $r\alpha, r\beta \in R_\sigma[X]$.*

Lemma 2.3. *Let G be a subgroup of $\mathcal{G}_n$. Then $\mathbb{C} * G$ is an Ore domain, and if I, J are nonzero left ideals of $\mathbb{C} * G$, then $I \cap J \neq 0$.*

Finally we require the following:

Lemma 2.4. *Let G be a discrete subgroup of $\mathcal{G}_n$, let $H \lhd G$ such that G/H is infinite cyclic, and let $x \in G$ such that Hx is a generator for G/H. If $\zeta \in \mathbb{C}$ and $|\zeta| = 1$, then there exists $y \in \mathcal{G}_n$ such that $\bar{y}\bar{h}\bar{y}^{-1} = \bar{h}$ for all $h \in H$ and $\bar{y}\bar{x}\bar{y}^{-1} = \zeta\bar{x}$ in $\mathbb{C} * \mathcal{G}_n$.*

Proof. Since G is discrete, we may choose $m \in \mathbb{Z}$ and a subset $\{h_1, \dots, h_m\}$ which generates H and is linearly independent over $\mathbb{R}$. Note that $\{h_1, \dots, h_m, x\}$ is also linearly independent over $\mathbb{R}$. Choose $t \in \mathbb{R}$ such that $e^{2\pi i t} = \zeta$, and define a bilinear form $\beta \colon \mathcal{G}_n \to \mathbb{R}$ by $\beta\big((a,b),(c,d)\big) = a \cdot d - b \cdot c$, where $a, b, c, d \in \mathbb{R}^n$. Note that in $\mathbb{C} * \mathcal{G}_n$, we have

$$(a,b)(c,d)(a,b)^{-1} = e^{2\pi i (a \cdot d - b \cdot c)}(c,d).$$

It is easily checked that β is nondegenerate, so there exists $y \in \mathcal{G}_n$ such that $\beta(y, h_i) = 0$ for all i and $\beta(y, x) = t$. This completes the proof. □

3. Faithful traces

In this section, we show that $W * G$ has a faithful weakly continuous tracial state, which in particular will establish that $W * G$ is a finite von Neumann algebra. Throughout this section, n will be a positive integer. The purpose of the next lemma is to reduce to the case when G is a lattice in $\mathbb{R}^{2n}$ such that $G \cap 1 \times \mathbb{R}^n = 1 \times \mathbb{Z}^n$; its proof is modelled on [4, §2, p. 2790].

We shall think of $\mathbb{R}^{2n}$ as $\mathbb{R}^n \oplus \mathbb{R}^n$, so we can view $\mathbb{R}^n$ as a subgroup of $\mathbb{R}^{2n}$ in the usual way via the map $x \mapsto (x, 0)$. We then have a monomorphism $\psi \colon \mathcal{G}_n \to \mathcal{G}_{2n}$ and this induces a monomorphism $\mathbb{C} * \mathcal{G}_n \to \mathbb{C} * \mathcal{G}_{2n}$, which we shall also call ψ.

Given $f, g \in L^2(\mathbb{R}^n)$, we can form the element $f \otimes g \in L^2(\mathbb{R}^{2n})$ defined by $(f \otimes g)(x, y) = f(x)g(y)$ for $x, y \in \mathbb{R}^n$, and then the functions of the form $\sum_{i=1}^m f_i \otimes g_i$ are dense in $L^2(\mathbb{R}^{2n})$. If $\theta \in \mathcal{B}(L^2(\mathbb{R}^n))$, then we have a well defined operator $\theta \otimes 1 \in \mathcal{B}(L^2(\mathbb{R}^{2n}))$ satisfying $(\theta \otimes 1)(f \otimes g) = (\theta f) \otimes g$ for all $f, g \in L^2(\mathbb{R}^n)$, and this yields a weakly continuous $*$-monomorphism $\theta \mapsto \theta \otimes 1 \colon \mathcal{B}(L^2(\mathbb{R}^n)) \to \mathcal{B}(L^2(\mathbb{R}^{2n}))$.

Note that when we view $\mathbb{C} * \mathcal{G}_n$ and $\mathbb{C} * \mathcal{G}_{2n}$ as subalgebras of $\mathcal{B}(L^2(\mathbb{R}^n))$ and $\mathcal{B}(L^2(\mathbb{R}^{2n}))$ respectively, then $\psi(\theta) = \theta \otimes 1$ for all $\theta \in \mathbb{C} * \mathcal{G}_n$. Furthermore, if $G \leq \mathcal{G}_n$, then ψ induces isomorphisms $W * G \to W * \psi G$ and (assuming $W * G$ is a finite von Neumann algebra) $U * G \to U * \psi G$, which means we may identify G with the subgroup ψG of $\mathcal{G}_{2n}$; we shall do this without further comment and without using ψ in the future.

Let $\{e_1, \dots, e_{2n}\}$ denote the standard basis for $\mathbb{R}^{2n}$, so e_i has a 1 in the ith position and zeros elsewhere, and $e_i \cdot e_j = \delta_{ij}$. If $G \leqslant \mathcal{G}_n$, then we define $\{\mathbb{C}G\} = \{\lambda g \mid \lambda \in \mathbb{C} \text{ and } g \in G\}$, a subset of $\mathbb{C} * G$.

Lemma 3.1. *Let G be a discrete subgroup of $\mathcal{G}_n$. Then there exists a lattice H in $\mathcal{G}_{2n}$ and a unitary operator $u \in \mathcal{B}(L^2(\mathbb{R}^{2n}))$, such that $H \cap 1 \times \mathbb{R}^{2n} = 1 \times \mathbb{Z}^{2n}, u\{\mathbb{C}\mathcal{G}_{2n}\}u^{-1} = \{\mathbb{C}\mathcal{G}_{2n}\}$ and $u\{\mathbb{C}G\}u^{-1} \subseteq \{\mathbb{C}H\}$.*

Proof. Choose an $\mathbb{R}$-basis $\{g_1, \dots, g_{2n}\}$ for $\mathcal{G}_n$ such that $\{g_1, \dots, g_r\}$ is a $\mathbb{Z}$-basis for G, where r is the rank of G. Let $\mathcal{E} = \{(e_1, 0), \dots, (e_{2n}, 0), (0, e_1), \dots, (0, e_{2n})\}$, let

$$\begin{aligned}\mathcal{F} = \{&(e_1, e_{n+1})/\sqrt{2}, (e_2, e_{n+2})/\sqrt{2}, \dots, (e_n, e_{2n})/\sqrt{2}, (e_{n+1}, e_1)/\sqrt{2},\\ &(e_{n+2}, e_2)/\sqrt{2}, \dots, (e_{2n}, e_n)/\sqrt{2}, (-e_{n+1}, e_1)/\sqrt{2}, (-e_{n+2}, e_2)/\sqrt{2}, \dots,\\ &(-e_{2n}, e_n)/\sqrt{2}, (-e_1, e_{n+1})/\sqrt{2}, (-e_2, e_{n+2})/\sqrt{2}, \dots, (-e_n, e_{2n})/\sqrt{2}\},\end{aligned}$$

and let

$$\begin{aligned}\mathcal{K} = \{&g_1, \dots, g_{2n}, (-e_{n+1}, e_1)/\sqrt{2}, (-e_{n+2}, e_2)/\sqrt{2}, \dots, (-e_{2n}, e_n)/\sqrt{2},\\ &(-e_1, e_{n+1})/\sqrt{2}, (-e_2, e_{n+2})/\sqrt{2}, \dots, (-e_n, e_{2n})/\sqrt{2}\},\end{aligned}$$

so $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{K}$ are $\mathbb{R}$-bases of $\mathcal{G}_{2n}$. For $i = 1, \dots, 4n$, we shall let $\hat{e}_i$, f_i, k_i denote the ith basis elements of $\mathcal{E}$, $\mathcal{F}$, $\mathcal{K}$ respectively, and we shall let K be the lattice in $\mathcal{G}_{2n}$ which has $\mathbb{Z}$-basis $\mathcal{K}$. Let A_i denote the coordinates of k_i with respect to the basis $\mathcal{F}$, and let a_{ji} denote the jth coordinate of A_i. Then for $2n+1 \le i \le 4n$, $a_{ji} = 1$ if $j = i$ and $a_{ji} = 0$ if $j \neq i$. Now define $h_i = \sum_{j=1}^{4n} a_{ji}\hat{e}_j \in \mathcal{G}_{2n}$, and let H be the subgroup of $\mathcal{G}_{2n}$ generated by the h_i. Then H is a lattice in $\mathcal{G}_{2n}$ such that $H \cap 1 \times \mathbb{R}^{2n} = 1 \times \mathbb{Z}^{2n}$.

Let T be the transition matrix from $\mathcal{E}$ to $\mathcal{F}$, and let $J = J_{2n} = \begin{pmatrix} 0_n & I_n \\ I_n & 0_n \end{pmatrix} \in \mathrm{M}_{2n}(\mathbb{R})$. Thus if T has entries t_{ij}, then $f_j = \sum_{i=1}^{4n} t_{ij}\hat{e}_i$, and if we think of the A_i as column vectors, then the coordinates of k_i with respect to $\mathcal{E}$ are TA_i. Also

$$\begin{aligned}T &= \begin{pmatrix} I_{2n}/\sqrt{2} & -J_{2n}/\sqrt{2} \\ J_{2n}/\sqrt{2} & I_{2n}/\sqrt{2} \end{pmatrix}\\ &= \begin{pmatrix} I_{2n} & 0_{2n} \\ -J_{2n} & I_{2n} \end{pmatrix} \begin{pmatrix} J_{2n}/\sqrt{2} & 0_{2n} \\ 0_{2n} & J_{2n}\sqrt{2} \end{pmatrix} \begin{pmatrix} 0_{2n} & -I_{2n} \\ I_{2n} & 0_{2n} \end{pmatrix} \begin{pmatrix} I_{2n} & 0_{2n} \\ -J_{2n} & I_{2n} \end{pmatrix}.\end{aligned}$$

Let $\tau, \alpha, \beta, \gamma \colon \mathcal{G}_{2n} \to \mathcal{G}_{2n}$ be the linear mappings determined by the matrices

$$T, \begin{pmatrix} J_{2n}/\sqrt{2} & 0_{2n} \\ 0_{2n} & J_{2n}\sqrt{2} \end{pmatrix}, \begin{pmatrix} I_{2n} & 0_{2n} \\ -J_{2n} & I_{2n} \end{pmatrix}, \begin{pmatrix} 0_{2n} & -I_{2n} \\ I_{2n} & 0_{2n} \end{pmatrix}$$

respectively, with respect to the basis $\mathcal{E}$, so $\tau\hat{e}_i = f_i$ for all i. Then $\tau H = K \supseteq G$, so it will be sufficient to show that there exists a unitary operator $u \in \mathcal{B}(L^2(\mathbb{R}^{2n}))$ such that $u^{-1}\mathbb{C}gu = \mathbb{C}\tau g$ for all $g \in \mathcal{G}_{2n}$. Since $\tau = \beta\alpha\gamma\beta$, it will be sufficient to do this with α, β, γ in place of τ. We now use metaplectic transformations [8, p. 578]. Write $g = (x, y)$ where $x, y \in \mathbb{R}^{2n}$, and then we have three cases to consider.

1. The matrix $\alpha = \begin{pmatrix} J_{2n}/\sqrt{2} & 0_{2n} \\ 0_{2n} & \sqrt{2}J_{2n} \end{pmatrix}$. For $f \in L^2(\mathbb{R}^{2n})$ and $t \in \mathbb{R}^{2n}$, we define $uf(t) = 2^{-n/2}f(Jt/\sqrt{2})$ (we are considering t as a column vector in $\mathbb{R}^{2n}$ here). Then u is $\mathbb{C}$-linear and $\|uf\|_2 = \|f\|_2$ for all $f \in L^2(\mathbb{R}^{2n})$, hence u is a unitary operator. Also $u^{-1}f(t) = 2^{n/2}f(\sqrt{2}Jt)$ because $J_{2n}^2 = I_{2n}$, consequently

$$\begin{aligned}u^{-1}guf(t) &= u^{-1}g2^{-n/2}f(Jt/\sqrt{2}) = u^{-1}e^{2\pi iy\cdot t}2^{-n/2}f(J(t+x)/\sqrt{2})\\ &= e^{2\pi i\sqrt{2}Jy\cdot t}f(t + Jx/\sqrt{2}) \quad \text{because } J_{2n} \text{ is symmetric}\\ &= (\alpha g)f(t)\end{aligned}$$

for all $t \in \mathbb{R}^{2n}$ and for all $f \in L^2(\mathbb{R}^{2n})$. Thus $u^{-1}gu = \alpha g$ as required.

2. The matrix $\beta = \begin{pmatrix} I_{2n} & 0_{2n} \\ -J_{2n} & I_{2n} \end{pmatrix}$. Here we define $uf(t) = e^{-\pi i Jt\cdot t}f(t)$. Then u is $\mathbb{C}$-linear and $\|uf\|_2 = \|f\|_2$, so u is a unitary operator. Since $u^{-1}f(t) = e^{\pi i Jt\cdot t}f(t)$,

$$\begin{aligned} u^{-1}guf(t) &= u^{-1}ge^{-\pi i Jt\cdot t}f(t) = u^{-1}e^{2\pi i y\cdot t}e^{-\pi i J(t+x)\cdot(t+x)}f(t+x) \\ &= e^{\pi i Jt\cdot t}e^{2\pi i y\cdot t}e^{-\pi i J(t+x)\cdot(t+x)}f(t+x) \\ &= e^{-\pi i Jx\cdot x}e^{2\pi i t\cdot(y-Jx)}f(t+x) \quad \text{because } J_{2n} \text{ is symmetric} \\ &= e^{-\pi i Jx\cdot x}(\beta g)f(t) \end{aligned}$$

for all $t \in \mathbb{R}^{2n}$, and we have shown that $u^{-1}gu \in \mathbb{C}(\beta g)$.

3. The matrix $\gamma = \begin{pmatrix} 0_{2n} & -I_{2n} \\ I_{2n} & 0_{2n} \end{pmatrix}$. Here we use the Fourier transform; specifically $uf(t) = \int_{\mathbb{R}^{2n}} e^{2\pi i t\cdot s}f(s)\,ds$, and then $u^{-1}f(t) = \int_{\mathbb{R}^{2n}} e^{-2\pi i t\cdot s}f(s)\,ds$. Observe that $(x,0)uf(t) = u(0,x)f(t)$ and $(0,y)uf(t) = u(-y,0)f(t)$, consequently

$$u^{-1}(x,y)u = u^{-1}(0,y)(x,0)u = (-y,0)(0,x) = e^{-2\pi i x\cdot y}(-y,x) = e^{-2\pi i x\cdot y}\gamma(x,y)$$

and we deduce that $u^{-1}\mathbb{C}gu = \mathbb{C}\gamma g$.

This completes the proof of Lemma 3.1 □

Lemma 3.2. *Let G be a discrete subgroup of $\mathcal{G}_n$, and define $\tau\colon \mathbb{C} * G \to \mathbb{C}$ by $\tau g = 0$ when $1 \neq g \in G$, and $\tau 1 = 1$. Then*

(i) *τ extends to a weakly continuous $\mathbb{C}$-linear map $W * G \to \mathbb{C}$.*
(ii) *If $\alpha, \beta \in W * G$, then $\tau(\alpha\beta) = \tau(\beta\alpha)$.*
(iii) *If $\alpha \in W * G$ and $x \in \mathcal{G}_n$, then $\tau(\bar{x}\alpha\bar{x}^{-1}) = \tau(\alpha)$.*
(iv) *If e is a nonzero projection in $W * G$, then $0 < \tau e \leq 1$.*
(v) *Let e, f be projections in $W * G$. If $eL^2(\mathbb{R}^n) \subseteq fL^2(\mathbb{R}^n)$, then $\tau e \leq \tau f$.*
(vi) *Let e, f be projections in $W * G$, and let h be the projection of $L^2(\mathbb{R}^n)$ onto $\overline{eL^2(\mathbb{R}^n) + fL^2(\mathbb{R}^n)}$. Then $h \in W * G$ and if $eL^2(\mathbb{R}^n) \cap fL^2(\mathbb{R}^n) = 0$, then $\tau e + \tau f = \tau h$.*

Proof. Since G is a discrete subgroup of $\mathcal{G}_n$, there is by Lemma 3.1, a lattice H in $\mathcal{G}_{2n}$ and a unitary operator $u \in \mathcal{B}(L^2(\mathbb{R}^{2n}))$ such that $H \cap 1 \times \mathbb{R}^{2n} = 1 \times \mathbb{Z}^{2n}, u\{\mathbb{C}\mathcal{G}_{2n}\}u^{-1} = \{\mathbb{C}\mathcal{G}_{2n}\}$ and $u\mathbb{C} * Gu^{-1} \subseteq \mathbb{C} * H$. If we can find a weakly continuous $\mathbb{C}$-linear map $\tau\colon \mathbb{C} * H \to \mathbb{C}$ with the required properties, then the weakly continuous $\mathbb{C}$-linear map $\alpha \mapsto \tau(u\alpha u^{-1})$ for $\alpha \in \mathbb{C} * G$ will suffice. Therefore we may assume that G is a lattice in $\mathcal{G}_n$ and $G \cap 1 \times \mathbb{R}^n = 1 \times \mathbb{Z}^n$. If a is a positive number, we shall let $\mathcal{C}(a)$ denote the standard unit cube in $\mathbb{R}^n$ with side of length a; thus $\mathcal{C}(a) = \{(a_1, \dots, a_n) \mid 0 \leq a_i \leq a \text{ for all } i\}$.

(i) Choose a positive integer b such that $h\mathcal{C}(1/b) \cap \mathcal{C}(1/b) = \emptyset$ whenever $(h,k) \in G \setminus (1 \times \mathbb{Z}^n)$, which is possible because G is a lattice in $\mathcal{G}_n$ such that $G \cap 1 \times \mathbb{R}^n = 1 \times \mathbb{Z}^n$, and set $\mathcal{C} = \mathcal{C}(1/b)$. Let $c = b^n$, let $\mathcal{C}_1, \dots, \mathcal{C}_c$ denote the c translates of $\mathcal{C}$ which are contained in the unit cube $\mathcal{C}(1)$, and for each i, let χ_i denote the characteristic function of $\mathcal{C}_i$. For $\theta \in W * G$, define

$$\tau\theta = \sum_{i=1}^{c} \langle \theta\chi_i, \chi_i \rangle = \sum_{i=1}^{c} \int_{\mathcal{C}_i} \theta\chi_i(t)\,dt.$$

Let $g \in G$ and write $g = (h,k)$ where $h, k \in \mathbb{R}^n$. Then $g\chi_i(t) = e^{2\pi i k\cdot t}\chi_i(t+h)$, so if $h \neq 0$ we have $g\chi_i(t) = 0$ for all $t \in \mathcal{C}_i$ and hence $\tau g = 0$. On the other hand if

$h = 0$ and $k \neq 0$, then $k \in \mathbb{Z}^n \setminus 0$ because G is a lattice such that $G \cap 1 \times \mathbb{R}^n = 1 \times \mathbb{Z}^n$, consequently

$$\tau g = \sum_{i=1}^{c} \int_{\mathcal{C}_i} e^{2\pi i k \cdot t}\, dt = \int_{\mathcal{C}} e^{2\pi i k \cdot t}\, dt = 0.$$

Finally $\tau 1 = 1$ and (i) is proven.

(ii) If $x, y \in G$, then $xy = 1$ if and only if $yx = 1$. Therefore $\tau \bar{x}\bar{y} = \tau \bar{y}\bar{x} = 0$ if $xy \neq 1$ and $\bar{x}\bar{y} = \bar{y}\bar{x}$ if $xy = 1$, hence $\tau \bar{x}\bar{y} = \tau \bar{y}\bar{x}$ for all $x, y \in G$ and we deduce that $\tau\alpha\beta = \tau\beta\alpha$ for all $\alpha, \beta \in \mathbb{C} * G$. Since τ is weakly continuous and $W * G$ is the weak closure of $\mathbb{C} * G$, we see that $\tau\alpha\beta = \tau\beta\alpha$ for all $\alpha, \beta \in W * G$, which proves (ii).

(iii) Define $\sigma\colon W * G \to \mathbb{C}$ by $\sigma(\alpha) = \tau(x\alpha x^{-1})$. Observe that $\sigma 1 = 1$ and if $1 \neq g \in G$, then $\sigma g = \tau(xgx^{-1}) = 0$ because $xgx^{-1} = \zeta g$ for some $\zeta \in \mathbb{C}$ with $|\zeta| = 1$. Thus $\sigma g = \tau g$ for all $g \in G$ and since σ is a weakly continuous $\mathbb{C}$-linear map, we deduce that $\sigma(\alpha) = \tau(\alpha)$ for all $\alpha \in W * G$, which is the required result.

(iv) Note that if e is a projection in $W * G$, then $e^* e = e$, and hence

$$\tau e = \sum_{i=1}^{c} \langle e^* e \chi_i, \chi_i \rangle = \sum_{i=1}^{c} \langle e\chi_i, e\chi_i \rangle \geq 0.$$

Let $K = \mathbb{Z}^n \times \mathbb{Z}^n \leqslant \mathcal{G}_n$. If $k \in K$, then $\tau e = \tau(k^{-1}ek)$ by (iii) and we deduce that

$$\tau e = \tau(k^{-1}ek) = \sum_{i=1}^{c} \langle k^{-1} e^* e k \chi_i, \chi_i \rangle = \sum_{i=1}^{c} \langle ek\chi_i, ek\chi_i \rangle.$$

Let χ denote the characteristic function of $\mathcal{C}(1)$, and suppose $\tau e = 0$. Then $ek\chi_i = 0$ for all i, hence $ek\chi = 0$ for all $k \in K$. Now the set $\{k\chi \mid k \in K\}$ forms a Hilbert basis for $L^2(\mathbb{R}^n)$ so if $\tau e = 0$, we see that $ef = 0$ for all $f \in L^2(\mathbb{R}^n)$ and we deduce that $e = 0$. Also $1 - e$ is a projection if e is a projection, so applying the above to $1 - e$ we obtain $0 \leq \tau(1 - e)$, hence $\tau e \leq 1$ and (iv) follows.

(v) Let h be the projection of $L^2(\mathbb{R}^n)$ onto the orthogonal complement of $eL^2(\mathbb{R}^n)$ in $fL^2(\mathbb{R}^n)$. Then $e + h = f$, hence $\tau e + \tau h = \tau f$. Thus $h \in W * G$ and the result follows from (iv).

(vi) Let u be a unitary operator in $(\mathbb{C} * G)'$. Then $ueu^{-1} = e$ and $ufu^{-1} = f$. Since h is the projection of $L^2(\mathbb{R}^n)$ onto $\overline{eL^2(\mathbb{R}^n) + fL^2(\mathbb{R}^n)}$, we see that uhu^{-1} is the projection of $L^2(\mathbb{R}^n)$ onto $\overline{ueu^{-1}L^2(\mathbb{R}^n) + ufu^{-1}L^2(\mathbb{R}^n)} = \overline{eL^2(\mathbb{R}^n) + fL^2(\mathbb{R}^n)}$ and we deduce that $uhu^{-1} = h$. Therefore $uh = hu$. Now $(\mathbb{C} * G)'$ is a von Neumann algebra, so any element of $(\mathbb{C} * G)'$ is a $\mathbb{C}$-linear sum of unitary elements, hence $xh = hx$ for all $x \in (\mathbb{C} * G)'$ and we conclude that $h \in W * G$.

We now claim that $h = e \cup f$ [2, p. 4]. Since $eL^2(\mathbb{R}^n) \subseteq hL^2(\mathbb{R}^n)$, we see that $e = he$ and hence $e \leq h$. Similarly $f \leq h$ and so $e \cup f \leq h$. Now let $g = e \cup f$. Then $gL^2(\mathbb{R}^n) \supseteq eL^2(\mathbb{R}^n), fL^2(\mathbb{R}^n)$ and hence $gL^2(\mathbb{R}^n) \supseteq hL^2(\mathbb{R}^n)$. We deduce that $g \geq h$, consequently $g = h$ and the claim is established.

If $eL^2(\mathbb{R}^n) \cap fL^2(\mathbb{R}^n) = 0$, then $(e \cap f)L^2(\mathbb{R}^n) \subseteq eL^2(\mathbb{R}^n) \cap fL^2(\mathbb{R}^n) = 0$. Since $W * G$ is a von Neumann algebra, we may apply the parallelogram law to deduce that $e \sim e \cup f - f$ [2, §1,§13]. Thus there is an element $w \in W * G$ such that $w^* w = e$ and $ww^* = e \cup f - f$. Since $\tau(w^* w) = \tau(ww^*)$, we deduce that $\tau e = \tau(e \cup f - f)$ and hence $\tau e + \tau f = \tau(e \cup f) = \tau h$. This completes the proof. $\square$

4. Proofs

Theorems 1.2 and 1.4 are now immediate consequences of the following result.

Lemma 4.1. *Let G be a discrete subgroup of $\mathbb{R}^n$ and let $\theta \in \mathbb{C} * G \setminus 0$. Then*

(i) *If $0 \neq f \in L^2(\mathbb{R}^n)$, then $\theta f \neq 0$.*
(ii) *θ is invertible in $U * G$.*
(iii) *The set $\{\gamma\delta^{-1} \mid \gamma \in \mathbb{C} * G, 0 \neq \delta \in \mathbb{C} * G\}$ is a division subring of $U * G$, and is equal to $\{\delta^{-1}\gamma \mid \gamma \in \mathbb{C} * G, 0 \neq \delta \in \mathbb{C} * G\}$.*

Proof. Since G is a discrete subgroup of $\mathcal{G}_n$, it is a free abelian group of rank at most $2n$. We shall prove the result by induction on the rank of G, the result being trivially true if the rank of G is zero, because then $G = 1$. Thus we may assume that the rank of G is strictly positive, and then there exists $H \lhd G$ such that $G/H \cong \mathbb{Z}$. Since H has strictly smaller rank than G, we may assume that the result is true for H. Let $\tau \colon W * G \to \mathbb{C}$ be the weakly continuous tracial state obtained from Lemma 3.2.

(i) For $\alpha \in \mathbb{C} * G$, let $\ker \alpha = \{f \in L^2(\mathbb{R}^n) \mid \alpha f = 0\}$, and let $\mathcal{N}(\alpha)$ be the projection from $L^2(\mathbb{R}^n)$ onto $\ker \alpha$. Suppose u is a unitary element in $(\mathbb{C} * G)'$. Then $u^{-1}\mathcal{N}(\alpha)u = \mathcal{N}(u^{-1}\alpha u) = \mathcal{N}(\alpha)$. Since $(\mathbb{C} * G)'$ is a von Neumann algebra, every element of $(\mathbb{C} * G)'$ is a linear combination of unitary elements of $(\mathbb{C} * G)'$ and we deduce that $\mathcal{N}(\alpha)$ commutes with every element of $(\mathbb{C} * G)'$. Therefore $\mathcal{N}(\alpha) \in W * G$.

Let $\nu = \sup\{\tau(\mathcal{N}(\alpha)) \mid 0 \neq \alpha \in \mathbb{C} * G\}$. If $\nu = 0$, then $\mathcal{N}(\theta) = 0$ by Lemma 3.2(iv), hence $\ker \theta = 0$ and the result follows, so we may assume that $0 < \nu \leq 1$. Therefore we may choose $\alpha \in \mathbb{C} * G$ such that $\tau\mathcal{N}(\alpha) > \nu/2$. Since G/H is infinite cyclic, there exists $x \in G$ such that Hx generates G/H, and then we may write $\alpha = \sum_{i=-\infty}^{i=\infty} \alpha_i x^i$ where $\alpha_i \in \mathbb{C} * H$ and $\alpha_i = 0$ for all but finitely many i. By replacing α with $x^m\alpha$ for some integer m, we may assume that $\alpha_0 \neq 0$ and $\alpha_i = 0$ for all $i < 0$.

By induction, there is a division subring D of $U * H$ containing $\mathbb{C} * H$ which is the division ring of fractions of $\mathbb{C} * H$. Let σ be the automorphism $\beta \mapsto x\beta x^{-1} \colon \mathbb{C} * H \to \mathbb{C} * H$. By Lemma 2.2 we may extend σ to an automorphism of D, which we shall also call σ. We now have a natural ring homomorphism $\theta \colon D_\sigma[X] \to U * G$, defined by $\theta X = x$ and $\theta d = d$ for all $d \in D$, which maps $(\mathbb{C} * H)_\sigma[X]$ into $\mathbb{C} * G$. By [6, lemma 16], there exists $\zeta \in \mathbb{C}$ with $|\zeta| = 1$ and $\beta', \gamma' \in D_\sigma[X]$ such that

$$\beta' \sum_i \alpha_0^{-1}\alpha_i X^i + \gamma' \sum_i \alpha_0^{-1}\alpha_i \zeta^i X^i = 1.$$

By Lemma 2.2, there exists $0 \neq r \in \mathbb{C} * H$ such that $r\beta'\alpha_0^{-1}, r\gamma'\alpha_0^{-1} \in (\mathbb{C} * H)_\sigma[X]$, so setting $\beta = r\beta'\alpha_0^{-1}$ and $\gamma = r\gamma'\alpha_0^{-1}$, we have $\beta, \gamma \in (\mathbb{C} * H)_\sigma[X]$ and

$$\beta \sum_i \alpha_i X^i + \gamma \sum_i \alpha_i \zeta^i X^i = r.$$

Set $\alpha' = \sum_i \alpha_i \zeta^i x^i$. Applying the homomorphism θ, we now have $\beta\alpha + \gamma\alpha' = r$. By Lemma 2.4 there exists $y \in \mathcal{G}_n$ such that $yhy^{-1} = h$ for all $h \in H$ and $yxy^{-1} = \zeta x$, and then we have $y\alpha y^{-1} = \alpha'$. Thus $\ker \alpha' = y(\ker \alpha)y^{-1}$, consequently $\mathcal{N}(\alpha') = y\mathcal{N}(\alpha)y^{-1}$ and using Lemma 3.2(iii), we deduce that $\tau\mathcal{N}(\alpha') = \tau N(\alpha) > \nu/2$.

Suppose $f \in \ker \alpha \cap \ker \alpha'$. Then $\alpha f = \alpha' f = 0$, hence $rf = 0$ because $r = \beta\alpha + \gamma\alpha'$, and we can invoke our inductive hypothesis to deduce that $f = 0$.

Therefore $\ker \alpha \cap \ker \alpha' = 0$. If π is the projection onto $\overline{\ker \alpha + \ker \alpha'}$, we now see from Lemma 3.2(vi) that

$$\tau\pi = \tau\mathcal{N}(\alpha) + \tau\mathcal{N}(\alpha') > \nu/2 + \nu/2 = \nu.$$

Using Lemma 2.3 we may choose δ so that $0 \neq \delta \in \mathbb{C} * G\alpha \cap \mathbb{C} * G\alpha'$, and then $\ker \delta \supseteq \ker \alpha + \ker \alpha'$, hence $\tau\mathcal{N}(\delta) \geq \tau\pi > \nu$ by Lemma 3.2(v). This contradicts the definition of ν and (i) is proven.

(ii) This follows from (i) and the remarks immediately preceding Theorem 1.4.

(iii) This follows from (ii), Lemma 2.3 and the comments immediately preceding Lemma 2.2. □

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