Remarks on the HRT Conjecture

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Abstract Motivated by a conjecture about time-frequency translations of functions, several properties of the Bargmann–Fock space $\mathscr{H}$ and the Segal–Bargmann transform $\mathscr{S}$ are investigated in this note. In particular, a characterization is given of those square integrable functions φ on $\mathbb{R}$ such that $z \in \mathbb{C} \longmapsto \mathscr{S}\varphi(z+\zeta) \in \mathbb{C}$ is in $\mathscr{H}$ for all $\zeta \in \mathbb{C}$.

1 Marc Yor

This article is dedicated to the memory of Marc Yor, a man whom I knew, liked, and admired for more than forty five years. Although this is not an entirely appropriate venue for reminiscences, I cannot resist the temptation to relate a vignette that epitomizes Marc for me. He and I were attending a conference in Japan. The conference was funded by the Taniguchi foundation and, by mathematical standards, quite lavish. Toward the end of the conference, the participants were invited to a formal dinner at a French restaurant in Kyoto. Having not anticipated such an occasion, Marc had failed to bring a tie and told me that he therefore could not attend the dinner. He was quite distraught about this and much relieved when I offered to lend him one of mine. Out of sartorial pride, nearly anyone else would have refused my offer, but Marc's habidashery was just as dowdy as my own and he had no reservations about accepting. At the end of the meal, the chef appeared and introduced himself, asking whether we had enjoyed the food. Apparently he had spent two years at a Parisian culinary school, and, when he discovered that there were French diners in our group, he began talking to us in what he considered to be French. However, none of the French present could understand a word of what he said and sat dumbfounded while he held forth. Sensing that this could become an embarrassing situation, Marc took it upon himself to respond and ended up accompanying the chef back into the kitchen.

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I met Marc at the Paris VI Laboratoire de Probabilités, where he held a position that provided him lots of time to do research and barely enough money to support his family. Like most young French probabilists at that time, Marc was deeply under the influence of P.A. Meyer and his Strasbourg school. However, although he understood Meyer's work as well as or better than anyone else whom I have known, he was launched on a research program that would take him in a quite different direction. Instead of the tomes of articles and books about abstract martingale theory, Marc had on his desk a large paperback book containing page after page of formulae, most of which he had already assimilated and the rest of which he soon would. He was spending more and more of his time mastering special functions and other material related to hard computation. As a result, Marc developed computational powers that very few probabilists possess, and these, combined with his thorough understanding of stochastic processes, became the tools that enabled him to make the remarkable contributions for which he is renowned.

The topic of this note has scant, if any, relationship to probability theory. Indeed, the only connection is the appearance of the Gaussian distribution and Hermite functions in the otherwise basic classical analysis that follows. Nonetheless, although I cannot pretend to share Marc's skills, I think that he might have found the computations here amusing if not profound, even though the conclusions drawn do not make a major contribution to topic under consideration. At best, this note will introduce a new audience to the topic and suggest an alternative approach. However, like Marc's life, it is sadly incomplete.

2 The HRT Conjecture

Let $\lambda_{\mathbb{R}}$ denote Lebesgue measure on $\mathbb{R}$. Given $(\xi, \eta) \in \mathbb{R}^2$, define the unitary map $\tau_{(\xi,\eta)} : L^2(\lambda_{\mathbb{R}}; \mathbb{C}) \longrightarrow L^2(\lambda_{\mathbb{R}}; \mathbb{C})$ by[1]

$$\tau_{(\xi,\eta)}\varphi(t) = e^{i\xi\eta} e^{i\eta t} \varphi(t + 2\xi).$$

The HRT (Heil, Ramanathan, Topiwala) conjecture in [3] states that for distinct points $(\xi_0, \eta_0), \dots, (\xi_n, \eta_n)$ in $\mathbb{R}^2$ and $\varphi \in L^2(\lambda_{\mathbb{R}}; \mathbb{C}) \setminus \{0\}$, $\tau_{(\xi_0,\eta_0)}\varphi, \dots, \tau_{(\xi_n,\eta_n)}\varphi$ are linearly independent elements of $L^2(\lambda_{\mathbb{R}}; \mathbb{C})$. Experts (of which I am not one) in time-frequency analysis, wavelets, and Gabor bases have expended a great deal of energy in attempts to prove this conjecture, and there is good deal of evidence that it is true.[2] However, as yet, only special cases have been verified, and the present

[1] It should be clear that there is nothing sacrosanct about the choice of either the constant factor or scaling in the definition of $\tau_{(\xi,\eta)}$. I made the choice that I did because it simplifies some of the expressions below.

[2] Evidence of their interest in the conjecture can be found by doing a Google search for "HRT conjecture".

article does little to lengthen the list of those cases. My own interest in the conjecture is its intimate relationship to questions about Weyl operators and Fock space.

To provide a brief introduction to the subject, I will begin by presenting some cases in which the conjecture is easily verified. Observe that

$$\tau_{(\xi,\eta)} \circ \tau_{(\xi',\eta')} = e^{i(\xi\eta'-\xi'\eta)}\tau_{(\xi+\xi',\eta+\eta')}. \tag{1}$$

Using this it is easy to check that the conjecture is true when $n = 1$. Indeed, linear dependence would imply the existence of $(c_0, c_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$ such that $c_0\tau_{(\xi_0,\eta_0)}\varphi = c_1\tau_{(\xi_1,\eta_1)}\varphi$. Clearly $\varphi = 0$ if either c_0 or c_1 is 0. Thus, by (1), we may assume that there exist a $c \neq 0$ and a $(\xi, \eta) \neq (0,0)$ such that $\varphi(t) = ce^{i\eta t}\varphi(t+\xi)$. If $\xi = 0$, then it is clear that this is possible only if $\varphi = 0$. On the other hand, if $\xi \neq 0$, then we would have that $|\varphi(t)|^2 = |c||\varphi(t+\xi)|^2$. Thus, either $\varphi = 0$ or $|c| = 1$. But if $|c| = 1$, then $|\varphi(t)|^2$ is a periodic, integrable function, and as such must be 0.

Using the following elementary lemma, another case for which one can easily verify their conjecture is when the (ξ_m, η_m)'s lie on either a vertical or a horizontal line.

Lemma 2.1 *Let $z_0, \dots, z_n \in \mathbb{C}$ be distinct and $c_0, \dots, c_n \in \mathbb{C}$ not all 0. Set $\psi(t) = \sum_{m=1}^{n} c_m e^{z_m t}$ for $t \in \mathbb{R}$. Then, for each $t \in \mathbb{R}$, $\varphi^{(k)}(t) \neq 0$ for some $0 \le k \le n$. In particular, the zeroes of φ are isolated and therefore φ vanishes at most countably often.*

Proof Suppose that $\varphi^{(k)}(t) = 0$ for $0 \le k \le n$, and set $c'_m = c_m e^{z_m t}$. Then $\sum_{m=0}^{n} c'_m z_m^k = 0$ for $0 \le k \le n$. But this would mean that the matrix $\big((z_m^k)\big)_{0\le k,m\le n}$ is degenerate, and therefore there must exist $b_0, \dots, b_n \in \mathbb{C}$, not all 0, such that the at most nth order polynomial $p(\zeta) = \sum_{m=0}^{n} b_k\zeta^k$ vanishes at the $(n+1)$ points $z_0, \dots, z_n$. □

To check the conjecture when the (ξ_m, η_m)'s lie on a vertical line, first note that, by (1), it suffices to handle the case when $\xi_m = 0$ for $0 \le m \le n$. In this case, linear dependence would imply the existence of $(c_0, \dots, c_n) \in \mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$ that such that $\varphi(t)\sum_{m=0}^{n} c_m e^{i\eta_m t} = 0$, and since $\sum_{m=0}^{n} c_m e^{i\eta_m t}$ can vanish only countably often, this would mean that $\varphi = 0$ almost everywhere. When the (ξ_m, η_m)'s lie on a horizontal line, again one can use (1) to reduce this time to the case when the η_m's are zero. But then linear dependence implies $\sum_{m=0}^{n} c_m\varphi(t+\xi_m) = 0$ for some $(c_0, \dots, c_n) \in \mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$. Since this would mean that $\hat{\varphi}(t)\sum_{m=0}^{n} c_m e^{-i\xi_m t} = 0$, we know that linear dependence implies $\varphi = 0$ in this case also.

In the next section I will develop some machinery that will allow us to recast the HRT in terms of Fock space.

3 Fock Space

Larry Baggett, who introduced me to the HRT conjecture, suggested that it would be helpful to reformulate the conjecture in Fock space, and, as we will see, his was a good suggestion because it allows us to introduce rotations into our arguments. What follows is a brief introduction to Fock space and an isometric, isomorphism between it and $L^2(\lambda_{\mathbb{R}};\mathbb{C})$.

Let $\gamma_{\mathbb{C}}$ be the Gaussian measure on the complex plane $\mathbb{C}$ given by $\gamma_{\mathbb{C}}(dz) = (\pi)^{-1}e^{-|z|^2}\,dz$, where dz here and elsewhere denotes Lebesgue measure on $\mathbb{C}$, and let $\mathscr{H}$ be the space of analytic functions f on $\mathbb{C}$ in $L^2(\gamma_{\mathbb{C}};\mathbb{C})$. Using the mean-value property for analytic functions, it is easy to show that $\mathscr{H}$ is closed in $L^2(\lambda_{\mathbb{R}};\mathbb{C})$, and the Hilbert space $\mathscr{H}$ with the inner product structure it inherits from $L^2(\gamma_{\mathbb{C}};\mathbb{C})$ is a called the either *Bargmann-Fock space*[3] or just *Fock space*.

In connection with his program to construct quantum fields, I.M. Segal [4] introduced an isometric isomorphism that takes $L^2(\lambda_{\mathbb{R}};\mathbb{C})$ onto $\mathscr{H}$. To describe the transformation, first observe that $\{z^n : n \geq 0\}$ is an orthogonal basis in $\mathscr{H}$. To check this, use polar coordinates to see that

$$\int z^m\bar{z}^n\,\gamma_{\mathbb{C}}(dz) = \delta_{m,n}2\int_0^\infty r^{2m+1}e^{-r^2}\,dr = \delta_{m,n}\int_0^\infty \rho^m e^{-\rho}\,d\rho = \delta_{m,n}m!,$$

which proves orthogonality and that $\|z^m\|_{\mathscr{H}} = \sqrt{m!}$. Next, define the operator $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, and note that $\partial_z^m e^{-|z|^2} = (-\bar{z})^m e^{-|z|^2}$. Hence, after integrating by parts, one sees that if $f \in \mathscr{H}$, then, by the mean value property for analytic functions,

$$\begin{aligned}\big(f, z^m\big)_{\mathscr{H}} &= \frac{1}{\pi}\int f^{(m)}(z)e^{-|z|^2}\,dz = \frac{1}{\pi}\int_0^\infty re^{-r^2}\left(\int_0^{2\pi} f^{(m)}\big(re^{i\theta}\big)\,d\theta\right)dr\\ &= 2f^{(m)}(0)\int_0^\infty re^{-r^2}\,dr = f^{(m)}(0),\end{aligned}$$

which means that

$$\sum_{m=0}^\infty \frac{(f, z^m)_{\mathscr{H}}}{m!}z^m = \sum_{m=0}^\infty \frac{f^{(m)}(0)}{m!}z^m = f(z).$$

Next define the Hermite function $h_n(t) = (-1)^n(2\pi)^{-\frac{1}{4}}e^{\frac{t^2}{4}}\partial_t^n e^{-\frac{t^2}{2}}$ for $n \geq 0$. Then it is a familiar fact that $\{h_m : m \geq 0\}$ is an orthogonal basis in $L^2(\lambda_{\mathbb{R}};\mathbb{C})$. In addition, $\|h_m\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} = \sqrt{m!}$. The *Segal-Bargmann transform* is the isometric,

[3] Before Segal's paper appeared in print, V. Bargmann discussed the same isomorphism in [1] without, in Segal's opinion, giving sufficient credit to its provenance. Segal never forgave Bargmann. For a more complete account of these ideas, see [2].

isomorphism $\mathscr{S}$ from $L^2(\lambda_{\mathbb{R}};\mathbb{C})$ onto $\mathscr{H}$ that takes $h_m \in L^2(\lambda_{\mathbb{R}};\mathbb{C})$ to z^m. That is,

$$\mathscr{S}\varphi(z) = \sum_{m=0}^{\infty} \frac{(\varphi, h_m)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}}{m!} z^m \quad \text{for } \varphi \in L^2(\lambda_{\mathbb{R}};\mathbb{C})$$

and

$$\mathscr{S}^{-1} f = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} h_m \quad \text{for } f \in \mathscr{H}.$$

There is another representation of $\mathscr{S}$. Namely, by Taylor's theorem one knows that

$$\sum_{m=0}^{\infty} \frac{h_m(t)}{m!} z^m = e^{-\frac{z^2}{2}} e^{zt-\frac{t^2}{4}}.$$

Hence,

$$\mathscr{S}\varphi(z) = (2\pi)^{-\frac{1}{4}} \int k(z,t)\varphi(t)\,dt \quad \text{where } k(z,t) = (2\pi)^{-\frac{1}{4}} e^{-\frac{z^2}{2}} e^{zt-\frac{t^2}{4}}. \tag{2}$$

Using this representation for $\mathscr{S}$ it is easy to check that if $(\xi,\eta) \in \mathbb{R}^2$ and $\zeta = \xi + i\eta$, then

$$\mathscr{S} \circ \tau_{(\xi,\eta)} = \mathscr{U}_\zeta \circ \mathscr{S} \quad \text{where } \mathscr{U}_\zeta f(z) = e^{-\frac{|\zeta|^2}{2} - \bar{\zeta} z} f(z+\zeta) \quad \text{for } f \in \mathscr{H}. \tag{3}$$

We will make use of the following elementary application of complex analysis.

Lemma 3.1 *Suppose that $\Phi : \mathbb{C} \longrightarrow \mathbb{C}$ is an analytic function, and for each $\zeta \in \mathbb{C}$ define $\varphi^\zeta : \mathbb{R} \longrightarrow \mathbb{C}$ by $\varphi^\zeta(t) = \Phi(t+\zeta)$. If*

$$\int_{\mathbb{R}\times[-R,R]} |\Phi(x+iy)|^2\,dxdy < \infty \quad \textit{for all } R \in (0,\infty),$$

then $\varphi^\zeta \in L^2(\lambda_{\mathbb{R}};\mathbb{C})$ and $\widehat{\varphi^\zeta}(\tau) = e^{-i\zeta\tau}\widehat{\varphi^0}(\tau)$ for all $\zeta \in \mathbb{C}$.

Proof Clearly the result for all $\zeta \in \mathbb{C}$ follows as soon as one proves it for purely imaginary ζ's. Thus we will restrict our attention to $\zeta = i\eta$ for $\eta \in \mathbb{R}$.

First observe that $\varphi^{i\eta} \in L^2(\lambda_{\mathbb{R}};\mathbb{C})$ for (Lebesgue) almost every $\eta \in \mathbb{R}$. Next, by the mean-value property for analytic functions,

$$|\Phi(z)| \le \frac{1}{\pi r^2} \int_{B(z,r)} |\Phi(x+iy)|\,dxdy \le \frac{1}{\sqrt{\pi}\,r} \left(\int_{B(z,r)} |\Phi(x+iy)|^2\,dxdy \right)^{\frac{1}{2}},$$

and so Φ is uniformly bounded on $\mathbb{R} \times [-R,R]$ for all $R > 0$.

Now set $\Phi_\epsilon(z) = e^{-\frac{\epsilon z^2}{2}}\Phi(z)$ for $\epsilon > 0$, and define $\varphi_\epsilon^{i\eta}(t) = \Phi_\epsilon(t + i\eta)$. Given $\eta_1 < \eta_2$ and $\tau \in \mathbb{R}$, apply Cauchy's formula to $z \rightsquigarrow e^{i\tau z}\Phi_\epsilon(z)$ on regions of the form $\{x + i\eta : x \in [-r, r] \ \&\ \eta \in [\eta_1, \eta_2]\}$ to obtain that $\widehat{\varphi_\epsilon^{i\eta_2}}(\tau) = e^{(\eta_2-\eta_1)\tau}\widehat{\varphi_\epsilon^{i\eta_1}}(\tau)$ after letting $r \to \infty$. In particular,

$$\widehat{\varphi^{i\eta_2}}(\tau) = e^{(\eta_2-\eta_1)\tau}\widehat{\varphi^{i\eta_1}}(\tau) \quad \text{if } \varphi^{i\eta_1}, \varphi^{i\eta_2} \in L^2(\lambda_{\mathbb{R}}; \mathbb{C}). \tag{*}$$

Thus, all that remains is to show that $\varphi^{i\eta} \in L^2(\lambda_{\mathbb{R}}; \mathbb{C})$ for all $\eta \in \mathbb{R}$. To this end, choose $\eta_- \le \eta + 2$, $\eta_+ \ge \eta + 2$, and and $\{\eta_k : k \ge 1\} \subseteq (\eta - 1, \eta + 1)$ so that $\varphi^{i\eta\pm} \in L^2(\lambda_{\mathbb{R}}; \mathbb{C})$, $\eta_k \longrightarrow \eta$, and $\varphi^{i\eta_k} \in L^2(\lambda_{\mathbb{R}}; \mathbb{C})$ for each $k \ge 1$. By (*),

$$\left(e^{(\eta_+-\eta_k)\tau} + e^{(\eta_--\eta_k)\tau}\right)\widehat{\varphi^{i\eta_k}} = \widehat{\varphi^{i\eta_+}} + \widehat{\varphi^{i\eta_-}},$$

and so $|\widehat{\varphi^{i\eta_k}}| \le |\widehat{\varphi^{i\eta_+}}| + |\widehat{\varphi^{i\eta_-}}|$. But, by Parseval's identity, this shows that

$$\sup_{k\ge1} \|\varphi^{i\eta_k}\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} < \infty$$

and therefore, by Fatou's lemma, that $\|\varphi^{i\eta}\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} < \infty$.

Our first application of this lemma provides another way of recovering φ from $\mathscr{S}\varphi$.

Lemma 3.2 *Given $\varphi \in L^2(\lambda_{\mathbb{R}}; \mathbb{C})$, set $f = \mathscr{S}\varphi$ and define $\Psi(z) = e^{-\frac{z^2}{2}} f(z)$ for $z \in \mathbb{C}$. If ψ is the restriction of Ψ to $\mathbb{R}$, then $\psi \in L^2(\lambda_{\mathbb{R}}; \mathbb{C})$,*

$$e^{\frac{\tau^2}{4}}\hat{\psi}(\tau) = \left(\frac{\pi}{2}\right)^{\frac14}\hat{\varphi}\left(\tfrac{\tau}{2}\right)$$

and so

$$\int e^{\frac{\tau^2}{2}}|\hat{\psi}(\tau)|^2\, d\tau = (2\pi)^{\frac32}\|f\|^2_{\mathscr{H}}.$$

Proof Since

$$\int e^{-2y^2}|\Psi(x + iy)|^2\, dxdy = \pi\|f\|^2_{\mathscr{H}},$$

Lemma 3.1 applies to Ψ and guarantees that $\psi \in L^2(\lambda_{\mathbb{R}}; \mathbb{C})$. Next observe that

$$\begin{aligned}\int e^{i\tau\xi}\hat{\psi}(\xi)\, d\xi &= (2\pi)^{-\frac14}\int e^{-\frac{t^2}{4}}\varphi(t)\left(\int e^{-\xi^2}e^{\xi(t+i\tau)}\, d\xi\right)dt \\ &= \left(\frac{\pi}{2}\right)^{\frac14} e^{-\frac{\tau^2}{4}}\int e^{\frac{i\tau t}{2}}\varphi(t)\, dt = \left(\frac{\pi}{2}\right)^{\frac14} e^{-\frac{\tau^2}{4}}\hat{\varphi}\left(\tfrac{\tau}{2}\right).\end{aligned}$$

Finally, the concluding assertion follows easily from this when one applies Parseval's identity and remembers that $\|f\|_{\mathscr{H}} = \|\varphi\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}$. $\square$

Let $\mathbb{D}$ denote the closed unit disk in $\mathbb{C}$, and for $\omega \in \mathbb{D}$ define $M_\omega f$ for $f \in \mathscr{H}$ by $M_\omega f(z) = f(\omega z)$. Then it is obvious that M_ω takes $\mathscr{H}$ into itself, M_ω is a contraction all $\omega \in \mathbb{D}$, and that M_ω is unitary if and only if $\omega \in \partial\mathscr{H}$. In addition, the operation H_ω on $L^2(\lambda_{\mathbb{R}};\mathbb{C})$ for which $M_\omega \circ \mathscr{S} = \mathscr{S} \circ H_\omega$ is

$$H_\omega\varphi = \sum_{m=0}^{\infty} \frac{\omega^m (\varphi, h_m)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}}{m!} h_m \quad \text{for } \omega \in \mathbb{D} \text{ and } \varphi \in L^2(\lambda_{\mathbb{R}};\mathbb{C}).$$

To develop an expression for H_ω as an integral operator, define

$$H(\omega, t, s) = \frac{1}{\sqrt{2\pi(1-\omega^2)}} \exp\left(-\frac{(1+\omega^2)t^2 - 4\omega ts + (1+\omega^2)s^2}{4(1-\omega^2)} \right) \tag{4}$$

for $(\omega, t, s) \in \operatorname{int}(\mathbb{D}) \times \mathbb{R} \times \mathbb{R}$. One can then check that $H_\omega h_n = \omega^n h_n$. Perhaps the easiest way to do this is to note that, for each $t \in \mathbb{R}$, both $H_\omega h_n(t)$ and $\omega^n h_n(t)$ are analytic functions on $\operatorname{int}(\mathbb{D})$. Hence, it suffices to check the equation when $\omega \in (0, 1)$. To this end, set $u(\tau, t) = \int H(e^{-\tau}, t, s) h_n(s)\, ds$. Then $u(\tau, t) \longrightarrow h_n(t)$ as $\tau \searrow 0$, and a straight forward computation shows that $\partial_\tau u = \left(\partial_t^2 - \frac{t^2}{2} + \frac{1}{2}\right)u$. At the same time, it is well known that $\left(\partial_t^2 - \frac{t^2}{2} + \frac{1}{2}\right)h_n = -nh_n$, and therefore $e^{-n\tau}h_n(t)$ is another solution to this initial value problem. Hence, by standard uniqueness results for solutions to the Cauchy initial value problem for parabolic equations, $u(\tau, t) = e^{-n\tau}h_n(t)$. When $\omega \in \partial\mathbb{D} \setminus \{1\}$, one can again express H_ω as an integral operator, although the integrand will no longer be integrable in general. Indeed, if

$$H(\omega, t, s) = \frac{1}{\sqrt{2\pi(1-\omega^2)}} \exp\left(-i\frac{\Im(\omega^2)t^2 - 4\Im(\omega)st + \Im(\omega^2)s^2}{2|1-\omega^2|^2} \right), \tag{5}$$

then, by the same argument as one uses to define the Fourier transform, one can show that

$$H_\omega\varphi(t) = \lim_{R\to\infty} \int_{-R}^{R} H(\omega, t, s)\varphi(s)\, ds,$$

where the limit is in $L^2(\lambda_{\mathbb{R}};\mathbb{C})$.

4 Rotations and the HRT Conjecture

Suppose $\omega \in \partial\mathbb{D}$. Then $M_\omega \circ \mathscr{U}_\zeta = \mathscr{U}_{\bar{\omega}\zeta} \circ M_\omega$, and so, because $M_\omega \circ \mathscr{S} = \mathscr{S} \circ H_\omega$, (3) implies that $H_\omega \circ \tau_{(\xi,\eta)} = \tau_{(\Re(\bar{\omega}\zeta),\Im(\bar{\omega}\zeta))} \circ H_\omega$, or, equivalently, that $H_\omega \circ \tau_{(\xi,\eta)} = \tau_{\rho_\omega(\xi,\eta)} \circ H_\omega$ where ρ_ω is the rotation given by the matrix $\begin{pmatrix} \Re(\omega) & \Im(\omega) \\ -\Im(\omega) & \Re(\omega) \end{pmatrix}$.

Lemma 4.1 *Let $(\xi_0, \eta_0), \dots, (\xi_\ell, \eta_\ell)$ be distinct points in $\mathbb{R}^2$, and assume that $(\xi_1, \eta_1), \dots, (\xi_\ell, \eta_\ell)$ lie on a line. Then there are real numbers $\alpha_0, \dots, \alpha_\ell$ and β_0 and an $\omega \in \partial\mathbb{D}$ such that, for any $\varphi \in L^2(\lambda_\mathbb{R}; \mathbb{C})$, $\tau_{(\xi_0,\eta_0)}\varphi, \dots, \tau_{(\xi_\ell,\eta_\ell)}\varphi$ are linearly dependent if and only if $\tau_{(\beta_0,\alpha_0)}H_\omega\varphi, \tau_{(0,\alpha_1)}H_\omega\varphi, \dots, \tau_{(0,\alpha_\ell)}H_\omega\varphi$ are.*

Proof Under the stated conditions, there exist $(\xi, \eta) \in \mathbb{R}^2$, $\alpha_0, \dots, \alpha_\ell \in \mathbb{R}$, $\beta_0 \in \mathbb{R}$, and $\theta \in [0, 2\pi)$ such that

$$\begin{aligned} (\xi_0, \eta_0) &= (\xi, \eta) + \alpha_0(-\sin\theta, \cos\theta) + \beta_0(\cos\theta, \sin\theta) \\ (\xi_k, \eta_k) &= (\xi, \eta) + \alpha_k(-\sin\theta, \cos\theta) \quad \text{for } 1 \le k \le \ell. \end{aligned} \tag{6}$$

Using (1), one can replace the (ξ_k, η_k)'s by $(\xi_k, \eta_k) - (\xi, \eta)$, and so we will assume that $(\xi, \eta) = (0, 0)$. Now take $\omega = \cos\theta + i\sin\theta$, and apply the preceding to see that $H_\omega \circ \tau_{(\xi_0,\eta_0)}\varphi = \tau_{(\beta_0,\alpha_0)}H_\omega\varphi$ and $H_\omega \circ \tau_{(\xi_k,\eta_k)}\varphi = \tau_{(0,\alpha_k)}H_\omega\varphi$ for $1 \le k \le \ell$. □

These simple observations allow us to prove the following two results about the HRT conjecture.

Theorem 4.2 *Refer to Lemma 4.1, and assume that, for each $1 \le k \le \ell$, $(\alpha_k - \alpha_0)\beta_0$ is a rational multiple of π. Then $\tau_{(\xi_0,\eta_0)}\varphi, \dots, \tau_{(\xi_\ell,\eta_\ell)}\varphi$ are linearly independent for any $\varphi \in L^2(\lambda_\mathbb{R}; \mathbb{C}) \setminus \{0\}$.*

Proof Assume that $\tau_{(\xi_0,\eta_0)}\varphi, \dots, \tau_{(\xi_\ell,\eta_\ell)}\varphi$ are linearly dependent. Then, by Lemma 4.1,

$$a_0 H_\omega\varphi(t + 2\beta_0) = \left(\sum_{k=1}^{\ell} a_k e^{i\alpha_k t}\right) H_\omega\varphi(t)$$

for some choice of $\omega \in \partial\mathbb{D}$ and $a_0, \dots, a_\ell \in \mathbb{C}$ that are not all 0. Set $\varphi_\omega = H_\omega\varphi$. If $a_0 = 0$ or $\beta_0 = 0$, then, by Lemma 2.1, φ_ω must vanish off a countable set, which means that, as an element of $L^2(\lambda_\mathbb{R}; \mathbb{C})$, it, and therefore φ itself, is 0. Thus, we can assume that $a_0 = 1$ and that $\beta_0 \ne 0$. In fact, because the argument is essentially the same when $\beta_0 < 0$, we will assume that $\beta_0 > 0$. In this case we have that

$$\varphi_\omega(t + 2\beta_0) = \chi(t)\varphi_\omega(t), \quad \text{where } \chi(t) = \sum_{k=1}^{\ell} a_k e^{i(\alpha_k - \alpha_0)t}. \tag{7}$$

Using induction on $n \geq 1$, it follows that

$$\varphi_\omega(t + 2n\beta_0) = \chi_n(t)\varphi_\omega(t) \quad \text{where } \chi_n(t) = \prod_{m=0}^{n-1} \chi\big(t + 2m\beta_0\big).$$

Thus, for any $n \geq 1$,

$$\int_{\mathbb{R}} |\varphi_\omega(t)|^2 \, dt = \int_{\mathbb{R}} |\varphi_\omega(t + 2n\beta_0)|^2 \, dt = \int_{\mathbb{R}} |\chi_n(t)|^2 |\varphi_\omega(t)|^2 \, dt. \tag{*}$$

Because of the rationality hypothesis, there exists a positive integer q such that $q(\alpha_k - \alpha_0)\beta_0$ is an integer multiple of π, and therefore $\chi_{nq}(t) = \chi_q(t)^n$. Hence, for any $m \in \mathbb{Z}$,

$$\begin{aligned}\int_{2mq\beta_0}^{\infty} |\varphi_\omega(t)|^2 \, dt &= \sum_{n=0}^{\infty} \int_{2mq\beta_0}^{2(m+1)q\beta_0} |\varphi_\omega(t + 2nq\beta_0)|^2 \, dt \\ &= \int_{2mq\beta_0}^{2(m+1)q\beta_0} \left(\sum_{n=0}^{\infty} |\chi_q(t)|^{2n} \right) |\varphi_\omega(t)|^2 \, dt,\end{aligned}$$

which is possible only if $|\chi_q(t)| < 1$ for almost every $t \in \mathbb{R}$ at which $\varphi_\omega(t) \neq 0$. Plugging this into (*) and applying Lebesgue's dominated theorem, we conclude that

$$\int_{\mathbb{R}} |\varphi_\omega(t)|^2 \, dt = \int_{\mathbb{R}} |\chi_q(t)|^{2n} |\varphi_\omega(t)|^2 \, dt \longrightarrow 0 \text{ as } n \to \infty.$$

□

Theorem 4.3 *Again refer to Lemma 4.1, let* $(\xi_0, \eta_0), \ldots, (\xi_\ell, \eta_\ell)$ *be given by* (6) *with some* $\beta_0 \neq 0$*, and determine* $(\tilde{\xi}_0, \tilde{\eta}_0)$ *by the first line of* (6) *with* $-\beta_0$ *replacing* β_0*. Assume that, for each* $1 \leq j, k \leq \ell$ *and* $1 \leq j', k' \leq \ell$, $\alpha_j + \alpha_k = \alpha_{j'} + \alpha_{k'}$ *if and only if* $\{j, k\} = \{j', k'\}$*. Then, for each* $\varphi \in L^2(\lambda_{\mathbb{R}}; \mathbb{C}) \setminus \{0\}$*, either* $\tau_{(\xi_0,\eta_0)}\varphi, \ldots, \tau_{(\xi_\ell,\eta_\ell)}\varphi$ *are linearly independent or* $\tau_{(\tilde{\xi}_0,\tilde{\eta}_0)}\varphi, \tau_{(\xi_1,\eta_1)}\varphi, \ldots, \tau_{(\xi_\ell,\eta_\ell)}\varphi$ *are linearly independent.*

Proof The only case needing comment is when, after taking the steps used in the proof of Theorem 4.2, one arrives at

$$\varphi_\omega(t + 2\beta_0) = \chi(t)\varphi_\omega(t) \quad \text{and} \quad \varphi_\omega(t - 2\beta_0) = \tilde{\chi}(t)\varphi_\omega(t),$$

where $\chi(t) = \sum_{k=1}^{\ell} a_k e^{i(\alpha_k - \alpha_0)t}$ and $\tilde{\chi}(t) = \sum_{k=1}^{\ell} \tilde{a}_k e^{i(\alpha_k - \alpha_0)t}$. But then

$$\varphi_\omega(t) = \chi(t - 2\beta_0)\varphi_\omega(t - 2\beta_0) = \chi(t - 2\beta_0)\tilde{\chi}(t)\varphi_\omega(t),$$

and, since $\varphi_\omega(t) \neq 0$ for uncountably many t's, one can apply Lemma 2.1 to conclude that $\chi(t-2\beta_0)\tilde{\chi}(t) = 1$ for all $t \in \mathbb{R}$. Taking $b_k = a_k e^{-i2(\alpha_k+\alpha_0)\beta_0}$ and $\tilde{b}_k = \tilde{a}_k$, we therefore have that

$$\sum_{k=1}^{\ell} b_k\tilde{b}_k e^{i2(\alpha_k-\alpha_0)t} + \frac{1}{2}\sum_{1\le j\neq k\le\ell}(b_j\tilde{b}_k + b_k\tilde{b}_j)e^{i(\alpha_j+\alpha_k-2\alpha_0)t} = 1. \qquad (*)$$

By Lemma 2.1, (*) cannot hold unless either $\alpha_m = \alpha_0$ for some $1 \le m \le \ell$ or $\alpha_m + \alpha_n = 2\alpha_0$ for some $1 \le m \neq n \le \ell$.

Suppose $\alpha_m = \alpha_0$. Then, by Lemma 2.1 and the distinctness of the $\alpha_1,\dots,\alpha_\ell$ and of the sums $\alpha_j + \alpha_k$, $b_k\tilde{b}_k = 0$ for $k \neq m$ and $b_j\tilde{b}_k = -b_k\tilde{b}_j$ for all $j \neq k$. Thus, if $b_m = \tilde{b}_m = 0$ but $\tilde{b}_k \neq 0$ for some $k \neq m$, then $b_k = 0$ and $b_j = -\frac{b_k\tilde{b}_j}{\tilde{b}_k} = 0$ for $j \neq k$, which means that χ must vanish. That is, either $\tilde{b}_k = 0$ for all k, in which case $\tilde{\chi}$ vanishes, or χ vanishes. Similarly, if b_m or $\tilde{b}_m$ is 0, then the equation $b_k\tilde{b}_m = -b_m\tilde{b}_k$ implies either χ or $\tilde{\chi}$ must vanish also. But if either χ or $\tilde{\chi}$ vanishes, then so does φ.

Finally, suppose that $\alpha_k \neq \alpha_0$ for any $1 \le k \le \ell$ but that $\alpha_m + \alpha_n = 2\alpha_0$ for some $m \neq n$. Then $b_k\tilde{b}_k = 0$ for all $1 \le k \le \ell$ and $b_j\tilde{b}_k = -b_k\tilde{b}_j$ if $j \neq k$ and $\{j,k\} \neq \{m,n\}$. Now assume that $\tilde{b}_k \neq 0$ for some k. Then, $b_j = 0$ for all j if $k \notin \{m,n\}$, $b_j = 0$ for $j \neq n$ if $k = m$, and $b_j = 0$ for $j \neq m$ if $k = n$. Hence, either $b_k \neq 0$ or $|\chi|$ would have to be constant, which would lead quickly to the conclusion that $\varphi = 0$. When $b_k \neq 0$, the same reasoning shows that $|\tilde{\chi}|$ is constant and therefore that $\varphi = 0$. □

It may be of some interest to point out that, under the conditions on the α_k's in Theorem 4.3, the conclusion can be formulated as the statement that not both the families $\tau_{(\beta_0,\alpha_0)}\varphi_\omega, \tau_{(0,\alpha_1)}\varphi_\omega \dots, \tau_{(0,\alpha_\ell)}\varphi_\omega$ and $\tau_{(\beta_0,\alpha_0)}\overline{\varphi_\omega}, \tau_{(0,\alpha_1)}\overline{\varphi_\omega} \dots, \tau_{(0,\alpha_\ell)}\overline{\varphi_\omega}$ can be linearly dependent unless $\varphi = 0$.

5 Baggett's Idea

The use to which Baggett put Fock space in connection with the HRT conjecture is the following. Suppose that $\zeta_0,\dots,\zeta_n$ are distinct elements of $\mathbb{C}$ and that $\mathscr{U}_{\zeta_0}f,\dots,\mathscr{U}_{\zeta_n}f$ are linearly dependent for some $f \in \mathscr{H}$. Then, without loss in generality, we may assume that, for some choice of $a_1,\dots,a_n \in \mathbb{C}$, $\mathscr{U}_{\zeta_0}f = \sum_{m=1}^n a_m\mathscr{U}_{\zeta_m}f$, where $\Re(\zeta_0) \le \Re(\zeta_m)$ and $\Re(\zeta_m) = \Re(\zeta_0) \implies \Im(\zeta_m) > \Im(\zeta_0)$ for each $1 \le m \le n$. Hence, if $\tilde{f} = \mathscr{U}_{-\zeta_0}f$, $\tilde{a}_m = e^{i\Im(\zeta_0\tilde{\zeta}_m)}a_m$, and $\tilde{\zeta}_m = \zeta_m - \zeta_0$ for $1 \le m \le n$, then $\tilde{f} = \sum_{m=1}^n \tilde{a}_m\mathscr{U}_{\tilde{\zeta}_m}f$ where $\tilde{\zeta}_m = \xi_m + i\eta_m$ have the property that

$$\xi_m \ge 0 \quad \text{and} \quad \xi_m = 0 \implies \eta_m > 0.$$

In particular, there exist $\alpha > 0$ and $\epsilon > 0$ such that $\alpha\xi_m + \eta_m \geq \epsilon$ for all $1 \leq m \leq n$. Now choose $\beta > 0$ so that $e^{\beta\epsilon} \geq 2\sum_{m=1}^n |\tilde{a}_m|$, and set $\zeta = \beta(\alpha + i)$. Then, the translate $T_\zeta \tilde{f}$ given by $T_\zeta \tilde{f}(z) = \tilde{f}(z + \zeta)$ satisfies

$$T_\zeta \tilde{f} = \sum_{m=1}^n e^{-\zeta\overline{\tilde{\zeta}_m}} \tilde{a}_m \mathscr{U}_{\tilde{\zeta}_m} T_\zeta \tilde{f}.$$

Since

$$\sum_{m=1}^n \left| e^{-\zeta\overline{\tilde{\zeta}_m}} \tilde{a}_m \right| \leq \frac{1}{2},$$

it follows that $\|T_\zeta \tilde{f}\|_{\mathscr{H}} \leq \frac{1}{2}\|T_\zeta \tilde{f}\|_{\mathscr{H}}$ and therefore that $f = 0$ if $T_\zeta \circ \mathscr{U}_{-\zeta_0} f \in \mathscr{H}$, which, since

$$\begin{aligned}\int |T_\zeta \circ \mathscr{U}_{-\zeta_0} f|^2 e^{-|z|^2}\, dz &= e^{2\Re(\bar{\zeta}_0\zeta)} \int |f(z - \zeta_0 + \zeta)|^2 e^{-|z-\zeta_0|^2}\, dz \\ &= e^{2\Re(\bar{\zeta}_0\zeta)} \int |f(z + \zeta)|^2 e^{-|z|^2}\, dz\end{aligned}$$

is tantamount to the assumption that $T_\zeta f \in \mathscr{H}$. Thus Baggett's argument proves that the Fock space formulation of HRT conjecture holds for $f \in \mathscr{H}$ with the property that, for each $\zeta \in \mathbb{C}$, the translate $T_\zeta f$ of f is again in $\mathscr{H}$.

Unfortunately, although this property holds for lots of $f \in \mathscr{H}$, it definitely does hold for all. For example, if $f(z) = e^{\frac{z^2}{2}} \frac{\sin z}{z}$, then $f \in \mathscr{H}$ but $T_\zeta f \notin \mathscr{H}$ unless $\Re(\zeta) = 0$. On the other hand, if $f \in \mathscr{H} \cap L^{2p}(\gamma_{\mathbb{C}}; \mathbb{C})$ for some $p > 1$, then $T_\zeta f \in \mathscr{H}$ for all $\zeta \in \mathbb{C}$. Indeed, writing $z = x + iy$ and $\zeta = \xi + i\eta$, one has that

$$\begin{aligned}\int |f(z + \zeta)|^2 e^{-|z|^2}\, dz &= e^{-|\zeta|^2} \int |f(z)|^2 e^{2(x\xi - y\eta)} e^{-|z|^2}\, dz \\ &\leq e^{-|\zeta|^2} \left(\int |f(z)|^{2p} e^{-|z|^2}\, dz \right)^{\frac{1}{p}} \left(\int e^{2p'(x\xi - y\eta)} e^{-|z|^2}\, dz \right)^{\frac{1}{p'}} \\ &= e^{(p'-1)|\zeta|^2} \|f\|^2_{L^p(\gamma_{\mathbb{C}}; \mathbb{C})},\end{aligned}$$

where $p' = \frac{p}{p-1}$ is the Hölder conjugate of p. In this connection, notice that if $\theta \in (0, 1)$, then $M_\theta f \in L^{2p}(\mu_{\mathbb{C}}; \mathbb{C})$ for $1 \leq p < \theta^{-2}$. Thus, since $M_\theta f \longrightarrow f$ in $\mathscr{H}$ as $\theta \nearrow 1$ and $M_\theta \circ \mathscr{S}\varphi = \mathscr{S} \circ H_\theta \varphi$, we know that the HRT conjecture holds for a dense set of $\varphi \in L^2(\lambda_{\mathbb{R}}; \mathbb{C})$.

In view of the preceding considerations, it would be interesting to characterize those $\varphi \in L^2(\lambda_{\mathbb{R}}; \mathbb{C})$ for which $T_\zeta \circ \mathscr{S}\varphi \in \mathscr{H}$ for all $\zeta \in \mathscr{H}$, and the rest of this article is devoted to finding such a characterization.

6 Translation in $\mathscr{H}$

Suppose that $f \in \mathscr{H}$, and set $\varphi = \mathscr{S}^{-1}f$. Then, after some elementary manipulations, one sees that

$$T_\zeta f(z) = (2\pi)^{-\frac{1}{4}} e^{\frac{\zeta^2}{4}} e^{-\frac{z^2}{2}} \int e^{z(t-\zeta)-\frac{(t-\zeta)^2}{4}} e_{\frac{\zeta}{2}}(t-\zeta)\varphi(t)\, dt, \tag{8}$$

where $e_\zeta(t) = e^{\zeta t}$. When $\zeta = \xi \in \mathbb{R}$, (8) implies that

$$T_\xi f(z) = e^{\frac{\xi^2}{4}} \int k(z,t) e_{\frac{\xi}{2}}(t)\varphi(t+\xi)\, dt,$$

and so $T_\xi f \in \mathscr{H}$ if and only if $e_{\frac{\xi}{2}}\varphi \in L^2(\lambda_{\mathbb{R}}; \mathbb{C})$, in which case one has that

$$T_\xi f = e^{\frac{\xi^2}{4}} \mathscr{S}\big(e_{\frac{\xi}{2}} \varphi^\xi\big) \quad \text{where } \varphi^\xi(t) = \varphi(\xi + t). \tag{9}$$

When ζ is not real, we will use Lemma 3.1 to justify the preceding change of variables.

Notice that the function $H(\omega, t, s)$ in (4) extends as an analytic function on $\mathrm{int}(\mathbb{D}) \times \mathbb{C} \times \mathbb{C}$ and that

$$\int_{\mathbb{R}} |H(\theta, w, s)|^2\, ds = \frac{1}{\sqrt{1+\theta^2}} \exp\left(\frac{1+\theta^2}{2(1-\theta^2)} v^2 - \frac{1-\theta^2}{2(1+\theta^2)} u^2\right) \tag{10}$$
$$\text{for } \theta \in (0,1) \text{ and } w = u + iv.$$

Hence, if

$$H_\theta \varphi(w) = \int H(\theta, w, s)\varphi(s)\, ds \quad \text{for } \varphi \in L^2(\lambda_{\mathbb{R}}; \mathbb{C}), \theta \in (0,1), \text{ and } w \in \mathbb{C},$$

then, for each $\theta \in (0,1)$, Lemma 3.1 applies to the function

$$\Phi(w) = e^{z(w-\zeta)-\frac{(w-\zeta)^2}{4}} e_{\frac{\zeta}{2}}(w-\zeta) H_\theta \circ \mathscr{S}^{-1} f(w)$$

and, together with (8), allows us to conclude that

$$T_\zeta \circ M_\theta f = e^{\frac{\zeta^2}{4}} \mathscr{S}\big(e_{\frac{\zeta}{2}}(H_\theta \circ \mathscr{S}^{-1} f\big) \tag{11}$$

for $f \in \mathscr{H}$ and $\theta \in (0,1)$.

Theorem 6.1 *Let $f \in \mathscr{H}$, and set $\varphi = \mathscr{S}^{-1}f$. Then $T_\zeta f \in \mathscr{H}$ if and only if*

$$\varliminf_{\theta \nearrow 1} \left\| e_{\frac{\zeta}{2}} (H_\theta \varphi)^\zeta \right\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} < \infty. \tag{12}$$

Moreover, if (12) holds, then $e_{\frac{\zeta}{2}}(H_\theta\varphi)^\zeta \longrightarrow e^{-\frac{\zeta^2}{4}} \mathscr{S}^{-1} \circ T_\zeta f$ in $L^2(\lambda_{\mathbb{R}};\mathbb{C})$ as $\theta \nearrow 1$.

Proof Set $f_\theta = M_\theta f$, $\varphi = \mathscr{S}^{-1}f$, and $\varphi_\theta = H_\theta\varphi$. By (11),

$$T_\zeta f_\theta = e^{\frac{\zeta^2}{4}} \mathscr{S}\big(e_{\frac{\zeta}{2}} \varphi_\theta^\zeta\big). \tag{*}$$

First suppose that $T_\zeta f \in \mathscr{H}$. If we show that $T_\zeta f_\theta \longrightarrow T_\zeta f$ in $\mathscr{H}$, then, by (*) we will know that $(e_{\frac{\zeta}{2}}\varphi_\theta)^\zeta \longrightarrow e^{-\frac{\zeta^2}{4}} \mathscr{S}^{-1} \circ T_\zeta f$ in $L^2(\lambda_{\mathbb{R}};\mathbb{C})$. Because $T_\zeta f_\theta(z) = f(\theta z + \theta\zeta) \longrightarrow T_\zeta f(z)$ for all $z \in \mathbb{C}$, $T_\zeta f_\theta \longrightarrow T_\zeta f$ in $\mathscr{H}$ will follow once we show that $\|T_\zeta f_\theta\|_{\mathscr{H}} \longrightarrow \|T_\zeta f\|_{\mathscr{H}}$. To this end, observe that

$$\int e^{-|z|^2} |f(\theta z + \theta\zeta)|^2 \, dz = \theta^{-2} \int e^{-|\theta^{-1}(z+(1-\theta)\zeta)|^2} |T_\zeta f(z)|^2 \, dz.$$

Thus, since

$$|\theta^{-1}(z + (1-\theta)\zeta)|^2 \geq |z|^2 - \frac{1-\theta}{1+\theta}|\zeta|^2,$$

the desired convergence follows from Lebesgue's dominated convergence theorem.

Finally, assume that (12) holds. Then, since $T_\zeta f_\theta \longrightarrow T_\zeta f$ pointwise, Fatou's lemma says that $T_\zeta f \in \mathscr{H}$.

□

Corollary 6.2 *Let $f \in \mathscr{H}$, and set $\varphi = \mathscr{S}^{-1}f$. If $\xi \in \mathbb{R}$, then $T_\xi f \in \mathscr{H}$ if and only if $e_{\frac{\xi}{2}}\varphi \in L^2(\lambda_{\mathbb{R}};\mathbb{C})$, in which case (9) holds. If $\eta \in \mathbb{R}$, then $T_{i\eta} f \in \mathscr{H}$ if and only if*

$$\int e^{2\eta\tau} |\hat{\varphi}(\tau)|^2 \, d\tau < \infty,$$

in which case $T_{i\eta} f = e^{-\frac{\eta^2}{4}} \mathscr{S}\big(e_{\frac{i\eta}{2}} \varphi^{i\eta}\big)$, where

$$\varphi^{i\eta}(t) = \frac{1}{2\pi} \int e^{-it\tau} e^{\eta\tau} \hat{\varphi}(\tau) \, d\tau$$

is the inverse Fourier transform of $e_\eta\hat{\varphi}$. *In particular, if* $\zeta = \xi + i\eta$, *then both* $T_\xi f$ *and* $T_\zeta f$ *are in* $\mathscr{H}$ *if and only if* $e_{\frac{\xi}{2}}\varphi \in L^2(\lambda_{\mathbb{R}};\mathbb{C})$ *and*

$$\int e^{2\eta\tau}\left|\widehat{e_{\frac{\xi}{2}}\varphi}(\tau)\right|^2 d\tau < \infty,$$

in which case $T_\zeta f = e^{\frac{\zeta^2}{4}}\mathscr{S}\big(e_{\frac{\zeta}{2}}\varphi^\zeta\big)$ *where*

$$\varphi^\zeta(t) = \frac{1}{2\pi}\int e^{-i(t+\zeta)\tau}\hat{\varphi}(\tau)\,d\tau$$

is the inverse Fourier transform of $e_{-i\zeta}\hat{\varphi}$.

Proof The first assertion was covered in the discussion leading to (9).

To prove the second assertion, first assume that $T_{i\eta}f \in \mathscr{H}$. Set $\Phi_\theta = H_\theta\varphi$ and $\varphi_\theta^\zeta(t) = \Phi_\theta(\zeta + t)$ for $\zeta \in \mathbb{C}$. Then, by Lemma 3.1, $\widehat{\varphi_\theta^\zeta}(\tau) = e^{-i\zeta\tau}\widehat{\varphi_\theta}(\tau)$, and so

$$\varphi_\theta^\zeta(t) = \frac{1}{2\pi}\int e^{-i(t+\zeta)\tau}\widehat{\varphi_\theta}(\tau)\,d\tau.$$

Hence, by (11) and Parseval's identity,

$$\|T_{i\eta}\circ M_\theta f\|_{\mathscr{H}}^2 = \frac{e^{-\frac{\eta^2}{2}}}{2\pi}\int e^{2\eta\tau}|\widehat{\varphi_\theta}(\tau)|^2\,d\tau,$$

and so, by Fatou's lemma and Theorem 6.1,

$$\int e^{2\eta\tau}|\hat{\varphi}(\tau)|^2\,d\tau \le \varliminf_{\theta\nearrow 1}\int e^{2\eta\tau}|\widehat{\varphi_\theta}(\tau)|^2\,d\tau = 2\pi e^{\frac{\eta^2}{2}}\varliminf_{\theta\nearrow 1}\|T_{i\eta}\circ M_\theta f\|_{\mathscr{H}} < \infty.$$

Finally, assume that $\eta \in (0,\infty)$ and that $\int e^{2\eta\tau}|\hat{\varphi}(\tau)|^2\,d\tau < \infty$. Because

$$\left|e^{-i\omega\tau}\hat{\varphi}(\tau)\right| \le \mathbf{1}_{[0,\infty)}(\tau)e^{|\Im(\omega)|\tau}|\hat{\varphi}(\tau)| + \mathbf{1}_{(-\infty,0]}(\tau)|\hat{\varphi}(\tau)|, \tag{*}$$

it is clear that, for each $0 < \delta < \frac{\eta}{2}$, $\tau \rightsquigarrow e^{-i\omega\tau}\hat{\varphi}(\tau)$ is uniformly integrable for ω with $\Im(\omega) \in [\delta, \eta-\delta]$ and therefore that

$$\omega \rightsquigarrow \Phi(\omega) = \frac{1}{2\pi}\int e^{-i\omega\tau}\hat{\varphi}(\tau)\,d\tau \quad \text{for } \omega \in \mathbb{C}\big([0,\eta)\big).$$

is analytic for ω with $\Im(\omega) \in [0,\eta)$. Furthermore, if $\varphi^\zeta(t) = \Phi(\zeta + t)$ for ζ with $\Im(\zeta) \in (0,\eta)$, then $\widehat{\varphi^{i\eta'}}(\tau) = e^{\eta'\tau}\hat{\varphi}(\tau)$ for $\eta' \in (0,\eta)$, and so by (*), Lebesgue's dominated convergence theorem, and Parseval's identity, $\varphi^{i\eta'} \longrightarrow \varphi^{i\eta}$ in $L^2(\lambda_{\mathbb{R}};\mathbb{C})$

as $\eta' \nearrow \eta$ and $\varphi^{i\eta'} \longrightarrow \varphi$ in $L^2(\lambda_{\mathbb{R}};\mathbb{C})$ as $\eta' \searrow 0$. Finally, if $0 < \eta_1 < \eta_2 < \eta$, then, by Lemma 3.1,

$$\int e^{z(t-i(\eta_2-\eta_1))-\frac{(t-i(\eta_2-\eta_1))^2}{4}} e_{\frac{i\eta}{2}}\big(t-i(\eta_2-\eta_1)\big)\varphi^{i\eta_1}(t)\,dt = \int e^{zt-\frac{t^2}{4}} e_{\frac{i\eta}{2}}(t)\varphi^{i\eta_2}(t)\,dt.$$

Thus, by (8), we get the desired result by letting $\eta_1 \searrow 0$ and $\eta_2 \nearrow \eta$. When $\eta < 0$, one can apply the preceding to $\omega \rightsquigarrow \Psi(-\omega)$.

□

The following corollary follows easily from the preceding and the fact that translates of $f \in \mathscr{H}$ are also in $\mathscr{H}$ if $f \in L^{2p}(\gamma_{\mathbb{C}};\mathbb{C})$ for some $p > 1$.

Corollary 6.3 *Suppose that $\varphi \in L^2(\lambda_{\mathbb{R}};\mathbb{C})$. Then $T_\zeta \circ \mathscr{S}\varphi \in \mathscr{H}$ for all $\zeta \in \mathbb{C}$ if and only if there exists an analytic function $\Phi : \mathbb{C} \longrightarrow \mathbb{C}$ such that (cf. the notation in Lemma 3.1) $\zeta \in \mathbb{C} \longmapsto e_{\frac{\zeta}{2}}\varphi^{\zeta} \in L^2(\lambda_{\mathbb{R}};\mathbb{C})$ is continuous and $\varphi = \varphi^0$, in which case $T_\zeta \circ \mathscr{S}\varphi = e^{\frac{\zeta^2}{4}}\mathscr{S}\big(e_{\frac{\zeta}{2}}\varphi^{\zeta}\big)$. In particular, if $\mathscr{S}\varphi \in L^{2p}(\gamma_{\mathbb{C}};\mathbb{C})$ for some $p > 1$, then such a Φ exists.*

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