Square-summable rank-one decomposition of nuclear operators

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Abstract

A problem posed by H. Feichtinger (and subsequently by C. Heil and D. Larson) asks whether a positive-definite integral operator with M_1 kernel admits a rank-one decomposition series that is also strongly square-summable in M_1. In this note, we approach this problem by considering its matrix (and finite-dimensional) variant and analyzing several functionals that measure the optimality of such decomposition. Some of the results are based on the joint work with Radu Balan.

Let X, Y be Banach spaces and let $x^* \in X^*, y \in Y$, we write $yx^* = x^* \otimes y : X \to Y$ to denote the rank-one operator specified by $yx^*(z) = x^*(z)y$. We define $\|z\|_p := (\sum_{k=1}^n |z(k)|^p)^{1/p}$ for $z \in \mathbb{R}^n$ or $\mathbb{C}^n$ with the usual modification for $p = \infty$. We use ℓ_p^n to denote $\mathbb{R}^n$ or $\mathbb{C}^n$ with $\|\cdot\|_p$.

1 The Main Problem

Problem 1.1 (Feichtinger '04, Heil-Larson '06)**.** *Given a positive semidefinite trace-class operator*

$$T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \quad f \mapsto \int_{\mathbb{R}^d} k(x,y)f(y)dy$$

with $k \in M_1(\mathbb{R}^{2d})$, can we find $(g_k)_{k\in\mathbb{N}}$ such that the operator-norm convergent series

$$T = \sum_{k\in\mathbb{N}} g_k g_k^* \text{ satisfies } \quad \sum_{k\in\mathbb{N}} \|g_k\|^2_{M_1(\mathbb{R}^d)} < \infty?$$

Here, $M_1(\mathbb{R}^d)$ is the L_1-modulation space, also called the Feichtinger algebra, and is normed by

$$\|f\|_{M_1(\mathbb{R}^d)} := \int_{\mathbb{R}^{2d}} |V_g f(\tau,\omega)|\, d\tau d\omega$$

where V_g is the windowed Fourier transform against a Gaussian window $g(x) = \exp(-\pi |x|^2)$, $V_g f(\tau,\omega) := \int_{\mathbb{R}^d} e^{-2\pi I\omega\cdot x} f(x)g(x-\tau)dx$.

By choosing a suitable ONB associated with $M_1(\mathbb{R}^d)$ (see [3]), the problem above is equivalent to the following.

Problem 1.2 (Heil-Larson '06)**.** *Let $\mathcal{E} = \{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$. Let $(c_{mn}) \in \ell_1(\mathbb{N}^2)$ such that $c_{mn} = \overline{c_{mn}}$ for all $m, n \in \mathbb{N}$. Define $T : \mathcal{H} \to \mathcal{H}$ by $T = \sum_{m,n\in\mathbb{N}} c_{mn} e_m^* \otimes e_n$, convergent in both in strong operator topology and absolutely in trace-class topology. Can we find*

$$h_k \in \mathcal{H}_1 := \left\{ h \in \mathcal{H} : \|h\|_1 := \sum_{n=1}^{\infty} |\langle h, e_n\rangle| < \infty \right\} \subset_{dense} \mathcal{H}$$

such that $T = \sum_{k=1}^{\infty} h_k^* \otimes h_k$ (in suitable topology) and

$$\sum_{k=1}^{\infty} \|h_k\|_1^2 = \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} |\langle h, e_n\rangle| \right)^2 < \infty. \tag{1.1}$$

We will see that $\sum \|h_k\|_1^2 < \infty$ guarantees convergence in operator norm topology.

Proposition 1.1 ([1]). *For $\mathcal{H} = \ell_2$ and $e_k = \delta_k$, $\mathcal{H} = \ell_1$.*

Therefore, we can state an even simpler version of the problem.

Problem 1.3. *Let $T : \ell_2 \to \ell_2$ be a Hermitian positive operator with $\sum_{kl} |\langle \delta_k, T\delta_l \rangle| < \infty$ (hence trace-class). Can we find $h_k \in \ell_1$ such that $T = \sum_k h_k h_k^*$ and $\sum_k \|h_k\|_1^2 < \infty$?*

We now consider a finite-dimensional variant. Let $\mathrm{Sym}^n(\mathbb{C})$ denote the space of Hermitian $n \times n$ matrices. For $A \in \mathrm{Sym}^n$, we define

$$\|A\|_{1,1} = \sum_{k,l=1}^{n} |A_{kl}| .$$

Let $\mathrm{PSD}^n(\mathbb{C})$ denote the cone of positive semidefinite matrices. For $A \in \mathrm{PSD}^n$, define

$$\gamma_+(A) := \inf \left\{ \sum_{k=1}^{N} \|z_k\|_1^2 : A = \sum_{k=1}^{N} z_k z_k^* \right\}.$$

Problem 1.4. *Is there a universal constant $C_0 > 0$ such that*

$$\gamma_+(A) \leq C_0 \|A\|_{1,1}?$$

Theorem 1.1. *If Problem 1.3 is answered in the positive, then Problem 1.4 is answered in the positive.*

Proof. We will prove the contrapositive statement. Let $A_n \in \mathrm{PSD}^{\phi(n)}(\mathbb{C})$ be a sequence of matrices such that $\|A_n\|_{1,1} = 1$ but $\gamma_+(A_n) \geq n \|A_n\|_{1,1}$. Consider an infinite block-diagonal matrix defined by $A := \bigoplus_{n=1}^{\infty} n^{-2} A_n$. The associated operator $A : \ell_2 \to \ell_2$ then satisfies the assumption of Problem 1.3

For $n \in \mathbb{N}$, let P_n denote the orthogonal projection from ℓ_2 to the range of A_n, $P_n = P_n^*$. We may then write

$$A = \sum_{m,n=1}^{\infty} P_m A P_n = \sum_{m,n=1}^{\infty} \delta_{mn} P_m A P_n = \sum_{n=1}^{\infty} P_n A P_n,$$

convergent in the strong operator topology.

Let $(h_k) \subset \ell_1$ be any decomposition of $A = \sum_k h_k h_k^*$. Then

$$A = \sum_{n=1}^{\infty} P_n \left(\sum_{k=1}^{\infty} h_k h_k^* \right) P_n = \sum_{n,k=1}^{\infty} (P_n h_k)(P_n h_k)^*$$

and

$$n^{-2} A_n = \sum_{k=1}^{\infty} (P_n h_k)(P_n h_k)^*.$$

Note that

$$\sum_{n=1}^{\infty} \|P_n h_k\|_1^2 = \sum_{n=1}^{\infty} \sum_{i,j=1}^{\infty} |(P_n h_k)(i)| \, |(P_n h_k)(i)| \leq \sum_{i,j=1}^{\infty} |h_k(i)| \, |h_k(j)| = \|h_k\|_1^2 .$$

As a consequence,

$$\sum_{k=1}^{\infty} \|h_k\|_1^2 \geq \sum_{n,k=1}^{\infty} \|P_n h_k\|_1^2 \geq \sum_{n=1}^{\infty} \frac{1}{n^2} \gamma_+(A_n) \geq \sum_{n=1}^{\infty} \frac{1}{n} = \infty. \tag{1.2}$$

∎

Problem 1.5. *Are Problems 1.4 and 1.3 equivalent?*

2 Some operator theory

Let $A \in \mathrm{M}^n(\mathbb{C})$, $A = \left[\mathrm{Col}_1 A | \cdots | \mathrm{Col}_n A\right]$. For $1 \le p, q < \infty$, define

$$\|A\|_{p,q} = \left(\sum_{k=1}^{n} \|\mathrm{Col}_k A\|_p^q\right)^{1/q} \tag{2.1}$$

with suitable modification for ℓ_∞^n. We also think of $A : \ell_q^n \to \ell_p^n$, and define the operator norm

$$\|A\|_{q\to p} := \sup_{\|z\|_q \le 1} \|Az\|_p .$$

Proposition 2.1. *Let $1 \le p \le q \le \infty$, $1/p + 1/q = 1$, and $A \in \mathrm{M}^n(\mathbb{C})$. The following hold.*

(A) $\|A\|_{q\to p} \le \|A\|_{p,p}$.

(B) $\|A\|_{q\to p} \le \|A\|_{\infty\to 1}$.

As a consequence, $\gamma_+(A) \ge \|A\|_{1,1} \ge \|A\|_{\infty\to 1} \ge \|A\|_{q\to p}$.

Proof. For the first statement, let $z \in \ell_q$. Then

$$\|Az\|_p = \left(\sum_{k=1}^{n} |\mathrm{Col}_k(A^*) \cdot z|^p\right)^{1/p} \le \left(\sum_{k=1}^{n} \|\mathrm{Col}_k A^*\|_p^p \|z\|_q^p\right)^{1/p} = \|A\|_{p,p} \|z\|_q .$$

Note that given $z \in \mathbb{C}^n$, $\|z\|_q \ge \|z\|_\infty$ and $\|z\|_1 \ge \|z\|_p$, so

$$\|A\|_{q\to p} = \inf_{\|z\|_q \le 1} \|Az\|_p \le \inf_{\|z\|_\infty \le 1} \|Az\|_1 = \|A\|_{\infty\to 1} .$$

□

Incidentally, we have the following.

Theorem 2.1 (Gluskin-Tanny '20)**.** *Let $A \in \mathrm{PSD}^n(\mathbb{R})$. Then*

$$\|A\|_{1,1} \le 3\kappa_G (\mathrm{rk} A)^{1/2} \|A\|_{\infty\to 1} . \tag{2.2}$$

The dependence on $(\mathrm{rk}A)^{1/2}$ is sharp, with $A = \begin{bmatrix} O_{k\times k} & 0 \\ 0 & 0 \end{bmatrix}$ where $O \in \mathcal{O}(k)$ satisfies $|O_{kl}| \lesssim k^{-1/2}$.

Definition 2.1. Let X and Y be Banach spaces. An operator $A : X \to Y$ is p-nuclear if there are sequences $(x_k^*) \subset B_{X^*}$, $(y_k) \subset B_Y$, and $(\lambda_k) \in \ell_p$, such that

$$A = \sum_{k=1}^{\infty} \lambda_k x_k^* \otimes y_k \tag{2.3}$$

where the series is convergent in the $\mathcal{L}(X, Y)$ topology. Moreover,

$$\gamma_p(A) := \inf \|\lambda\|_p . \tag{2.4}$$

We use $\mathcal{N}_p(X, Y)$ to denote the Banach space of nuclear operators from X to Y with norm $\gamma_p = \nu_p$. When $p = 1$, we drop the 1 in all the notation.

We mostly care about the case $X = \ell_\infty^n$, $Y = \ell_1^n$, and $p = 1$. In this case, $(\ell_\infty^n)^* = \ell_1^n$. In the infinite-dimensional case, we use $X = c_0 \subset \ell_\infty$, so $X^* = \ell_1$.

Remark 2.1. Some facts:

- From construction, regardless of the Banach spaces X and Y and the moment $1 \le p < \infty$, every operator in $\mathcal{N}_p(X,Y)$ is the γ_p-limit of finite-rank operators, and so is compact.
- Let H_1 and H_2 be Hilbert spaces, then $\mathcal{N}_2(H_1,H_2) = \mathcal{S}_2(H_1,H_2)$ isometrically. The latter consists of 2-Schatten operators or Hilbert-Schmidt operators, with norm $\|A\|_{\mathcal{S}_2}^2 = \sum_k \|Ae_k\|_{H_2}^2$.

Proposition 2.2 (Lemma 2.7 in [2])**.** *Let $A \in \mathrm{M}^n(\mathbb{C})$, viewed as an operator from $\ell_\infty^n \to \ell_1^n$. Then $\gamma(A) = \|A\|_{1,1}$.*

Proof. Fix $A \in \mathrm{M}^n(\mathbb{C})$. To show $\gamma \le \|\cdot\|_{1,1}$, we can write

$$A = \left[\mathrm{Col}_1(A)|\cdots|\mathrm{Col}_n(A)\right] = \sum_{k=1}^n \delta_k^* \otimes \mathrm{Col}_k(A).$$

Therefore,

$$\gamma(A) \le \sum_{k=1}^n \|\mathrm{Col}_k(A)\|_1 \|\delta_k\|_{\ell_1} = \sum_{k=1}^n \|\mathrm{Col}_k(A)\|_1 = \|A\|_{1,1}.$$

For the other direction, let $\epsilon > 0$, and let $(x_k),(y_k) \subset \ell_1^n$ with $\|x_k\|_1, \|y_k\|_1 \le 1$, and $\lambda \in \ell_1$, such that $A = \sum_k \lambda_k x_k y_k^*$ and $\|\lambda\|_1 \le \gamma(A) + \epsilon$. Then

$$\|A\|_{1,1} = \left\|\sum_k \lambda_k x_k y_k^*\right\|_{1,1} \le \sum_k |\lambda_k| \|x_k y_k^*\|_{1,1} \le \sum_k |\lambda_k| \|x_k\|_1 \|y_k\|_1 \le \gamma(A) + \epsilon.$$

□

3 Preliminary properties for γ_+

Proposition 3.1. *γ_+ is sub-additive and positive-homogeneous on PSD^n.*

Proof. Let $A, B \in \mathrm{PSD}^n$, and let $(z_k),(w_k) \subset \ell_1^n$ satisfies

$$\sum_k \|z_k\|_1^2 \le \gamma_+(A) + \epsilon \quad \text{and} \quad \sum_k \|w_k\|_1^2 \le \gamma_+(B) + \epsilon.$$

Concatenate and re-index z_k and w_k to form $(x_k) \subset \ell_1^n$, so $A + B = \sum x_k x_k^*$. Moreover,

$$\gamma_+(A+B) \le \sum_k \|x_k\|_1^2 = \sum_k \|z_k\|_1^2 + \sum_k \|w_k\|_1^2 \le \gamma_+(A) + \gamma_+(B) + 2\epsilon.$$

Positive homogeneity is proved similarly. □

Definition 3.1. Let $u : X \to X$ be a finite rank operator. Then we can write $u = \sum_{k=1}^N x_k^* \otimes \tilde{x}_k$, $x_k^* \in X^*$ and $\tilde{x}_k \in X$. We can define the trace of u to be $\mathrm{tr}(u) = \sum_k x_k^*(\tilde{x}_k)$. This definition is invariant of representation.

Proposition 3.2. *Given $A \in \mathrm{PSD}^n$, we have $\gamma(A) \le \gamma_+(A) \le n\mathrm{tr}(A) \le n\|A\|_{1,1} = n\gamma(A)$.*

Proof. The only nontrivial inequality is $\gamma_+(A) \le n\mathrm{tr}(A)$. Fix $A \in \mathrm{PSD}^n(\mathbb{C})$ and let $(z_k)_{k=1}^N$ be any factorization $A = \sum_k z_k z_k^*$. Since

$$\|z_k\|_1 = \|z_k \cdot 1_n\|_1 \le \|1_n\|_2 \|z_k\|_2 = \sqrt{n}\|z_k\|_2,$$

we have

$$\gamma_+(A) \le \sum_k \|z_k\|_1^2 \le n \sum_k \|z_k\|_2^2 = n \sum_k z_k^*(z_k) = n\mathrm{tr}(A).$$

□

Proposition 3.3. *Given* $A \in \mathrm{PSD}^n$, *we have* $\gamma_+(A) \le \mathrm{rank}(A) n^{1/2} \|A\|_{2\to 1} \le \mathrm{rank}(A) n^{1/2} \|A\|_{\infty\to 1} \le \mathrm{rank}(A) n^{1/2} \|A\|_{1,1}$.

Proof. Let $A = \sum_k^r z_k z_k^*$ be the spectral factorization, $r = \mathrm{rank}(A)$, and $Az_k = \lambda_k z_k$ with $\lambda_k > 0$. Let $e_k = \lambda_k^{-1/2} z_k$. Then

$$\lambda_k \|z_k\|_1 = \|Az_k\|_1 \le \|A\|_{2\to 1} \|z_k\|_2 .$$

Rearrange, we see that

$$\|z_k\|_1 \le \|A\|_{2\to 1} \lambda_k^{-1/2} \|e_k\|_2 = \|A\|_{2\to 1} \lambda_k^{-1/2}.$$

For the other copy of $\|z_k\|_1$, use Hölder's inequality again to get $\|z_k\|_1 \le \sqrt{n} \|z_k\|_2$. Therefore,

$$\begin{aligned}\gamma_+(A) &\le \sum_{k=1}^r \|z_k\|_1^2 \le \sqrt{n} \|A\|_{2\to 1} \sum_{k=1}^r \|e_k\|_2^2 = \sqrt{n} \|A\|_{2\to 1} \mathrm{tr}\left(\sum_{k=1}^r e_k e_k^*\right) \\ &= \mathrm{rk}(A)\sqrt{n} \|A\|_{2\to 1} \le \mathrm{rk}(A)\sqrt{n} \|A\|_{1,1} .\end{aligned}$$

□

4 Duality

Let S_1^n denote the unit sphere in ℓ_1^n. Let $\mathcal{M}(S_1^n)$ denote the cone of positive Borel measures on S_1^n, and let $\mathcal{M}_\pm(S_1^n)$ denote the space of signed Borel measures on S_1^n. By the Riesz-Markov-Kakutani representation theorem,

$$C(S_1^n)^* \cong \mathcal{M}_\pm(S_1^n).$$

We also have the duality of the positive cones

$$C_+(S_1^n)^* \cong \mathcal{M}(S_1^n).$$

Theorem 4.1 (R. Balan)**.** *For any* $A \in \mathrm{PSD}^n(\mathbb{C})$,

$$\gamma_+(A) = \inf\left\{\int_{S_1^n} d\mu : A = \int_{S_1^n} zz^* d\mu,\ \mu \in \mathcal{M}(S_1^n)\right\}$$

Consider the dual pairs $(\mathcal{M}_\pm(S_1^n), C(S_1^n))$ and $(\mathrm{Sym}^n(\mathbb{C}), \mathrm{Sym}^n(\mathbb{C}))$, where the bilinear forms are given by the natural duality pairing, so that we equip $\mathcal{M}_\pm$ with the weak star topology, $C(S_1^n)$ with the weak topology, and $\mathrm{Sym}^n(\mathbb{C})$ with the Euclidean topology.

Consider the map

$$\Phi : \mathcal{M}_\pm(S_1^n) \to \mathrm{Sym}^n(\mathbb{C}) \quad \mu \mapsto \Phi(\mu) = \int_{S_1^n} zz^* d\mu.$$

The adjoint of Φ (with respect to the duality pairing) is given by

$$\Phi^* : \mathrm{Sym}^n(\mathbb{C}) \to C(S_1^n) \quad T \mapsto \Phi^*(T)(z) = \mathrm{tr}(T \cdot zz^*) = \langle z, Tz\rangle.$$

By linear duality theory, the following functional on $\mathrm{PSD}^n(\mathbb{C})$

$$\delta_+(A) := \sup\left\{\mathrm{tr}(AT) : T \in \mathrm{Sym}^n(\mathbb{C}) \text{ and } \langle z, Tz\rangle \le 1 \,\forall z \in S_1^n\right\}$$

is the dual linear program of that associated with γ_+.

Theorem 4.2. *For all* $A \in \mathrm{PSD}^n(\mathbb{C})$, $\delta_+(A) = \gamma_+(A)$.

One direction is simple.

Lemma 4.1. $\delta_+ \leq \gamma_+$.

Proof. Fix $A \in \mathrm{PSD}^n$ and let $\epsilon > 0$. Let $A = \sum_{k=1}^N z_k z_k^*$ be a decomposition such that $\sum_k \|z_k\|_1^2 \leq \gamma_+(A) + \epsilon$. Note that for any $T = T^*$ with $\langle z, Tz\rangle \leq 1$ for all $z \in S_1^n$,

$$\mathrm{tr}(AT) = \mathrm{tr}\left(\sum_k z_k z_k^* \circ T\right) = \sum_k \langle z_k, T z_k\rangle = \sum_k \|z_k\|_1^2 \langle \frac{z_k}{\|z_k\|_1}, \frac{T z_k}{\|z_k\|_1}\rangle \leq \sum_k \|z_k\|_1^2 \leq \gamma_+(A) + \epsilon.$$

□

Theorem 4.3 (Hahn-Banach Separation Theorem)**.** *Let V be a topological vector space over $\mathbb{R}$ and let $K, L \subset V$ be disjoint convex subsets of V with L compact. Then there exists a bounded linear functional $\phi \in V^*$ and a real number α such that*

$$\phi(x) \leq \alpha < \phi(y) \quad \text{for all } x \in K, y \in L.$$

For the reverse direction, consider the convex body

$$\mathcal{K} = \{(\Phi(\mu), \langle \mu, 1\rangle + r) : \mu \in \mathcal{M}(S_1^n),\ r \geq 0\} \subset \mathrm{Sym}^n(\mathbb{C}) \times \mathbb{R}$$

equipped with the induced topology.

Lemma 4.2. *Let $A \in \mathrm{PSD}^n(\mathbb{C})$. Suppose $\delta_+(A) = M$, then $(A, M) \in \overline{\mathcal{K}}$.*

Proof. Suppose towards a contradiction, that $(A, M) \notin \overline{\mathcal{K}}$. Then (A, M) is separated from $\mathcal{K}$ by a hyperplane, i.e., there exists some $(T_0, \lambda) \in \mathrm{Sym}^n(\mathbb{C}) \times \mathbb{R}$ such that

$$\langle (A, M), (T_0, \lambda)\rangle < \langle \mathcal{K}, T_0\rangle.$$

More explicitly,

$$\mathrm{tr}(AT_0) + \lambda M < \mathrm{tr}\left(T_0 \cdot \int_{S_1^n} zz^* d\mu\right) + \lambda \int_{S_1^n} d\mu + \lambda r \text{ for all } \mu \in \mathcal{M}(S_1^n) \text{ and } r \geq 0. \tag{4.1}$$

By setting $\mu = 0$ and $r = 0$, we see that

$$\mathrm{tr}(AT_0) + \lambda M < 0. \tag{4.2}$$

Note that we must also have

$$\mathrm{tr}\left(T_0 \cdot \int_{S_1^n} zz^* d\mu\right) + \lambda \int_{S_1^n} d\mu + \lambda r \geq 0 \quad \text{for all } \mu \in \mathcal{M}(S_1^n) \text{ and } r \geq 0. \tag{4.3}$$

By setting $r = 0$ in (4.3) and let μ range over all possible Borel measures, we have

$$\Phi^*(T)(z) = \langle z, T_0 z\rangle \geq -\lambda \quad \text{for all } z \in S_1^n. \tag{4.4}$$

By setting $\mu = 0$ in (4.3), we have

$$\lambda \geq 0.$$

- Suppose $\lambda > 0$. Then $-\lambda^{-1}T_0 \in \mathrm{Sym}^n(\mathbb{C})$ and $\langle z, -\lambda^{-1}T_0 z\rangle = -\lambda^{-1}\langle z, T_0 z\rangle \leq 1$ by (4.4). Therefore, $-\lambda^{-1}T_0$ is feasible for δ_+. By (4.2),
$$\mathrm{tr}(A(-\lambda^{-1}T_0)) > M,$$
contradicting M being the supremum.

- Suppose $\lambda = 0$. Let $\tilde{T} \in \mathrm{Sym}^n(\mathbb{C})$ with $\langle z, Tz \rangle \leq 1$ for all $z \in S_1^n$. It is clear that $\tilde{T} - \alpha T_0 \in \mathrm{Sym}^n(\mathbb{C})$ for all $\alpha \geq 0$. Since $\Phi(T_0)(z) \geq 0$ for all $z \in S_1^n$, we have $\langle z, (\tilde{T} - \alpha T)z \rangle \leq 1$ for all $z \in S_1^n$ and $\alpha \geq 0$. Therefore, $\tilde{T} - \alpha T_0$ is feasible for δ_+ for all $\alpha \geq 0$.

  Since $\mathrm{tr}(A(\tilde{T} - \alpha T_0)) \leq \delta_+(A) = M < \infty$, we must have $\mathrm{tr}(AT_0) \geq 0$. This contradicts (4.2).

In either case, we arrive at a contradiction. Therefore, $(A, M) \in \overline{\mathcal{K}}$.

□

Lemma 4.3. *$\mathcal{K}$ is closed.*

Proof. Let $(A_j, M_j)_j$ be a sequence in $\mathcal{K}$ converging to $(A, M) \in \mathrm{Sym}^n(\mathbb{C}) \times \mathbb{R}$. By definition, there exists a sequence $\mu_j \in \mathcal{M}(S_1^n)$ and $r_j = M_j - \int_{S_1^n} d\mu_j \in \mathbb{R}$, such that $A_j = \int_{S_1^n} zz^* d\mu_j$. We may, without loss of generality, assume that $\int_{S_1^n} d\mu_j \leq 100M$. By weak-star compactness, we can replace μ_j by a subsequence, such that $\mu_j \to \mu \in \mathcal{M}(S_1^n)$ in the weak-star topology. Integrate against zz^*, we have $\int_{S_1^n} zz^* d\mu = A$. Integrate against the constant function 1, we have $M = \int_{S_1^n} d\mu + \lim_j r_j$. Since $r_j \geq 0$, $\lim_j r_j \geq 0$. Therefore, $(A, M) \in \mathcal{K}$. □

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