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On the symplectic geometry of deformations of a hyperbolic surface

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Let R be a Riemann surface. In this manuscript we consider a geometry on the moduli space $X(R)$ for R, which we regard as the space of equivalence classes of constant curvature metrics on the underlying smooth manifold of R. Classically the space of flat metrics for a torus is the locally symmetric space $O(2) \setminus \mathrm{SL}(2; \mathbf{R})/\mathrm{SL}(2; \mathbf{Z})$. We shall describe a symplectic geometry for the space of hyperbolic metrics on a surface of negative Euler characteristic.

The Teichmüller space $T(R)$, a covering of the moduli space, is a complex Kähler manifold. A Kähler metric for $T(R)$, defined in terms of the Petersson product for automorphic forms, was introduced by Weil, [1]. The Weil-Petersson metric is invariant under the covering transformations and so projects to the moduli space $X(R)$. The metric provides a link between the function theory of R and the geometry of $X(R)$.

In the Fenchel-Nielsen manuscript [8] a deformation, based on an amalgamation construction for Fuchsian groups, is introduced. The deformation is defined geometrically by cutting the surface along a simple closed geodesic, rotating one *side* of the cut relative to the other, and attaching the *sides* in their new position. The hyperbolic metric in the complement of the cut extends to a hyperbolic metric on the new surface. Choose a free homotopy class $[\alpha]$ on the surface R; then for each marked surface $\check{R}$ realize $[\alpha]$ by the closed geodesic $\alpha_{\check{R}}$. The Fenchel-Nielsen deformations for the $\alpha_{\check{R}}$ then define a 1-parameter group of diffeomorphisms of $T(R)$, whose infinitesimal generator by definition is the Fenchel-Nielsen vector field t_α. In [21] the Fenchel-Nielsen deformation was described in terms of quasiconformal mappings and an investigation of the vector fields t_* was begun. The Fenchel-Nielsen vector fields were found to be related to the geodesic length functions l_*, introduced by Fricke-Klein to provide coordinates for $T(R)$.

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Our discussion of the symplectic geometry for $T(R)$ begins in Sections 2 and 4 with the duality formula

$$\omega(t_\alpha, \ ) = -dl_\alpha,$$

where ω is the Weil-Petersson Kähler form. As a consequence we have that the Fenchel-Nielsen vector fields are Hamiltonian for the symplectic form ω; i.e. the Lie derivative $L_{t_\alpha}\omega$ is zero. The Hamiltonian potential of the flow generated by t_α is the geodesic length function $-l_\alpha$. A second consequence is the cosine formula

$$\omega(t_\alpha, t_\beta) = \sum_{p \in \alpha \# \beta} \cos \theta_p,$$

where the sum is over the cosines of the angles at the intersections of the geodesics α and β in the geometry of $\check{R}$. Illustrated by this formula is the correspondence between the symplectic geometry on $T(R)$ and the hyperbolic geometry of $\check{R}$.

From this we turn in Section 4 to consideration of the Lie bracket $[t_\alpha, t_\beta]$. First the normalized Fenchel-Nielsen vector fields T_* are introduced. The integral span of the vector fields T_* is found to be a Lie algebra which has a purely topological description. In the course of the discussion we consider the Lie derivatives $t_\alpha l_\beta$ and $t_\alpha t_\beta l_\gamma$, which are evaluated in terms of the hyperbolic geometry of the geodesics α, β and γ.

The manuscript is divided into four sections. We begin with a review of the relevant analytic theory of Teichmüller space. References for this material are the articles of Ahlfors [1], [2], [3], Bers [4] and Earle [7]. The Fenchel-Nielsen vector fields t_*, length differentials dl_*, and Weil-Petersson metric are introduced in the second section. The third section is devoted to the calculation of the Lie derivatives $t_\alpha l_\beta$ and $t_\alpha t_\beta l_\gamma$. Our approach consists of first calculating the derivative for the case of the hyperbolic plane and then forming the sum, a Poincaré series, over the uniformization group of the Riemann surface. Finally in the last section we explore the Weil-Petersson symplectic geometry of Teichmüller space.

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1. Preliminaries

The Teichmüller space of a Riemann surface S, where S has a hyperbolic metric of finite area, is a complex manifold. We begin by discussing its tangent and cotangent spaces. The universal cover of S is the upper half plane H endowed with the hyperbolic metric and the uniformization group is a finitely generated Fuchsian group. The isometry group of H is identified with PSL(2; $\mathbf{R}$); PSL(2; $\mathbf{R}$) acts by Möbius transformations. A discontinuous group of isometries of

H is Fuchsian. Specifically we shall consider finitely generated Fuchsian groups with torsion having limit set $\mathbf{R}$, the real axis. All considerations unless otherwise indicated are for groups satisfying this description.

Introduce $M(\Gamma)$, the vector space of Γ invariant Beltrami differentials. A Beltrami differential $\mu \in M(\Gamma)$ is a complex valued, measurable, essentially bounded function $\mu(z)$ on H. The function μ satisfies the Γ transformation law $\mu(\gamma(z))\overline{\gamma'(z)}/\gamma'(z) = \mu(z)$, for all $\gamma \in \Gamma$ and thus defines a tensor of type $(-1, 1)$ on the Riemann surface H/Γ. The vector space $Q(\Gamma)$ of integrable holomorphic quadratic differentials will also be considered. An element $\varphi \in Q(\Gamma)$ is a holomorphic function $\varphi(z)$ on H satisfying $\varphi(\gamma(z))\gamma'(z)^2 = \varphi(z)$, for all $\gamma \in \Gamma$ and $\int_\Delta|\varphi|$ is finite, where Δ is a Γ fundamental domain. An element of $Q(\Gamma)$ defines a symmetric tensor of type $(2, 0)$ on H/Γ. Given $\mu \in M(\Gamma)$ and $\varphi \in Q(\Gamma)$ the product $\mu\varphi$ is a Γ invariant area form; the integral pairing $(u, \varphi) = \int_\Delta \mu\varphi$ is well defined. The null space $Q(\Gamma)^\perp \subset M(\Gamma)$ of the pairing is of special interest; define $N(\Gamma) = Q(\Gamma)^\perp$. The finite dimensional vector spaces $M(\Gamma)/N(\Gamma)$ and $Q(\Gamma)$ are dual.

The holomorphic tangent and cotangent spaces at Γ of the Teichmüller space are identified respectively with $M(\Gamma)/N(\Gamma)$ and $Q(\Gamma)$. Solving the Beltrami equation is the fundamental construction. Denote by $M_1(\Gamma)$ the open unit ball of $M(\Gamma)$ endowed with the L^∞ norm. Given $\mu \in M_1(\Gamma)$ the Beltrami equation

$$\begin{cases} w_{\bar z} = \mu w_z, & z \in H \\ w_{\bar z} = \bar\mu(\bar z) w_z, & z \in \mathbf{C} - H \end{cases} \tag{1.1}$$

has a unique homeomorphism solution $w = w^\mu$ with normalization: w fixes 0, 1 and ∞ ([3], [7]). The map $w(z)$ is quasiconformal; an elementary argument provides that $w\Gamma w^{-1}$ is again a Fuchsian group. A new conformal structure is defined for the topological surface underlying H/Γ by declaring w to be an isothermal coordinate. The new structure is conformally equivalent to $H/w\Gamma w^{-1}$.

A quasiconformal solution to (1.1) defines an isomorphism of Fuchsian groups: ρ_w: $\Gamma \to w\Gamma w^{-1}$ by $\rho_w(\gamma) = w\gamma w^{-1}$, for all $\gamma \in \Gamma$. An equivalence relation for isomorphisms is defined by $\rho_w \sim \rho_{w_0}$, provided a Möbius transformation A exists such that $\rho_{A \circ w} = \rho_{w_0}$. The Teichmüller space of Γ, $T(\Gamma)$, is the set of equivalence classes. $T(\Gamma)$ inherits the structure of a complex manifold by declaring the natural projection Φ: $M_1(\Gamma) \to T(\Gamma)$ to be holomorphic. The equivalence relation $\sim$ and complex structure are independent of the choice of Γ. The kernel at the origin of the differential $d\Phi$ of the map Φ: $M_1(\Gamma) \to T(\Gamma)$ is $N(\Gamma) \subset M(\Gamma)$. Accordingly, the quotient space $M(\Gamma)/N(\Gamma)$ represents the holomorphic tangent space of $T(\Gamma)$ at $\Phi(0)$. The dual of $M(\Gamma)/N(\Gamma)$ is $Q(\Gamma)$ which then represents the holomorphic cotangent space of $T(\Gamma)$ at $\Phi(0)$. The integral pairing $\mathrm{Re}(\mu, \varphi) = \mathrm{Re}\int_{H/\Gamma}\mu\varphi$, $\mu \in M(\Gamma)$, $\varphi \in Q(\Gamma)$, induces the natural pairing of the underlying (real) tangent and cotangent spaces.

A natural Hermitian structure on $T(\Gamma)$ is given by the Hermitian form $h(\varphi, \psi) = \int_{H/\Gamma}\varphi\bar{\psi}(\mathrm{Im}\, z)^{-2}$, $\varphi, \psi \in Q(\Gamma)$. The associated inner product $\langle \varphi, \psi \rangle = \frac{1}{2}\mathrm{Re}\int_{H/\Gamma}\varphi\bar{\psi}(\mathrm{Im}\, z)^{-2}$ is the Weil-Petersson cometric for $T(\Gamma)$. The metric is Kähler, has negative holomorphic sectional curvature and is invariant under the action of the Teichmüller modular group ([1], [2]). We shall consider in Section 4 the symplectic structure of the Kähler form.

Each coset of $M(\Gamma)/N(\Gamma)$ has a unique harmonic representative. In particular the Beltrami differentials $(\mathrm{Im}\, z)^{-2}\bar{\varphi}$, $\varphi \in Q(\Gamma)$ are the harmonic tensors of type $(-1, 1)$ with respect to the hyperbolic Laplacian. Denote by $B(\Gamma)$ the space of harmonic tensors of type $(-1, 1)$. The projection P: $M(\Gamma) \to B(\Gamma)$ is given by the integral

$$P[\mu](z) = (\mathrm{Im}\, z)^2 \frac{12}{\pi} \int_H \frac{\mu(\zeta)}{(\zeta - \bar{z})^4} d\sigma(\zeta) \tag{1.2}$$

for $\mu \in M(\Gamma)$, where $d\sigma$ is the Euclidean area form, [2]. We note for reference that the kernel of P (with the hypothesis on Γ in effect) is the subspace $N(\Gamma)$; indeed $\int_{H/\Gamma}\mu\varphi = \int_{H/\Gamma}P[\mu]\varphi$, for all $\mu \in M(\Gamma)$, for all $\varphi \in Q(\Gamma)$. The Weil-Petersson Riemannian structure is given on the tangent space by the quadratic form

$$g(\mu, \nu) = 2\,\mathrm{Re}\int_{H/T} P[\mu]\overline{P[\nu]}(\mathrm{Im}\, z)^{-2} \tag{1.3}$$

for all $\mu, \nu \in M(\Gamma)$. We close our general discussion of the tangent and cotangent spaces with the observation that $2(\mathrm{Im}\, z)^{-2}\overline{P[\mu]} \in Q(\Gamma)$ is the Riemannian dual of the real tangent vector given by μ, $\mu \in M(\Gamma)$.

Now we consider the action on $\mathbf{R}$ of a solution of the Beltrami equation (1.1). Let ρ_w and ρ_{w_0} be isomorphisms of Γ as described above. A fundamental point is that the relation $\rho_w \sim \rho_{w_0}$ is equivalent to the condition $w|_{\mathbf{R}} = w_0|_{\mathbf{R}}$. Consequently the assignment $\mu \to w^{\mu}|_{\mathbf{R}}$, $\mu \in M_1(\Gamma)$, induces an embedding of Teichmüller space into the set of homeomorphisms of $\mathbf{R}$. We shall study this embedding by considering the restriction of w to a finite number of points. First let us discuss the differential of the solution map $\mu \to w^{\mu}$.

Let $\mu(\varepsilon)$, a Beltrami differential defined for small ε, satisfy $\|\mu(\varepsilon)\|_{\infty} < 1$ and $\mu(\varepsilon) = \varepsilon\mu_0 + o(\varepsilon)$ in $L^{\infty}(H)$. Equation (1.1) has a unique normalized solution w^{ε} for $\mu(\varepsilon)$. An expansion $w^{\varepsilon}(z) = z + \varepsilon\dot{w}(z) + o(\varepsilon)$, where $o(\varepsilon)$ is uniform on compact sets, is valid. The first variation $\mathring{w} = \left.\dfrac{dw^{\varepsilon}}{d\varepsilon}\right|_{\varepsilon=0}$ is characterized by

i)
$$\begin{cases} \mathring{w}_{\bar{z}} = \mu_0 & \text{in } H, \\ \mathring{w}_{\bar{z}} = \overline{\mu_0(\bar{z})} & \text{in } \mathbf{C} - H \end{cases}$$

in the sense of weak L^2 derivatives and

ii) $\mathring{w}$ vanishes at 0, 1 and is $o(|z|^2)$ at ∞, [3].

Alternately $\mathring{w}$ is given by the potential integral

$$\mathring{w}[\mu_0](z) = \frac{-1}{\pi}\int \mu_0(\zeta)P(\zeta, z) + \overline{\mu_0(\zeta)}P(\bar{\zeta}, z)\, d\sigma(\zeta) \tag{1.4}$$

where $P(\zeta, z) = \dfrac{1}{\zeta - z} + \dfrac{z-1}{\zeta} - \dfrac{z}{\zeta - 1}$, [3]. The linear map $\mu \to \mathring{w}[\mu]$ represents the differential at the origin of the solution map $\mu \to w^{\mu}$, [3]. Evaluation of the integral (1.4) is fundamental to the analytic approach to Teichmüller theory.

We shall be considering n-tuples of points on the extended boundary $\hat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ of the half plane H. The action of the isometry group of the hyperbolic plane H extends to $\hat{\mathbf{R}}$. The action is effective and transitive on the set of oriented triples of points. Consequently four points of $\mathbf{R}$ form a geometric configuration. The cross ratio $(p, q, r, s) = \dfrac{(p-r)(q-s)}{(p-s)(q-r)}$ is the invariant of the quadruple. We adopt the convention for all considerations involving the cross ratio that if any of p, q, r or s is the point at infinity then the value is determined by continuity. We focus our attention on invariants of H/Γ given by the cross ratio of tuples from $\hat{\mathbf{R}}$. Such a quantity is a function on the Teichmüller space and hence has a differential given by an element of $Q(\Gamma)$.

LEMMA 1.1. *Let $\mu \in M(\Gamma)$ and $w^{\varepsilon\mu}$, $\|\varepsilon\mu\|_{\infty} < 1$ be the solution of* (1.1). *Given p, q, r, and $s \in \hat{\mathbf{R}}$, distinct, then*

$$\left.\frac{d}{d\varepsilon}(w^{\varepsilon\mu}(p), w^{\varepsilon\mu}(q), w^{\varepsilon\mu}(r), w^{\varepsilon\mu}(s))\right|_{\varepsilon=0}$$
$$= -\frac{2}{\pi}(p, q, r, s)\operatorname{Re}\int_H \mu(\zeta)\frac{pr + qs - ps - qr}{(\zeta - p)(\zeta - q)(\zeta - r)(\zeta - s)}\, d\sigma(\zeta).$$

Proof. The first variation (1.4) $\mathring{w}[\mu](z)$ of $w^{\varepsilon\mu}(z)$ is given for $z \in \mathbf{R}$ by

$$\mathring{w}[\mu](z) = \frac{-2}{\pi}\operatorname{Re}\int_H \mu(\zeta)P(\zeta, z)\, d\sigma(\zeta).$$

Now proceeding formally, we have

$$\begin{aligned}\frac{d}{d\varepsilon}&(w^{\varepsilon\mu}(p), w^{\varepsilon\mu}(q), w^{\varepsilon\mu}(r), w^{\varepsilon\mu}(s))\Big|_{\varepsilon=0}\\ &= (p,q,r,s)\left(\frac{(\mathring{w}[\mu](p)-\mathring{w}[\mu](r))}{p-r}+\frac{(\mathring{w}[\mu](q)-\mathring{w}[\mu](s))}{q-s}\right.\\ &\qquad\left.-\frac{(\mathring{w}[\mu](p)-\mathring{w}[\mu](s))}{p-s}-\frac{(\mathring{w}[\mu](q)-\mathring{w}[\mu](r))}{q-r}\right)\\ &= -\frac{2}{\pi}(p,q,r,s)\operatorname{Re}\int_H \mu(\zeta)\frac{pr+qs-ps-qr}{(\zeta-p)(\zeta-q)(\zeta-r)(\zeta-s)}\,d\sigma(\zeta).\end{aligned}$$

The argument is completed by transforming the calculation to the unit disc, where the convergence is readily verified.

Now given a holomorphic function $h \in L^1(H)$, by an argument of Poincaré, the sum (a Poincaré series)

$$\Theta h = \sum_{A\in\Gamma} h \circ A A'^2$$

converges uniformly and absolutely on compacts and indeed $\Theta h \in Q(\Gamma)$. If we observe that H is tessellated by a Γ fundamental domain, it is elementary that

$$\int_H \mu h\, d\sigma = \int_{H/\Gamma} \mu\Theta h$$

for $h \in L^1(H)$ and $\mu \in M(\Gamma)$.

Definition 1.2. Given $p, q, r, s \in \hat{\mathbf{R}}$, distinct, then

$$K(\zeta; p,q,r,s) = -\frac{2}{\pi}(p,q,r,s)\frac{(pr+qs-ps-qr)}{(\zeta-p)(\zeta-q)(\zeta-r)(\zeta-s)}.$$

From the above remarks $\Theta K(p,q,r,s) \in Q(\Gamma)$ and represents the differential of the invariant (p,q,r,s). We record this observation in the following.

LEMMA 1.3. *Let $p, q, r, s \in \hat{\mathbf{R}}$ and $w^{\varepsilon\mu}$, $\|\varepsilon\mu\|_\infty < 1$, $\mu \in M(\Gamma)$, be a solution of* (1.1). *Then*

$$\frac{d}{d\varepsilon}(w^{\varepsilon\mu}(p), w^{\varepsilon\mu}(q), w^{\varepsilon\mu}(r), w^{\varepsilon\mu}(s))\Big|_{\varepsilon=0} = \operatorname{Re}(\mu, \Theta K) = \operatorname{Re}\int_{H/\Gamma}\mu\Theta K.$$

Observe that because of the invariance of the cross ratio we need not require $w^{\varepsilon\mu}$ to be normalized. We shall consider three applications of the above lemma.

Example 1. The length of a closed geodesic. Consider a Möbius transformation A: $z \to \lambda z$, $\lambda > 1$, an element of the Fuchsian group Γ. The projection to

H/Γ of the axis of A, the imaginary axis, is a closed periodic geodesic. This geodesic has length $\log \lambda = \log(At, t, 0, \infty)$, where t is arbitrary in $\mathbf{R} - \{0\}$. More generally, assume A is a hyperbolic transformation, $A \in \Gamma$, with repelling (resp. attracting) fixed point r_A (resp. a_A). The length of the geodesic α associated to A is $\log(As, s, r_A, a_A)$ for $s \in \hat{\mathbf{R}} - \{r_A, a_A\}$. The length of α, l_α, is a geometric invariant of H/Γ (the geodesic length function). Given $\mu \in M(\Gamma)$, the differential dl_α evaluated on the tangent vector μ is

$$\operatorname{Re}(\mu, dl_\alpha) = (As, s, r_A, a_A)^{-1} \operatorname{Re} \int_{H/\Gamma} \mu \Theta K(As, s, r_A, a_A).$$

The variational formula for the length of α was given in a different form by Gardiner, [10]. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and define

$$\omega_A = (\operatorname{tr}^2 A - 4)\left(c\zeta^2 + (d-a)\zeta - b\right)^{-2},$$

where tr denotes the trace. The identity $\omega_{D^{-1}AD} = \omega_A \circ D D'^2$ is elementary. Denote by $\langle A \rangle$ the cyclic group generated by A and define the relative Poincaré series

$$\Theta_\alpha = \sum_{B \in \langle A \rangle \backslash \Gamma} \omega_{B^{-1}AB} \tag{1.5}$$

first considered by Petersson, as noted in [11]. Gardiner's formula for dl_α evaluated on the tangent vector μ, $\mu \in M(\Gamma)$, is

$$\operatorname{Re}(\mu, dl_\alpha) = \frac{2}{\pi} \operatorname{Re} \int_{H/\Gamma} \mu \Theta_\alpha.$$

We note that $\dfrac{2}{\pi}\Theta_\alpha = (As, s, r_A, a_A)^{-1} \Theta K(As, s, r_A, a_A)$, an identity first noted by Hejhal, [11].

Example 2. The angle of intersection of geodesics. Assume the points $q, r \in \hat{\mathbf{R}}$ separate $p, s \in \hat{\mathbf{R}}$. Then $\cos\theta = 2(p, q, r, s) - 1$, where θ is the angle formed by $\widehat{vp}$ and $\widehat{vq}$ and $v = \widehat{ps} \cap \widehat{qr}$ (given $a, b \in H \cup \mathbf{R}$, $\widehat{ab}$ denotes the unique hyperbolic geodesic with endpoints a and b). Given Γ acting on H, the angle $2(p, q, r, s) - 1$ is a geometric invariant of H/Γ. The differential $d\cos\theta$ has value

$$\operatorname{Re}(\mu, 2\Theta K(p, q, r, s)) = 2 \operatorname{Re} \int_{H/\Gamma} \mu \Theta K$$

on the tangent vector μ, $\mu \in M(\Gamma)$.

Hejhal considered [11, p. 356, Example 1] the Poincaré series $\Theta K(r_A, r_B, a_B, a_A)$ where A and B were of a special form. As will be indicated in the third section, his period table [11, p. 357] gives the twist derivatives of the cosine of the angle between the axes of A and B.

Example 3. The twist parameter. Let a Riemann surface H/Γ and a simple closed geodesic γ be given. The surface is cut open along γ, one *side* of the cut is rotated relative to the other and then the *sides* are attached in their new position. The hyperbolic structure in the complement of the cut extends naturally to a hyperbolic structure on the new surface. That the metric naturally extends is suggested by observing that an ε, $\varepsilon > 0$, sufficiently small, neighborhood of γ is a cylinder and supports a one dimensional group of rotation isometries. The deformation preserves the orbits of this one dimensional group. In the case that γ separates H/Γ, the magnitude of the twist deformation is measured by the location of a fixed point of an element of Γ, given the normalization of three points to 0, 1 and ∞, [21]. If γ does not separate, the magnitude of the twist can be measured by the length of a second geodesic, [21]. These parameters are expressible in terms of the cross ratio. If an initial value is fixed for the parameter, then the subsequent value for a surface H/Γ defines a function on Teichmüller space. Its differential is given in terms of ΘK.

2. The Fenchel-Nielsen vector fields

A one parameter family of deformations is defined by the construction of Example 3. In particular, given a marked hyperbolic surface H/Γ, fix a simple closed geodesic α. Cut the surface along α, rotate one *side* of the cut relative to the other, and then attach the *sides* in their new position. A geodesic intersecting the cut is deformed to a broken geodesic. The hyperbolic structure in the complement of the cut extends naturally to a hyperbolic structure on the new surface. By varying the amount of rotation, a one parameter family of deformations, the Fenchel-Nielsen (twist) deformations, is defined. We shall study the tangent vector field of this family. By the identification of fundamental groups, α represents a free homotopy class on each marked Riemann surface. For a marked surface $\check{R}$, consider the geodesic $\alpha_{\check{R}}$ freely homotopic to α; let t_α be the initial tangent vector of the Fenchel-Nielsen deformation about $\alpha_{\check{R}}$. The assignment $\check{R} \to t_\alpha$ defines a vector field on Teichmüller space.

The basic idea for the deformation already appears in the work of Fricke-Klein, [9]. The deformation as such is introduced in the work of Dehn and Fenchel-Nielsen, [8]. In the Fenchel-Nielsen manuscript the deformation is considered extensively. The formal definition is given by an amalgamation construction for Fuchsian groups. In [21], we obtain a description of the deformation in terms of quasiconformal mappings. Accordingly here we begin with the analytic definition in terms of Beltrami differentials. The deformation is normalized such that the hyperbolic displacement, measured between two points on opposite *sides* of the geodesic α, increases at unit speed.

The prototype of the Fenchel-Nielsen deformation is for the case of $\langle B \rangle$ a cyclic group, generated by the transformation B: $z \to \lambda z$, $\lambda > 0$. Choose $\phi(\theta)$, $\theta = \arg z$, smooth with compact support in $(0, \pi)$ and $\int_0^\pi \phi \, d\theta = \frac{1}{2}$. Define $\Phi(\theta) = \int_0^\theta \phi \, d\theta$. The formula

$$w = z \exp(2\varepsilon \Phi(\theta)) \tag{2.1}$$

defines a quasiconformal automorphism of H. The Fenchel-Nielsen deformation is the induced map of $H/\langle B \rangle$ to $H/w\langle B \rangle w^{-1}$. Now we make the formal definition. Assume $A \in \Gamma$ is a hyperbolic transformation corresponding to the geodesic α on H/Γ and that Θ_α is the associated Poincaré series of Petersson (see (1.5)).

Definition 2.1. The Beltrami differential $t_\alpha = \dfrac{i}{\pi}(\operatorname{Im} z)^2 \overline{\Theta}_\alpha$ is the tangent vector to the Fenchel-Nielsen deformation about α.

If α is a simple closed geodesic then t_α is the tangent vector to the deformation discussed above, [21]. The Fenchel-Nielsen deformation for a non-simple geodesic has not previously been considered. We shall describe a geometric characterization in Lemma 2.6. Now we take up the definition of the Fenchel-Nielsen deformation in the large.

A point of Teichmüller space is a marked Fuchsian group. Fix a reference group Γ_0 once and for all. A type-preserving isomorphism j of Fuchsian groups Γ_1, Γ_2 satisfies: $j(\gamma)$ is parabolic if and only if γ is parabolic, for all $\gamma \in \Gamma$. Consider pairs (Γ, j) where j: $\Gamma_0 \to \Gamma$ is an orientation and type-preserving isomorphism of Fuchsian groups. Pairs (Γ_1, j_1) and (Γ_2, j_2) are equivalent provided C, a Möbius transformation, exists with $C\Gamma_1 C^{-1} = \Gamma_2$, $Cj_1C^{-1} = j_2$. The equivalence class $\check{\Gamma} = \{(\Gamma, j)\}$ of a pair is a marked Fuchsian group. The isomorphisms j are used to identify the marked groups. A geodesic α on H/Γ_0 defines a conjugacy class $[A]$ in Γ_0. The conjugacy class $[j(A)]$ for $\check{\Gamma}$ determines a geodesic $j(\alpha)$ on H/Γ. Associate to the marked quotient $H/\check{\Gamma}$ the tangent vector $t_{j(\alpha)}$ of Definition 2.1. By this assignment the vector field t_α is defined on the Teichmüller space.

Definition 2.2. The Fenchel-Nielsen deformation about α of the marked quotient $H/\check{\Gamma}$ is the integral curve of t_α with initial point $H/\check{\Gamma}$.

We observe that the Petersson series Θ_α, the vector field t_α, and the differential form dl_α are all invariants of the geodesic α and the corresponding transformation $A \in \Gamma$. These invariants are obviously related.

LEMMA 2.3. *Given $A \in \Gamma$, hyperbolic, then $(t_\alpha)^* = -idl_\alpha$, where * indicates the dual with respect to the Weil-Petersson metric.*

Proof. By the preliminary discussion we have

$$(t_\alpha)^* = 2(\operatorname{Im} z)^{-2}\overline{P\left[\frac{i}{\pi}(\operatorname{Im} z)^2\overline{\Theta}_\alpha\right]} = \frac{2}{\pi i}\Theta_\alpha = -idl_\alpha.$$

The functions l_α and Weil-Petersson metric are known to be real analytic; by the above the vector fields, t_α must also be real analytic. We shall use juxtaposition $t_\alpha l_\beta$ to indicate the natural action of the tangent vector t_α on the function l_β.

THEOREM 2.4. *Linear reciprocity of twist derivatives. Let α and β be geodesics on the quotient H/Γ. Then*

$$t_\beta l_\alpha + t_\alpha l_\beta = 0.$$

Proof. The argument proceeds formally given Lemma 2.3; $t_\beta l_\alpha + t_\alpha l_\beta = \langle t_\beta, -it_\alpha\rangle + \langle t_\alpha, -it_\beta\rangle$. Since the Weil-Petersson metric is Hermitian the identity follows.

We now consider the question of providing a geometric interpretation for the Fenchel-Nielson deformation about a non-simple curve.

Definition 2.5. Let $A \in \Gamma$ be a hyperbolic transformation. A is Γ-elementary if no axis of a conjugate of A separates the fixed points of A, and the axis of A contains no elliptic fixed points.

A primitive, Γ-elementary transformation corresponds to a simple closed geodesic. Given $\Gamma_1 \subset \Gamma$, a subgroup of finite index, then $N(\Gamma) = N(\Gamma_1)$ and there is the natural inclusion of the tangent spaces, $M(\Gamma)/N(\Gamma)$ into $M(\Gamma_1)/N(\Gamma_1)$.

LEMMA 2.6. *Let $A \in \Gamma$ be a hyperbolic transformation. A finite index normal subgroup $\Gamma_1 \subset \Gamma$ and integer m exist such that A^m is Γ_1-elementary. Let $\tilde{A}_1, \ldots, \tilde{A}_n$ be the $\Gamma_1 \setminus \Gamma$ orbit of $A^m \in \Gamma_1$. Then*

$$t_A = \frac{1}{m}\sum_{j=1}^{n} t_{\tilde{A}_j} \quad \textit{in } M(\Gamma_1)/N(\Gamma_1)$$

and $\tilde{A}_1, \ldots, \tilde{A}_n$ are Γ_1-elementary.

Proof. By a theorem of Scott [17], a finite index subgroup $\Gamma_0 \subset \Gamma$ exists with $A \in \Gamma_0$, and A is Γ_0-elementary. Choose a finite index Γ-normal subgroup $\Gamma_1 \subset \Gamma_0$. Let m be the smallest positive integer such that $A^m \in \Gamma_1$; then A^m is Γ_1-elementary. The group Γ acts by conjugation on Γ_1; each Γ conjugate of A^m is Γ_1-elementary. Choose the representatives $\tilde{A}_1, \ldots, \tilde{A}_n$ for the $\Gamma_1 \setminus \Gamma$ conjugates of A^m; necessarily $mn = [\Gamma\colon \Gamma_1]$. We shall establish the result for the Petersson series Θ_A,

$$\begin{aligned}\Theta_A &= \frac{1}{m} \sum_{C \in \langle A^m \rangle \backslash \Gamma} \omega_{C^{-1}A^mC} \\ &= \frac{1}{m} \sum_{D \in \Gamma_1 \backslash \Gamma} \sum_{C \in \langle A^m \rangle \backslash \Gamma_1} \omega_{D^{-1}C^{-1}A^mCD} \\ &= \frac{1}{m} \sum_{D \in \Gamma_1 \backslash \Gamma} \sum_{B \in \langle D^{-1}A^mD \rangle \backslash \Gamma_1} \omega_{B^{-1}D^{-1}A^mDB} \\ &= \frac{1}{m} \sum_{D \in \Gamma_1 \backslash \Gamma} \Theta_{D^{-1}A^mD}.\end{aligned}$$

We close this section with the observation that the identities for the traces of the elements of a matrix product are dual to identities for the vector fields t_α.

Example 4. Twist identities. If $A \in \Gamma$ is hyperbolic, then $|\operatorname{tr} A|$ defines in a natural way a function on the Teichmüller space. Using the identity $2\cosh l_A/2 = |\operatorname{tr} A|$ we see that the differentials $d|\operatorname{tr} A|$ and dl_A are related: $d \operatorname{tr} A = \operatorname{sgn}(\operatorname{tr} A) \sinh l_A/2\, dl_A$, where sgn denotes the sign of a real number. Note that $\operatorname{tr} A$ is locally well defined on Teichmüller space. Now for $A, B \in SL(2; \mathbf{R})$ the trace identity $\operatorname{tr} A \operatorname{tr} B = \operatorname{tr} AB + \operatorname{tr} A^{-1}B$ is elementary. Forming differentials we have $\operatorname{tr} B d \operatorname{tr} A + \operatorname{tr} A d \operatorname{tr} B = d \operatorname{tr} AB + d \operatorname{tr} A^{-1}B$. Assume A, B, AB and $A^{-1}B$ are hyperbolic; then applying Lemma 2.3 we have the dual identity

$$\begin{aligned}(2.2)\quad & 2\operatorname{sgn}(\operatorname{tr} A \operatorname{tr} B)\cosh l_B/2 \sinh l_A/2\, t_A \\ & \quad + 2\operatorname{sgn}(\operatorname{tr} A \operatorname{tr} B)\cosh l_A/2 \sinh l_B/2\, t_B \\ & = \operatorname{sgn}(\operatorname{tr} AB)\sinh l_{AB}/2\, t_{AB} + \operatorname{sgn}(\operatorname{tr} A^{-1}B)\sinh l_{A^{-1}B}/2\, t_{A^{-1}B}.\end{aligned}$$

A special case of the above is for A, B hyperbolic with parabolic commutator. Then $H/\Gamma(A, B)$, where $\Gamma(A, B)$ is the group generated by A and B, is a torus with one puncture. In this case A, B, AB and $A^{-1}B$ all correspond to simple geodesics and hence the deformations are geometric. The vector fields t_A, t_B, t_{AB} and $t_{A^{-1}B}$ satisfy the identity (2.2).

3. The first and second Lie derivatives of a geodesic length function

The deformation discussed in the previous sections is an isometry in the complement of the cut. A geodesic α intersecting the cut is deformed to a broken geodesic α_b. One expects that the associated periodic geodesic on the new surface is obtained by *sliding* pairs of endpoints of the arcs of α_b along the cut until they meet with common tangents. In this event the derivatives $t_\beta l_\alpha$ and

$t_\gamma t_\beta l_\alpha$ would involve the angles of the intersections $\alpha \# \beta$ and possibly the lengths of the arcs of α_b. Indeed we find that this is exactly the situation.

We consider several applications of the formulas. A second reciprocity identity is discovered. Furthermore, for the unique hyperbolic structure on the complement of three points in the Riemann sphere $\hat{\mathbf{C}}$ we find that pairs and triples of geodesics satisfy geometric identities. The relation of these formulas to the Weil-Petersson Kähler form will be taken up in the next section.

Hejhal introduced periods for holomorphic differentials and used them to study Poincaré series, [11]. Our method was inspired by Hejhal's investigations; we shall elaborate. Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, hyperbolic, define the expression $\Omega_A = (cz^2 + (d-a)z - b)^{-1}$. The quantity Ω_A is an Abelian differential for the cyclic group $\langle A \rangle$. Now for $\varphi \in Q(\Gamma)$ and A corresponding to the geodesic α on H/Γ Hejhal introduced the period

$$\mathcal{P}(\varphi; A) = \tfrac{1}{2}\int_{z_0}^{Az_0} \varphi \Omega_A^{-1}$$

which is independent of z_0 and of the integration path. The period $\mathcal{P}$ and Fenchel-Nielsen tangent t_α for α are related as follows, [21].

LEMMA 3.1.

$$\int_{H/\Gamma} t_\alpha \varphi = \frac{\operatorname{sgn}(\operatorname{tr} A)}{2i} \operatorname{csch} l_A/2\mathcal{P}(\varphi; A).$$

To calculate the t_α twist derivative of a cross ratio we must evaluate the integral $\operatorname{Re} \int_{H/\Gamma} t_\alpha \Theta K$, where K is the variational kernel of Definition 1.2. Hejhal by comparison considered the periods $\operatorname{Im} \mathcal{P}(\Theta R; A)$ for R a rational function, [11]. The above lemma relates the two. An effective procedure for computation of the periods was given by Hejhal. Two steps are involved: determination of the separation properties of axes in H of elements of Γ, and the computation of residues. Our considerations are for the cross ratio and the method is similar to Hejhal's. By virtue of the following lemma the computation of residues is reduced to a combinatorial matter. Our approach exploits the geometry of the problem.

We begin with the considerations for the trivial group $\Gamma = \{I\}$. Denote by $t(\widehat{s_1 s_2})$ the tangent vector to the twist along the single geodesic $\widehat{s_1 s_2}$; recall that $\widehat{s_1 s_2}$ is the hyperbolic geodesic with endpoints $s_1, s_2 \in \hat{\mathbf{R}}$. We wish to evaluate $t(\widehat{s_1 s_2})(z_1, z_2, z_3, z_4)$. The tangent $t(\widehat{s_1 s_2})$ is defined by an appropriate conjugate of (2.1); a normalization is not required because of the invariance of the cross ratio. In particular note that $t(\widehat{s_1 s_2}) = t(\widehat{s_2 s_1})$. The geodesic $\widehat{s_1 s_2}$ separates H into two hyperbolic half planes; the ordering s_1 *then* s_2 is used to define a left and right half plane.

LEMMA 3.2. *Assume $z_j \in \hat{\mathbf{R}}$, $1 \leq j \leq 4$, are distinct and that $s_1, s_2 \in \hat{\mathbf{R}}$ are distinct. Then*

$$t(\widehat{s_1 s_2})(z_1, z_2, z_3, z_4)$$
$$= (z_1, z_2, z_3, z_4) \sum_{j=1}^{4} \chi_L(z_j)\left[(z_{\sigma(j)}, s_1, s_2, z_j) - (z_{\tau(j)}, s_1, s_2, z_j)\right]$$

where

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

are elements of S_4, the permutation group on four letters, and χ_L is the characteristic function of the left component of $\hat{\mathbf{R}} - \{s_1, s_2\}$. The formula remains valid if χ_L is replaced by $-\chi_R$; χ_R is the characteristic function of the right component of $\hat{\mathbf{R}} - \{s_1, s_2\}$.

Proof. By the invariance properties of the cross ratio we may assume $s_1 = 0$, $s_2 = \infty$. Also we assume the deformation fixes 0, 1 and ∞. The map inducing the deformation is given in (2.1); in particular

$$w^{\varepsilon}(z) = \begin{cases} z, & z \in \mathbf{R}, \quad z \geq 0 \\ e^{\varepsilon} z, & z \in \mathbf{R}, \quad z < 0 \end{cases}$$

where ε is the parameter of the twist. An elementary calculation of $\frac{d}{d\varepsilon}(w^{\varepsilon}(z_1), w^{\varepsilon}(z_2), w^{\varepsilon}(z_3), w^{\varepsilon}(z_4))|_{\varepsilon=0}$ verifies the formula. The invariance of the cross ratio allows us to replace w^{ε} by $e^{-\varepsilon} w^{\varepsilon}$, establishing the remark for χ_R.

Observing that the twist derivative of a cross ratio is given in terms of cross ratios, we can in particular compute derivatives of all orders. To compute the effect of a tangent vector t_{α}, we first lift the deformation to the universal cover H. The twist tangent t_{α} lifts to a sum of twists; the sum is over the components in H of the preimage of α. We begin by calculating $t_A l_B$.

It will be convenient to introduce notation. Given geodesics α and β on H/Γ, the intersection locus $\alpha \# \beta$ is defined in terms of the parametrized arcs α and β. Specifically if $\alpha = \alpha(s)$ and $\beta = \beta(s')$, $s, s' \in S^1$, are parametrizations, then $\alpha \# \beta$ corresponds to the points (s_0, s_0') of the torus $S^1 \times S^1$ such that $\alpha(s_0) = \beta(s_0')$ is a transverse intersection of α and β. Assume $\alpha(s_0) = \beta(s_0') = p$ and $\tilde{\alpha}$ (resp. $\tilde{\beta}$) is the branch of α (resp. β) defined by a neighborhood of s_0 (resp. s_0') in S^1. A trigonometric expression $V(p)$ is defined by the geometry of the ordered pair $(\tilde{\alpha}, \tilde{\beta})$ at p:

i) Provided the covering is not ramified at p, $V(p) = \cos\theta$ where $0 \leq \theta \leq \pi$ is the angle at p measured from $\tilde{\alpha}$ to $\tilde{\beta}$;

ii) Provided the covering is ramified at p of order m, m even, $V(p) = 2\csc\frac{\pi}{m}\sin\left(\frac{\pi}{m} - \theta\right)$ where $0 \leq \theta \leq \frac{2\pi}{m}$ is the angle at p measured *from* $\tilde{\alpha}$ *to* $\tilde{\beta}$ (the complete angle at p is $\frac{2\pi}{m}$);

iii) Provided the covering is ramified at p of order m, m odd, $V(p) = \csc\frac{\pi}{2m}\sin(\frac{\pi}{2m} - \theta)$ where $0 \leq \theta \leq \frac{\pi}{m}$ is the angle at p measured *from* $\tilde{\alpha}$ *to* $\tilde{\beta}$. Note that in the second case the branches $\tilde{\alpha}$ and $\tilde{\beta}$ *end* at p, while in the third case they *pass through* p. Also observe that the quantity $V(p)$ is antisymmetric in α and β.

THEOREM 3.3. *The cosine formula. Let the geodesics α and β on H/Γ correspond respectively to A and B in Γ. Then*

$$t_A l_B = \sum_{p \in \alpha \# \beta} V(p).$$

Proof. By Example 1 the length of β is $\log(Bt, t, r_B, a_B)$. Choose $t \in \hat{\mathbf{R}}$ such that it is not fixed by any element of Γ. Assume $\widehat{a_1a_2}$ is in fact the axis of A and $\widehat{a_1a_2}$ separates r_B, a_B with r_B to its left. Translates of $\widehat{a_1a_2}$ are naturally identified with the cosets $\langle A\rangle \backslash \Gamma$ by the rule: $\langle A\rangle C \in \langle A\rangle \backslash \Gamma$ is associated to $C^{-1}(\widehat{a_1a_2})$. Let $\mathcal{G}$ be the subset of elements $C \in \Gamma$ such that $C^{-1}(\widehat{a_1a_2})$ separates r_B, a_B. If $C \in \mathcal{G}$ then $CB^n \in \mathcal{G}$, $n \in \mathbf{Z}$; $\langle B\rangle$ acts on $\mathcal{G}$ by right multiplication. In the natural manner the double cosets $\langle A\rangle \backslash \mathcal{G}/\langle B\rangle$ are identified with the intersection locus $\alpha \# \beta$. First we consider the contribution to $t_A l_B$ by the axes of $\langle A\rangle \backslash \mathcal{G}$. Each $\langle B\rangle$ orbit will be considered separately; it will suffice to consider the $\langle B\rangle$ orbit of A. Its contribution to $t_A l_B$ is the sum

$$\sum_{n=-\infty}^{\infty} t(B^{-n}(\widehat{a_1a_2}))\log(Bt, t, r_B, a_B). \tag{3.1}$$

The description of the twist derivative as the integral $\mathrm{Re}\int_{H/\Gamma} t_\alpha \Theta K$ is used to establish the absolute convergence of (3.1). Replacing A by $B^{-m}AB^m$ if necessary, we may assume t is in the strip bounded by $\widehat{a_1a_2}$ and $B^{-1}(\widehat{a_1a_2})$. The summation (3.1) is divided into three parts: $n \geq 1$, $n = 0$ and $n \leq -1$. Now for $n \geq 1$, r_B is the only point of $\{Bt, t, r_B, a_B\}$ to the left of $B^{-n}(\widehat{a_1a_2})$. Using Lemma 3.2, we have

$$\sum_{n=1}^{\infty} t(B^{-n}(\widehat{a_1a_2}))\log(Bt, t, r_B, a_B)$$

$$= \sum_{n=1}^{\infty} (Bt, B^{-n}a_1, B^{-n}a_2, r_B) - (t, B^{-n}a_1, B^{-n}a_2, r_B),$$

and by the invariance of the cross ratio

$$= \sum_{n=1}^{\infty} \left(B^{n+1}t, a_1, a_2, r_B\right) - \left(B^n t, a_1, a_2, r_B\right)$$

$$= \left(a_B, a_1, a_2, r_B\right) - \left(Bt, a_1, a_2, r_B\right).$$

Now for the single term $n = 0$ we have

$$t(\widehat{a_1a_2})\log(Bt, t, r_B, a_B)$$

$$= \left(Bt, a_1, a_2, r_B\right) - \left(t, a_1, a_2, r_B\right) + \left(a_B, a_1, a_2, t\right) - \left(r_B, a_1, a_2, t\right).$$

With the identity $(z, a, b, c) + (c, a, b, z) = 1$, this simplifies to $(Bt, a_1, a_2, r_B) - (t, a_1, a_2, a_B)$. Finally for $n \leq -1$, a_B is the unique point in the right half plane of $B^{-n}(\widehat{a_1a_2})$. Thus

$$= \sum_{n=-\infty}^{-1} t(B^{-n}(\widehat{a_1a_2}))\log(Bt, t, r_B, a_B)$$

$$= \sum_{n=-\infty}^{-1} \left(Bt, B^{-n}a_1, B^{-n}a_2, a_B\right) - \left(t, B^{-n}a_1, B^{-n}a_2, a_B\right)$$

$$= \sum_{n=-\infty}^{-1} \left(B^{n+1}t, a_1, a_2, a_B\right) - \left(B^n t, a_1, a_2, a_B\right)$$

$$= \left(t, a_1, a_2, a_B\right) - \left(r_B, a_1, a_2, a_B\right).$$

The total contribution of the three parts is

$$\left(a_B, a_1, a_2, r_B\right) - \left(r_B, a_1, a_2, a_B\right) = 2\left(a_B, a_1, a_2, r_B\right) - 1.$$

Each $\langle B \rangle$ orbit in $\langle A \rangle \setminus \mathcal{G}$ contributes such a term; hence

$$\sum_{C \in \langle A \rangle \setminus \mathcal{G}} t\left(C^{-1}(\widehat{a_1a_2})\right)\log(Bt, t, r_B, a_B) = \sum_{C \in \langle A \rangle \setminus \mathcal{G}/\langle B \rangle} 2\left(a_B, c_1, c_2, r_B\right) - 1$$

where $\widehat{c_1c_2}$ the axis of $C^{-1}AC$ has r_B to its left.

The quantity $2(a_B, c_1, c_2, r_B) - 1$ is the cosine of the angle formed by the intersection at $\tilde{p}$ of $\widehat{c_1c_2}$ and $\widehat{r_Ba_B}$. The total contribution of this intersection to the derivative is expressible in terms of the local geometry at p, the projection of $\tilde{p}$ to H/Γ. The expression $V(p)$ is immediate if $\tilde{p}$ is not an elliptic fixed point. If $\tilde{p}$ is an elliptic fixed point of even order (the normalizer of $\langle A \rangle$ is infinite dihedral) we sum over the stabilizer of $\tilde{p}$ and obtain the contribution at p,

$$2 \sum_{j=0}^{\frac{m}{2}-1} \cos\left(\theta + j\frac{2\pi}{m}\right) = 2\csc\frac{\pi}{m}\sin\left(\frac{\pi}{m} - \theta\right).$$

Finally if $\tilde{p}$ is an elliptic fixed point of odd order (the normalizer of $\langle A \rangle$ is infinite cyclic), then the contribution at p is

$$\sum_{j=0}^{m-1} \cos\left(\theta + j\frac{\pi}{m}\right) = \csc\frac{\pi}{2m}\sin\left(\frac{\pi}{2m} - \theta\right).$$

The proof will be completed if we show that

$$\sum_{C \in \langle A \rangle \backslash \Gamma - \mathcal{G}} t\left(C^{-1}(\widehat{a_1 a_2})\right)\log(Bt, t, r_B, a_B)$$

vanishes. First consider those terms such that Bt, t, r_B and a_B are in a common half plane of $C^{-1}(\widehat{s_1 s_2})$. Recall that the twist deformation (2.1) is an isometry in each of its half planes. Each of the terms under present consideration vanishes. It now remains to consider those cosets $\langle A \rangle C \in \langle A \rangle \setminus \Gamma$ such that $C^{-1}(\widehat{a_1 a_2})$ separates Bt or t from $\widehat{r_B a_B}$. Certainly $(CB)^{-1}(\widehat{a_1 a_2})$ separates t from $r_B a_B$ if and only if $C^{-1}(\widehat{a_1 a_2})$ separates Bt from $\widehat{r_B a_B}$. With this in mind the remaining terms of the sum are grouped in ordered pairs $(\langle A \rangle C, \langle A \rangle CB)$, where C^{-1} separates t from $\widehat{r_B a_B}$. In the special case of an axis $C^{-1}(\widehat{a_1 a_2})$ separating both t and Bt from $\widehat{r_B a_B}$, the coset $\langle A \rangle C$ will occur in two distinct pairs. Now group the terms for the pair $(\langle A \rangle C, \langle A \rangle CB)$:

$$\left(t\left(C^{-1}(\widehat{a_1 a_2})\right) + t\left((CB)^{-1}(\widehat{a_1 a_2})\right)\right)\log(Bt, t, r_B, a_B).$$

The $C^{-1}(\widehat{a_1 a_2})$ term is

$$\left(r_B, C^{-1}a_1, C^{-1}a_2, Bt\right) - \left(a_B, C^{-1}a_1, C^{-1}a_2, Bt\right),$$

while the $(CB)^{-1}(\widehat{a_1 a_2})$ term is

$$\left(a_B, (CB)^{-1}a_1, (CB)^{-1}a_2, t\right) - \left(r_B, (CB)^{-1}a_1, (CB)^{-1}a_2, t\right).$$

Their sum is zero and consequently if we group the remaining terms of the sum $\sum_{\langle A \rangle \backslash \Gamma - \mathcal{G}}$, the vanishing follows. The calculation of $t_A l_B$ is complete.

Now we turn to the consideration of the second twist derivative. Our method is the same. In order to simplify the calculations, elliptic elements will not be considered.

THEOREM 3.4. *The sine length formula. Let the geodesics α, β and γ on H/Γ correspond respectively to A, B and C in Γ. Assume the axes of A and C contain no elliptic fixed points. Then*

$$t_C t_A l_B = \sum_{(p,q) \in (\gamma \# \beta) \times (\alpha \# \beta)} \frac{e^{l_1} + e^{l_2}}{2(e^{l_B} - 1)} \sin\theta_p \sin\theta_q - \sum_{(r,s) \in (\gamma \# \alpha) \times (\alpha \# \beta)} \frac{e^{m_1} + e^{m_2}}{2(e^{l_A} - 1)} \sin\theta_r \sin\theta_s, \tag{3.2}$$

where θ_* is the angle at the indicated intersection. The quantities l_1, l_2 (respec-

tively m_1, m_2) are the distances along β between p and q (respectively along α between r and s).

Proof. We begin with the formula $t_A l_B = \Sigma_{q\in\alpha\#\beta}\cos\theta_q$. By linearity it suffices to consider $t_C\cos\theta_q$, $q \in \alpha\#\beta$. As much as possible we continue with the notation of the above discussion. Again we refer to the integral Re $\int_{H/\Gamma} t_\gamma \Theta K$ and have that $\Sigma_{D\in\langle C\rangle\backslash\Gamma} t(D^{-1}(\widehat{a_1 a_2}))2(a_B, a_1, a_2, r_B)$ can be computed term by term. The situation requires a careful tally of the axes separating a_B, r_B, a_1 and a_2. Note that $a_1 a_2$ is the axis of A. Recall that if the points z_1, z_2, z_3 and z_4 are in a common half plane of $\widehat{s_1 s_2}$ then $t(\widehat{s_1 s_2})(z_1, z_2, z_3, z_4) = 0$. Accordingly we focus our attention on $\mathcal{S}$ the set of axes in the orbit of $\widehat{c_1 c_2}$, the axis of C, which separate a_B, r_B, a_1 and a_2; denote by $\widehat{s_1 s_2}$ an element of $\mathcal{S}$. Now $\mathcal{S}$ is partitioned into three subsets:

i) $\widehat{s_1 s_2} \in \mathcal{S}$, such that $\widehat{s_1 s_2}$ intersects $\widehat{r_B a_B}$ and for all $n \in \mathbf{Z}$, $B^{-n}(\widehat{s_1 s_2})$ does not separate a_1, a_2;

ii) $\widehat{s_1 s_2} \in \mathcal{S}$, such that $\widehat{s_1 s_2}$ intersects $\widehat{a_1 a_2}$ and for all $n \in \mathbf{Z}$, $A^{-n}(\widehat{s_1 s_2})$ does not separate r_B, a_B;

iii) $\widehat{s_1 s_2} \in \mathcal{S}$, such that $D \in \langle A\rangle \cup \langle B\rangle$ exists with $D^{-1}(\widehat{s_1 s_2})$ intersecting both $\widehat{r_B a_B}$ and $a_1 a_2$.

In the manner of the $t_A l_B$ calculation we shall sum over the $\langle A\rangle$ and $\langle B\rangle$ orbits. Note that D in case iii) may not be unique. We begin with case i) and orient $\widehat{s_1 s_2}$ to place r_B to the left. By invariance we may assume $a_B = \infty$, $r_B = 0$ and if necessary $\widehat{s_1 s_2}$ is replaced by $B^{-n}(\widehat{s_1 s_2})$ to ensure that $\widehat{s_1 s_2}$ is contained in the strip bounded by $\widehat{a_1 a_2}$ and $B^{-1}(\widehat{a_1 a_2})$. The contribution to the derivative from the $\langle B\rangle$ orbit of $\widehat{s_1 s_2}$ is $\Sigma_{n=-\infty}^{\infty} t(B^{-n}(\widehat{s_1 s_2}))2(\infty, a_1, a_2, 0)$. We decompose this sum into two parts: $n \geq 0$ and $n \leq -1$. For $n \geq 0$, the only element of $\{0, a_1, a_2, \infty\}$ to the left of $B^{-n}(\widehat{s_1 s_2})$ is 0. Consequently we have

$$\sum_{n=0}^{\infty} t(B^{-n}(\widehat{s_1 s_2}))2(\infty, a_1, a_2, 0)$$

$$= \sum_{n=0}^{\infty} 2(\infty, a_1, a_2, 0)\left[(a_1, B^{-n}s_1, B^{-n}s_2, 0) - (\infty, B^{-n}s_1, B^{-n}s_2, 0)\right]$$

$$= \sum_{n=0}^{\infty} 2(\infty, a_1, a_2, 0)\left(\frac{-s_1 B^{-n}s_2}{a_1(s_1 - s_2)}\right)$$

$$= 2(\infty, a_1, a_2, 0)(0, s_1, s_2, \infty)(s_1, a_1, 0, \infty)\sum_{n=0}^{\infty} \lambda(B)^{-n},$$

where $\lambda(B)(> 1)$ is the multiplier of B. For $n \leq -1$, the only element of

$\{0, a_1, a_2, \infty\}$ to the right of $B^{-n}(\widehat{s_1 s_2})$ is ∞. We calculate

$$\sum_{n=-\infty}^{-1} t(B^{-n}(\widehat{s_1 s_2}))2(\infty, a_1, a_2, 0)$$

$$= \sum_{n=-\infty}^{-1} 2(\infty, a_1, a_2, 0)[(0, B^{-n}s_1, B^{-n}s_2, \infty) - (a_2, B^{-n}s_1, B^{-n}s_2, \infty)]$$

$$= \sum_{n=-\infty}^{-1} 2(\infty, a_1, a_2, 0)\left(\frac{-a_2 s_2}{s_2(B^{-n}s_1 - B^{-n}s_2)}\right)$$

$$= 2(\infty, a_1, a_2, 0)(0, s_1, s_2, \infty)(a_2, s_2, 0, \infty) \sum_{n=-\infty}^{-1} \lambda(B)^n.$$

The total contribution of the $\langle B \rangle$ orbit of $\widehat{s_1 s_2}$ is

$$\frac{2}{\lambda(B) - 1}(\infty, a_1, a_2, 0)(0, s_1, s_2, \infty)[\lambda(B)(s_1, a_1, 0, \infty) + (a_2, s_2, 0, \infty)].$$

Without the normalization of r_B and a_B, the expression becomes

$$\frac{2}{\lambda(B) - 1}(a_B, a_1, a_2, r_B)(r_B, s_1, s_2, a_B)[\lambda(B)(s_1, a_1, r_B, a_B) + (a_2, s_2, r_B, a_B)].$$

In particular if $A = C$ then $\widehat{a_1 a_2} \in \mathcal{S}$ is considered under case i) and contributes the term

$$2\frac{\lambda(B) + 1}{\lambda(B) - 1}(a_B, a_1, a_2, r_B)(r_B, a_1, a_2, a_B).$$

We now derive an intrinsic expression for

$$\lambda(B)(s_1, a_1, r_B, a_B) + (a_2, s_2, r_B, a_B).$$

An arc $\widehat{v_1 v_2}$, $v_1, v_2 \in \mathbf{R}$, is contained in the locus of $(2x - v_1 - v_2)^2 + 4y^2 = (v_1 - v_2)^2$. If $\widehat{v_1 v_2}$ intersects $\widehat{0\infty}$ the ordinate of the intersection is $y = (-v_1 v_2)^{1/2}$. If a second arc $\widehat{u_1 u_2}$, $u_1, u_2 \in \mathbf{R}$ intersects $\widehat{0\infty}$ then $l = \frac{1}{2}\log u_1 u_2 / v_1 v_2$ is the signed distance along $\widehat{0\infty}$ *from* $v_1 v_2 \cap \widehat{0\infty}$ *to* $\widehat{u_1 u_2} \cap \widehat{0\infty}$. Now $e^l((u_2, u_1, 0, \infty)(v_1, v_2, 0, \infty))^{1/2} = |(u_2, v_2, 0, \infty)|$ and thus

$$\lambda(B)(s_1, a_1, r_B, a_B) + (a_2, s_2, r_B, a_B)$$

$$= ((a_2, a_1, r_B, a_B)(s_1, s_2, r_B, a_B))^{1/2}[e^{l_1} + e^{l_2}],$$

where l_1, l_2 are the distances on H/Γ along β between the projections of $\widehat{s_1 s_2} \cap \widehat{r_B a_B}$ and $\widehat{a_1 a_2} \cap \widehat{r_B a_B}$. Finally we note the symmetry $(z, a, b, c) = ((b, z, a, c) - 1)/(b, z, a, c)$.

We now treat case ii) by reducing to case i). In the discussion thus far it has not been necessary to distinguish A from A^{-1}. Choose A such that $a_2 = a_A$ and

$a_1 = r_A$. Interchanging the roles of A and B in the calculation for case i), we have the formula

$$\sum_{n=-\infty}^{\infty} t(A^{-n}(\widehat{s_1 s_2}))2(a_A, a_B, r_B, r_A) = \frac{2}{\lambda(A)-1}(a_A, a_B, r_B, r_A)(r_A, s_1, s_2, a_A)$$

$$\cdot((r_B, a_B, r_A, a_A)(s_1, s_2, r_A, a_A))^{1/2}[e^{l_1} + e^{l_2}],$$

where l_1, l_2 are the distances along α between the projections of $\widehat{r_B a_B} \cap \widehat{a_1 a_2}$ and $\widehat{a_1 a_2} \cap \widehat{s_1 s_2}$. Thus by substitution of the symmetry $(a_B, a_1, a_2, r_B) = 1 - (a_2, a_B, r_B, a_1)$, the contribution of the $\langle A \rangle$ orbit of $\widehat{s_1 s_2}$ (case ii)) is

$$\sum_{n=-\infty}^{\infty} t(A^{-n}(\widehat{s_1 s_2}))2(a_B, a_1, a_2, r_B) = \frac{-2}{\lambda(A)-1}(a_B, a_2, a_1, r_B)(a_1, s_1, s_2, a_2)$$

$$\cdot((r_B, a_B, a_1, a_2)(s_1, s_2, a_1, a_2))^{1/2}[e^{l_1} + e^{l_2}].$$

Case iii) is also treated by reducing to case i). A representative $\widehat{s_1 s_2}$ is chosen to intersect $\widehat{r_B a_B}$; separate sums are then considered for the orbit $\langle B \rangle(\widehat{s_1 s_2})$ and the orbit $\langle A \rangle B^{-k}(\widehat{s_1 s_2})$ where $B^{-k}(\widehat{s_1 s_2})$ separates $\widehat{a_1 a_2}$. For an axis $B^{-k}(\widehat{s_1 s_2})$ intersecting $\widehat{a_1 a_2}$, the expression for $t(B^{-k}(\widehat{s_1 s_2}))(a_B, a_1, a_2, r_B)$ can be written as the sum of a term from the orbit $\langle A \rangle B^{-k}(\widehat{s_1 s_2})$ and a term from the orbit $\langle B \rangle(\widehat{s_1 s_2})$. The total contribution of axes from case iii) is a finite sum of distinct $\langle B \rangle$ orbits, similar to case i), and distinct $\langle A \rangle$ orbits, similar to case ii). A moment's reflection shows that each point of $\gamma \# \beta$ contributes a term as in case i), and each point of $\gamma \# \alpha$ contributes a term as in case ii). The final formula is obtained after elementary trigonometric simplifications.

Now we turn to the applications of Theorems 3.3 and 3.4.

THEOREM 3.5. *Quadratic reciprocity of twist derivatives. Let Γ be a torsion free Fuchsian group. Let α, β and γ be geodesics on the surface H/Γ. Then*

$$t_\alpha t_\beta l_\gamma + t_\gamma t_\alpha l_\beta + t_\beta t_\gamma l_\alpha = 0.$$

Proof. First we observe that the sums

$$\sum_{(\alpha\#\gamma)\times(\beta\#\gamma)} \quad \text{and} \quad \sum_{(\alpha\#\beta)\times(\beta\#\gamma)}$$

occurring in the formula (3.2) are uniquely characterized by their support. With this in mind the left hand side of the identity is given as

$$\sum_{(\alpha\#\gamma)\times(\beta\#\gamma)} - \sum_{(\alpha\#\beta)\times(\beta\#\gamma)} + \sum_{(\gamma\#\beta)\times(\alpha\#\beta)} - \sum_{(\gamma\#\alpha)\times(\alpha\#\beta)}$$

$$+ \sum_{(\beta\#\alpha)\times(\gamma\#\alpha)} - \sum_{(\beta\#\gamma)\times(\gamma\#\alpha)}.$$

The vanishing is immediate.

A natural question is the behaviour of the geodesic length function l_β along an integral curve of t_α. In the geometric case, α a simple closed geodesic, the formula (3.2) shows that $(t_\alpha)^2 l_\beta$ is nonnegative and is actually positive if $\alpha \# \beta \neq \varnothing$. In particular an intersection $\alpha \# \gamma$ is by definition transverse; hence if $\alpha = \gamma$ is simple then $\alpha \# \gamma$ is empty and the second sum of (3.2) is likewise empty. The intersection angles θ_* are in the interval $(0, \pi)$ and thus the first sum of (3.2) is positive provided it is nontrivial. The convexity of the functions l_β along integral curves of t_α was first observed by Kerckhoff, [13], [14]. This fact is utilized in his proof of the Hurwitz-Nielsen conjecture. Using the convexity observation we obtain a result on the nonvanishing of Poincaré series.

THEOREM 3.6. *Assume Γ is finitely generated with infinite limit set, not necessarily $\hat{\mathbf{R}}$. Let α correspond to the Γ-elementary transformation A. Let c_1 and c_2 be arbitrary, one from each component of $\hat{\mathbf{R}} - \{a_1, a_2\}$, where $\widehat{a_1 a_2}$ is the axis of A. The Poincaré series of*

$$\frac{1}{(\zeta - a_1)(\zeta - a_2)(\zeta - c_1)(\zeta - c_2)}$$

is nontrivial.

Proof. First we consider the case of H/Γ having finite area ($\hat{\mathbf{R}}$ is the limit set of Γ). We observe that axes $\widehat{b_{1,n} b_{2,n}}$ of transformations $B_n \in \Gamma$ exist converging to $\widehat{c_1 c_2}$. Specifically recall that for H/Γ of finite area the closed geodesics are dense in the unit tangent bundle. Consequently a tangent vector of $\widehat{c_1 c_2}$ is necessarily the limit of tangents of closed geodesics; axes $\widehat{b_{1,n} b_{2,n}}$ for these geodesics converge to $\widehat{c_1 c_2}$. Define θ_n to be the intersection angle of $\widehat{b_{1,n} b_{2,n}}$ and $\widehat{a_1 a_2}$; necessarily θ_n converges to θ_0, the intersection angle of $\widehat{c_1 c_2}$ and $\widehat{a_1 a_2}$. We have by Theorem 3.4 for an axis $\widehat{b_1 b_2}$ of an element B, intersecting $\widehat{a_1 a_2}$ at q,

$$4\,\mathrm{Re}\big(t_\alpha, \Theta K(b_1, a_1, a_2, b_2)\big) = 2 t_\alpha \cos\theta_q$$

$$= \sum_{p \in \alpha \# \beta} \frac{e^{l_1} + e^{l_2}}{(e^{l_B} - 1)} \sin\theta_p \sin\theta_q \geq \frac{\lambda(B) + 1}{\lambda(B) - 1} \sin^2\theta_q.$$

The inequality

$$4\,\mathrm{Re}\big(t_\alpha, \Theta K(c_1, a_1, a_2, c_2)\big) \geq \sin^2\theta_0 > 0$$

is obtained on forming the limit of the sequence $\widehat{b_{1,n} b_{2,n}}$. We note that

$$K(\zeta; c_1, a_1, a_2, c_2) = -\frac{2}{\pi}(c_1, a_1, a_2, c_2)\frac{(a_1c_2 + a_2c_1 - a_1a_2 - c_1c_2)}{(\zeta - c_1)(\zeta - a_1)(\zeta - a_2)(\zeta - c_2)}.$$

The argument is complete in the case where H/Γ is of finite area.

The case of Γ finitely generated with limit set a proper subset of $\hat{\mathbf{R}}$ will be reduced to the above. An ideal boundary component of H/Γ is a point or Jordan curve. If it is a Jordan curve, there is a unique geodesic in its free homotopy class. The finite-area subdomain bounded by these geodesics is the Nielsen kernel $NK(\Gamma)$ of H/Γ. The conformal double of $NK(\Gamma)$ is a Riemann surface $H/\tilde{\Gamma}$ of finite hyperbolic area. Indeed, $H/\tilde{\Gamma}$ is the metric double of $NK(\Gamma)$. Accordingly α can also be considered as a geodesic on $H/\tilde{\Gamma}$ and corresponds to a $\tilde{\Gamma}$-elementary transformation. Expressing the summation for a $\tilde{\Gamma}$ Poincaré series as $\Sigma_{\tilde{\Gamma}} = \Sigma_{\Gamma\backslash\tilde{\Gamma}}\Sigma_{\Gamma}$, we observe that the nonvanishing of the $\tilde{\Gamma}$ sum certainly implies the nonvanishing of the Γ sum. Now the $\tilde{\Gamma}$ Poincaré series of $(1/(\zeta - a_1)(\zeta - a_2)(\zeta - c_1)(\zeta - c_2))$ is nontrivial by the first case of our argument, and the conclusion follows.

Example 5. Geodesics on Schwarz surfaces. Let $\Delta_{a,b,c}$ be the hyperbolic triangle with vertex angles π/a, π/b and π/c where a, b and c are positive integers or infinity and $1/a + 1/b + 1/c < 1$. The reflections in the sides of $\Delta_{a,b,c}$ generate a discontinuous group of hyperbolic motions. The index 2 subgroup of conformal transformations is the Schwarz triangle group $\Gamma_{a,b,c}$. The geometry of a Schwarz surface $H/\Gamma_{a,b,c}$ is very special; its Teichmüller space $T(\Gamma_{a,b,c})$ is a singleton.

The geodesics of the quotient $H/\Gamma_{a,b,c}$ will satisfy trigonometric identities as a natural consequence of this rigidity. In particular let α and β be any two closed geodesics on $H/\Gamma_{a,b,c}$; then

$$\sum_{p\in\alpha\#\beta} V(p) = 0. \tag{3.3}$$

We have a second class of identities for the group $\Gamma_{\infty,\infty,\infty}$. The group of the ideal triangle $\Delta_{\infty,\infty,\infty}$ is the principal congruence subgroup $\Gamma(2)$ of the modular group. Let α, β and γ be any three closed geodesics on $H/\Gamma(2)$; then

$$\sum_{(p,q)\in(\gamma\#\beta)\times(\alpha\#\beta)} \frac{e^{l_1} + e^{l_2}}{(e^{l_B} - 1)} \sin\theta_p \sin\theta_q$$
$$- \sum_{(r,s)\in(\gamma\#\alpha)\times(\alpha\#\beta)} \frac{e^{m_1} + e^{m_2}}{(e^{l_A} - 1)} \sin\theta_r \sin\theta_s = 0$$

with the notation of (3.2).

We indicate the proof of (3.3). The key observation is that $Q(\Gamma_{a,b,c}) = \{0\}$. Now $\Theta K(Bt, t, r_B, a_B) \in Q(\Gamma_{a,b,c})$, hence vanishes identically. By the calculation of Theorem 3.3, we note that

$$\sum_{p\in\alpha\#\beta} V(p) = (Bt, t, r_B, a_B)^{-1}\operatorname{Re}(t_\alpha, \Theta K(Bt, t, r_B, a_B))$$

and observe that the integrand is trivial.

4. The symplectic structure of Teichmüller space

The Weil-Petersson Kähler form ω_{wp} provides an example of a symplectic form. A vector field X is Hamiltonian for the symplectic form ω provided the Lie derivative $L_X\omega$ vanishes. Equivalently ω is invariant under the local flow generated by X. Our main result is that the Fenchel-Nielsen vector fields t_α are Hamiltonian for the Weil-Petersson Kähler form. Hence the t_α are infinitesimal motions of ω_{wp}. We find in particular that the symplectic geometry of the vector fields $t_\alpha, \dots, t_\kappa$ is determined by the hyperbolic geometry of the geodesics $\alpha, \dots, \kappa$. Perhaps the correspondence is best illustrated by the cosine formula $\omega_{wp}(t_\alpha, t_\beta) = \Sigma_{p\in\alpha\#\beta}V(p)$ and in the computation of the Lie bracket $[t_\alpha, t_\beta]$. The Lie bracket is evaluated by the law of cosines from hyperbolic trigonometry. Furthermore, we observe that ω_{wp}, being a closed exterior 2-form, is equivalent to the first and second reciprocity identities.

We begin with a review of the Weil-Petersson Hermitian structure. The holomorphic tangent space at Γ of the Teichmüller space $T(\Gamma)$ is represented by $M(\Gamma)/N(\Gamma)$. Multiplication by i is the automorphism J of $M(\Gamma)/N(\Gamma)$ defining the complex structure of $T(\Gamma)$. Given $\mu \in M(\Gamma)$, denote by $(\partial/\partial z(\mu))$ the associated holomorphic tangent vector. For a complex manifold there is a natural isomorphism between the holomorphic tangent space and the underlying real tangent space. A holomorphic vector $\partial/\partial z$ is associated to the tangent direction $\partial/\partial z + \partial/\partial \bar{z}$. With this isomorphism a metric tensor g can be extended to the $(1,0)$ vectors in the complexification. The Hermitian condition is that the extension be the real part of a Hermitian product. The Hermitian product for the Weil-Petersson metric is

$$h\left(\frac{\partial}{\partial z(\mu)}, \frac{\partial}{\partial z(\nu)}\right) = \int_{H/\Gamma} P[\mu]\overline{P[\nu]}(\operatorname{Im} z)^{-2}, \quad \text{for } \mu, \nu \in M(\Gamma).$$

Accordingly the real symmetric 2-form is

$$g(\mu, \nu) = 2\operatorname{Re} h\left(\frac{\partial}{\partial z(\mu)}, \frac{\partial}{\partial z(\nu)}\right) = \langle \mu, \nu \rangle,$$

for $\mu, \nu \in M(\Gamma)$. And finally the Kähler form is

$$\omega_{wp}(\mu, \nu) = g(J\mu, \nu) = 2\operatorname{Re} h\left(\frac{\partial}{\partial z(i\mu)}, \frac{\partial}{\partial z(\nu)}\right) = -2\operatorname{Im} h\left(\frac{\partial}{\partial z(\mu)}, \frac{\partial}{\partial z(\nu)}\right)$$

for $\mu, \nu \in M(\Gamma)$.

A closed exterior 2-form ω is symplectic provided it has maximal rank at each tangent space. A symplectic form induces an isomorphism Ω between the tangent and cotangent bundles. If v is a tangent vector then $\Omega(v) = \omega(v, \ )$ is a cotangent vector. Similarly if X is a vector field, $\Omega(X)$ is a differential 1-form. A vector field X is Hamiltonian provided the Lie derivative $L_X\omega$ vanishes identically. By the Cartan equation for the Lie derivative, $L_X\omega(\ ,\ ) = d\omega(X,\ ,\ ) + d(\omega(X,\ ))$ and as ω is symplectic, $L_X\omega(\ ,\ ) = d\Omega(X)$. We shall use this characterization: X is Hamiltonian if $\Omega(X)$ is closed. The Hamiltonian vector fields form a Lie algebra. In fact for X and Y Hamiltonian vector fields

$$[X, Y] = \Omega^{-1}(d\omega(Y, X)), \tag{4.1}$$

from which it is immediate that the bracket $[X, Y]$ is Hamiltonian, [18]. A symplectic structure induces an isomorphism between the Hamiltonian vector fields and the closed 1-forms. The isomorphism is used to define a Lie bracket, referred to as the Poisson bracket, for closed 1-forms. Teichmüller space is a cell; consequently in the present situation a form is closed if and only if it is exact. Finally a function f is called a Hamiltonian potential for the vector field X provided $X = \Omega^{-1}(df)$.

We will now discuss the symplectic geometry of the Weil-Petersson Kähler form. For the remainder of the section, the Weil-Petersson Kähler form and the basic isomorphism will be denoted respectively as ω and Ω. The discussion begins with the observation that $-l_\alpha$ is a Hamiltonian potential for the vector field t_α.

LEMMA 4.1. *Let α be a geodesic on H/Γ. Then*

$$\Omega(t_\alpha) = -dl_\alpha.$$

Proof. By definition $\Omega(t_\alpha)$ is the cotangent vector $\omega(t_\alpha,\ ) = g(Jt_\alpha,\ )$, hence is the Riemannian dual of Jt_α. The tangent vector Jt_α is given by $-(1/\pi)(\operatorname{Im} z)^2\overline{\Theta}_\alpha$, whose dual (see section 1) is $-\dfrac{2}{\pi}\Theta_\alpha = -dl_\alpha$.

In the following the Kähler form ω is evaluated in terms of the hyperbolic geometry of closed geodesics on H/Γ.

LEMMA 4.2. *Let α and β be geodesics on H/Γ. Then*

$$\omega(t_\alpha, t_\beta) = t_\alpha l_\beta = \sum_{p \in \alpha \# \beta} V(p).$$

Proof. By the above lemma, $\omega(t_\alpha, t_\beta) = -t_\beta l_\alpha$. The second equation is Theorem 3.3.

The above result suggests that the Kähler form is completely determined by the geometry of geodesics. This is valid provided the Fenchel-Nielsen vector fields everywhere span the tangent space. By Lemma 2.3 it suffices to show that the differentials dl_α everywhere span the cotangent space. This last statement is a classical result and appears in the work of Fricke-Klein [9], Fenchel-Nielsen [8], Ahlfors [1] and Keen [12]. Now given this, we use the reciprocity identities to show that ω is a closed exterior 2-form, providing a new proof that the metric is Kähler. It suffices by the above to consider ω evaluated on the Fenchel-Nielsen vector fields. We begin with

$$\omega(t_\alpha, t_\beta) + \omega(t_\beta, t_\alpha) = t_\alpha l_\beta + t_\beta l_\alpha = 0,$$

a consequence of Lemma 4.2 and the linear reciprocity identity; hence ω is an exterior 2-form. From the definition of the exterior derivative we have

$$\begin{aligned} d\omega(t_\alpha, t_\beta, t_\gamma) = {} & t_\alpha\omega(t_\beta, t_\gamma) - t_\beta\omega(t_\alpha, t_\gamma) + t_\gamma\omega(t_\alpha, t_\beta) \\ & - \omega([t_\alpha, t_\beta], t_\gamma) + \omega([t_\alpha, t_\gamma], t_\beta) - \omega([t_\beta, t_\gamma], t_\alpha). \end{aligned}$$

We evaluate, using Lemmas 4.1 and 4.2

$$\begin{aligned} d\omega(t_\alpha, t_\beta, t_\gamma) = {} & t_\alpha t_\beta l_\gamma + t_\beta t_\gamma l_\alpha + t_\gamma t_\alpha l_\beta \\ & - [t_\alpha, t_\beta] l_\gamma + [t_\alpha, t_\gamma] l_\beta - [t_\beta, t_\gamma] l_\alpha, \end{aligned}$$

and the last expression vanishes by the quadratic reciprocity identity. We conclude that ω is closed.

A natural question is to consider the relation if any between the vector fields t_α and the Killing vector fields of the Weil-Petersson metric. We provide an answer in the geometric case: α simple. It is necessary to review certain elementary facts concerning vector fields. A vector field X is complete provided it can be integrated; it is natural to interpret completeness as a growth condition at infinity. A complete vector field X is Killing for a Riemannian metric g, provided the integrated transformations are g-isometries. Similarly a complete vector field X on a complex manifold M is analytic provided the integrated transformations are biholomorphisms of M. Now the geometric construction for the Fenchel-Nielsen deformation in the case of a simple geodesic α defines the integrated transformation of t_α. The vector fields t_α, α simple, are complete. The following theorem is standard.

THEOREM 4.3. *Let M be a Kähler manifold with metric g and associated Kähler form ω. If X, a complete vector field, is g-Killing and ω-Hamiltonian, then X is analytic.*

COROLLARY 4.4. *Let α be a simple geodesic on H/Γ. Then t_α is not a Killing vector field for the Weil-Petersson metric.*

Proof. From the above it is sufficient to show that t_α is not analytic. Royden established that the full group of biholomorphisms of $T(\Gamma)$ is the Teichmüller modular group, a discrete group, [16]. Hence the only complete analytic vector field of Teichmüller space is the trivial vector field.

Fricke-Klein established that local coordinates for $T(\Gamma)$ are given by an appropriate choice of geodesic length functions l_α. Coordinates of this type have been used for many calculations in Teichmüller theory. A characterization of the complex structure for $T(\Gamma)$ in the length coordinates has not been obtained. Nevertheless the expression for the Kähler form is elementary. Let $l_1,\dots, l_n$ be geodesic length functions giving local coordinates at a point of Teichmüller space; consider the Fenchel Nielsen vector fields $t_j = -\Omega^{-1}(dl_j)$, $1 \le j \le n$.

LEMMA 4.5, [23]. *Let $l_1,\dots, l_n$ provide local coordinates for $T(\Gamma)$. Let $\omega_{jk} = \omega(t_j, t_k)$ and denote by (W_{jk}) the inverse of (ω_{jk}); then in the local coordinates $l_1,\dots, l_n$,*

$$\omega = -\sum_{j<k} W_{jk}\, dl_j \wedge dl_k.$$

In [23] this formula is used to determine the volume of the moduli space of once punctured tori; the result is $\dfrac{\pi^2}{6}$.

Now we consider a characterization of the Lie algebra of Hamiltonian vector fields. First we note a result of Fricke-Klein: geodesic length functions $l_1,\dots, l_q$ can be chosen such that the induced map of $T(\Gamma)$ to $\mathbf{R}^q$ is a real analytic embedding. Using this, we have the following, as usual: $t_j = -\Omega^{-1}(dl_j)$, $1 \le j \le q$.

THEOREM 4.6. *A smooth vector field X on $T(\Gamma)$ is Hamiltonian if and only if there exists a smooth function f: $\mathbf{R}^q \to \mathbf{R}$ such that $X = \dfrac{\partial f}{\partial x_j}(l_1,\dots, l_q)t_j$.*

Proof. Teichmüller space is a cell; thus X is Hamiltonian if and only if $\Omega(X)$ is exact. The exactness of $\Omega(X)$ is equivalent to the existence of a smooth function f with $\Omega(X) = df$. The function f can be expressed as the composition $f(l_1,\dots, l_q)$; the differential df is $\dfrac{\partial f}{\partial x_j} dl_j$. Finally by Lemma 4.1, $\Omega\left(\dfrac{\partial f}{\partial x_j} dl_j\right) = -\dfrac{\partial f}{\partial x_j} t_j$, establishing the result.

Finally we consider the Lie bracket $[t_\alpha, t_\beta]$ of the Fenchel-Nielsen vector fields t_α and t_β. The formula reveals a Lie algebra $\mathfrak{T}$, the twist lattice, defined over the integers. We find the twist lattice to be determined by the isomorphism class of the Fuchsian group Γ.

Definition 4.7. Given $A \in \Gamma$, hyperbolic, define the normalized Fenchel-Nielsen vector field $T_A = 4 \sinh l_A/2\ t_A$. The twist lattice $\mathfrak{T}$ is the $\mathbf{Z}$ span of the vector fields T_A, $A \in \Gamma$, hyperbolic.

Note that by Lemma 4.1 the Hamiltonian potential for T_A is the function $4|tr A|$. Because of the elementary formula $[fX, kY] = fk[X, Y] + f(Xk)Y -$

$k(Yf)X$ for f, k smooth functions and X, Y smooth vector fields, the Lie bracket for Fenchel-Nielsen vector fields is completely determined by its restriction to the lattice $\mathfrak{T}$.

THEOREM 4.8. *The twist lattice $\mathfrak{T}$ is a Lie algebra over* $\mathbf{Z}$. *Given $A, B \in \Gamma$, hyperbolic, let $\mathfrak{I}$ be those elements C of Γ such that the axis of $C^{-1}AC$ intersects transversely the axis of B. Then*

$$[T_A, T_B] = \sum_{C \in \langle A \rangle \backslash \mathfrak{I} / \langle B \rangle} T_{A_c B^{-1}} - T_{A_c B} \tag{4.2}$$

where $A_c = C^{-1}AC$ or $A_c = C^{-1}A^{-1}C$ is determined such that r_B will lie to the left of its axis.

Proof. We begin with trigonometry; consider the following figure.

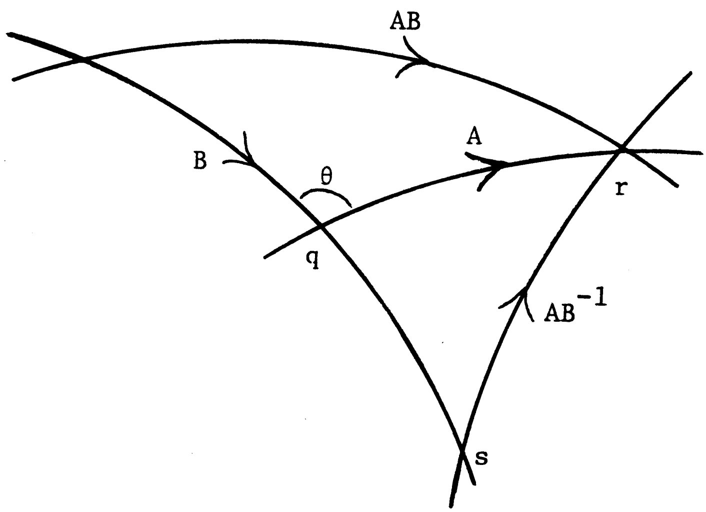

FIGURE 1

Let q, r and s be the rotations thru angle π at the indicated intersections. We obtain the expressions

$$A = rq, \qquad B = sq, \qquad AB = rqsq \quad \text{and} \quad AB^{-1} = rs.$$

The axes of these transformations are labeled and the sense of each is indicated. The distance between consecutive vertices on an axis is one-half the translation length. Using the law of cosines we have

$$\cos\theta = \frac{\cosh l_A/2 \cosh l_B/2 - \cosh l_{AB}/2}{\sinh l_A/2 \sinh l_B/2}$$

and

$$\cos \pi - \theta = \frac{\cosh l_A/2 \cosh l_B/2 - \cosh l_{AB^{-1}}/2}{\sinh l_A/2 \sinh l_B/2}$$

or

$$(4.3) \qquad 2\cos\theta = \frac{\cosh l_{AB^{-1}}/2 - \cosh l_{AB}/2}{\sinh l_A/2 \sinh l_B/2}.$$

Now using (4.1) and Lemma 4.2, we have

$$[T_A, T_B] = -\Omega^{-1}(d(16 \sinh l_A/2 \sinh l_B/2\; t_A l_B));$$

then by Theorem 3.3, and (4.3),

$$= -8\Omega^{-1}\Big(\sum_{C \in \langle A \rangle \backslash \mathcal{G}/\langle B \rangle} d\big(\cosh l_{A_cB^{-1}}/2 - \cosh l_{A_cB}/2\big)\Big)$$

$$= -4\Omega^{-1}\Big(\sum_{C \in \langle A \rangle \backslash \mathcal{G}/\langle B \rangle} \sinh l_{A_cB^{-1}}/2\, dl_{A_cB^{-1}} - \sinh l_{A_cB}/2\, dl_{A_cB}\Big)$$

and finally by Lemma 4.1,

$$= \sum_{C \in \langle A \rangle \backslash \mathcal{G}/\langle B \rangle} T_{A_cB^{-1}} - T_{A_cB}.$$

That $\mathfrak{T}$ is a Lie algebra over $\mathbf{Z}$ is an immediate consequence of the formula.

Remarks. The separation property of axes in a group Γ is known to be a topological invariant; this fact is suggested by formula (4.2). The Lie bracket $[T_A, T_B]$ is evaluated by an arbitrary choice of Γ and consideration of the transformations $A, B \in \Gamma$. We conclude that the twist lattice is an isomorphism invariant of Γ. In particular if Γ is torsion free, $\mathfrak{T}$ is an invariant of the fundamental group. Finally, by considering the Hamiltonian potentials $4|\operatorname{tr} A|$, we can show that the $\mathbf{R}$ span of $\mathfrak{T}$ is infinite dimensional.

The vector fields t_α, T_α and the Lie bracket may also be considered for finitely generated groups where H/Γ has infinite area. Many of the present formulas extend to this case by virtue of the doubling construction discussed in the proof of Theorem 3.6. This is possible even though the Teichmüller space does not have a complex structure.

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