

Weil-Petersson perspectives

Scott A. Wolpert

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1 Introduction.

We highlight recent progress in the study of the Weil-Petersson (WP) geometry of finite dimensional Teichmüller spaces. For recent progress on and the understanding of infinite dimensional Teichmüller spaces the reader is directed to [TT04a, TT04b]. As part of the highlight, we also present possible directions for future investigations. Recent works on WP geometry involve new techniques which present new opportunities.

We begin with background highlights. The reader should see [Ahl61, Ber74, Nag88, Roy75], as well as [Wol03, Sec. 1 and 2] for particulars of our setup and notation for the augmented Teichmüller space $\overline{\mathcal{T}}$, the mapping class group MCG , Fenchel-Nielsen (FN) coordinates and the complex of curves $C(F)$ for a surface F . We describe the main elements.

For a reference topological surface F of genus g with n punctures and negative Euler characteristic, a Riemann surface R (homeomorphic to F) with complete hyperbolic metric, a marked Riemann/hyperbolic surface is the equivalence class of a pair $\{(R, f)\}$ for $f : F \rightarrow R$ an orientation preserving homeomorphism; equivalence for post composition with a conformal homeomorphism. Teichmüller space \mathcal{T} is the space of equivalence classes, [Ahl61, IT92, Nag88]. The mapping class group $MCG = Homeo^+(F)/Homeo_0(F)$ acts properly discontinuously on \mathcal{T} by taking $\{(R, f)\}$ to $\{(R, f \circ h^{-1})\}$ for a homeomorphism h of F .

\mathcal{T} is a complex manifold with the MCG acting by biholomorphisms. The cotangent space at $\{(R, f)\}$ is $Q(R)$, the space of integrable holomorphic quadratic differentials, [Ahl61, Nag88]. For $\phi, \psi \in Q(R)$ the Teichmüller (Finsler metric) conorm of ϕ is $\int_R |\phi|$, while the WP dual Hermitian pairing is $\int_R \phi \overline{\psi} (ds^2)^{-1}$, where ds^2 is the R complete hyperbolic metric. The WP metric is Kähler, not complete, with negative sectional curvature and the MCG acts by isometries (see Section 3 below), [Ahl61, Mas76, Roy75, Tro86,

Wol86]. The *complex of curves* $C(F)$ is defined as follows. The vertices of $C(F)$ are free homotopy classes of homotopically nontrivial, nonperipheral, simple closed curves on F . A k -simplex of $C(F)$ consists of $k + 1$ distinct homotopy classes of mutually disjoint simple closed curves. A maximal set of mutually disjoint simple closed curves, a *pants decomposition*, has $3g - 3 + n$ elements for F a genus g , n punctured surface.

A free homotopy class γ on F determines the geodesic-length function ℓ_γ on \mathcal{T} ; $\ell_\gamma(\{R, f, \})$ is defined to be the hyperbolic length of the unique geodesic freely homotopic to $f(\gamma)$. The geodesic-length functions are convex along WP geodesics, [Wol87b, Wol]. On a hyperbolic surface the geodesics for a pants decomposition decompose the surface into geometric pairs of pants: subsurfaces homeomorphic to spheres with a combination of three discs or points removed, [Abi80, IT92]. The hyperbolic geometric structure on a pair of pants is uniquely determined by its boundary geodesic-lengths. Two pants boundaries with a common length can be abutted to form a new hyperbolic surface (a complete hyperbolic structure with possible further geodesic boundaries.) The common length ℓ for the joined boundary and the offset, or *twist* τ , for adjoining the boundaries combine to provide parameters (ℓ, τ) for the construction. The twist τ is measured as displacement along boundaries. The Fenchel-Nielsen (FN) parameters $(\ell_{\gamma_1}, \tau_{\gamma_1}, \dots, \ell_{\gamma_{3g-3+n}}, \tau_{\gamma_{3g-3+n}})$ valued in $(\mathbb{R}_+ \times \mathbb{R})^{3g-3+n}$ provide global real-analytic coordinates for \mathcal{T} , [Abi80, Mas01, Wol82]. Each pants decomposition determines a global coordinate.

A bordification of \mathcal{T} , the *augmented Teichmüller space*, is introduced by extending the range of the parameters. For an ℓ_γ equal to zero, the twist is not defined and in place of the geodesic for γ there appears a pair of cusps. Following Bers [Ber74] the extended FN parameters describe marked (possibly) noded Riemann surfaces (marked stable curves.) An equivalence relation is defined for marked noded Riemann surfaces and a construction is provided for adjoining to \mathcal{T} frontier spaces (where subsets of geodesic-lengths vanish) to obtain the augmented Teichmüller space $\overline{\mathcal{T}}$, [Abi77, Abi80]. $\overline{\mathcal{T}}$ is not locally compact since in a neighborhood of ℓ_γ vanishing the FN angle $\theta_\gamma = 2\pi\tau_\gamma/\ell_\gamma$ has values filling \mathbb{R} . The leading-term expansion for the WP metric is provided in [DW03, Yam01, Wol03]. Following Masur the WP metric extends to a complete metric on $\overline{\mathcal{T}}$, [Mas76].

The group MCG acts (not properly discontinuously) as a group of homeomorphisms of $\overline{\mathcal{T}}$ and $\overline{\mathcal{T}}/MCG$ is topologically the Deligne-Mumford compactified moduli space of stable curves $\overline{\mathcal{M}}$. We will include the empty set as a -1 -simplex of $C(F)$. The complex is partially ordered by inclusion of simplices. For a marked noded Riemann surface $\Lambda(\{(R, f)\}) \in C(F)$ is the

simplex of free homotopy classes on F mapped to nodes of R . The level sets of Λ are the strata of $\overline{\mathcal{T}}$.

$(\overline{\mathcal{T}}, d_{WP})$ is a $CAT(0)$ metric space; see [DW03, Yam01, Wol03] and the attribution to Benson Farb in [MW02]. The general structure of $CAT(0)$ spaces is described in detail in [BH99]. $CAT(0)$ spaces are complete metric spaces of *nonpositive curvature*. In particular for $p, q \in \overline{\mathcal{T}}$ there is a unique length-minimizing path \widehat{pq} connecting p and q . An additional property is that WP geodesics do not *refract*; an open WP geodesic segment $\widehat{pq} - \{p, q\}$ is contained in the (open) stratum $\Lambda(p) \cap \Lambda(q)$ [DW03, Yam01, Wol03]; an open segment is a solution of the WP geodesic differential equation on a product of Teichmüller spaces. The stratum of $\overline{\mathcal{T}}$ (the level sets of Λ) are totally geodesic complete subspaces. Strata also have a metric-intrinsic description: a stratum is the union of all open length-minimizing paths containing a given point, [Wol03, Thrm. 13].

A classification of *flats*, the locally Euclidean isometrically embedded subspaces, of $\overline{\mathcal{T}}$ is given in [Wol03, Prop. 16]. Each flat is contained in a proper substratum and is the Cartesian product of geodesics from component Teichmüller spaces. A structure theorem characterizes limits of WP geodesics on $\overline{\mathcal{T}}$ modulo the action of the mapping class group, [Wol03, Sec. 7]. Modulo the action the general limit is the unique length-minimizing piecewise geodesic path connecting the initial point, a sequence of strata and the final point. An application is the construction of axes in $\overline{\mathcal{T}}$ for elements of the MCG , [DW03, Wol03]. The geodesic limit behavior is suggested by studying the sequence $\widehat{pT_\gamma^n p}$ for a Dehn twist $T_\gamma \in MCG$: modulo MCG the limit is two copies of the length-minimizing path from p to $\{\ell_\gamma = 0\}$.

Jeffrey Brock established a collection of important results on WP synthetic geometry, [Bro03, Bro02]. Brock introduced an approach for approximating WP geodesics. A result is that the geodesic rays from a point of \mathcal{T} to the maximally noded Riemann surfaces have initial tangents dense in the initial tangent space. We used the approach in [Wol03, Coro. 19] to find that $\overline{\mathcal{T}}$ is the closed convex hull of the discrete subset of marked maximally noded Riemann surfaces. We also showed that the geodesics connecting the maximally noded Riemann surfaces have tangents dense in the tangent bundle of \mathcal{T} .

Howard Masur and Michael Wolf established the WP counterpart to H. Royden's celebrated theorem: each WP isometry of \mathcal{T} is induced by an element of the extended MCG , [MW02]. A simplified proof of the Masur-Wolf result is given as follows: [Wol03, Thrm. 20], the elements of $Isom_{WP}(\mathcal{T})$ extend to isometries of $\overline{\mathcal{T}}$; the extensions preserve the metric-intrinsic stratum

of $\overline{\mathcal{T}}$ and so correspond to simplicial automorphisms of $C(F)$; the simplicial automorphisms of $C(F)$ are in general induced by the elements of the *MCG* from the work of N. Ivanov, M. Korkmaz and F. Luo, [Iva97, Kor99, Luo00]; and finally $\overline{\mathcal{T}} - \mathcal{T}$ is a uniqueness set for WP isometries. Brock and Dan Margalit have recently extended available techniques to include the special (g, n) types of $(1, 1)$, $(1, 2)$ and $(2, 0)$, [BM04].

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2 Classical metrics for \mathcal{T} .

A puzzle of Teichmüller space geometry is that the classical Teichmüller and Weil-Petersson metrics in combination have desired properties: Kähler, complete, finite volume, negative curvature, and a suitable sphere at infinity, yet individually the metrics already lack the basic properties of completeness and non-positive curvature. In fact N. Ivanov showed that $\mathcal{M}_{g,n}$, $3g - 3 + n \geq 2$, admits no complete Riemannian metric of pinched negative sectional curvature, [Iva88]. Recently authors have considered asymptotics of the WP metric and curvature with application to introducing modifications to obtain *designer metrics*.

Curt McMullen considered adding a small multiple of the Hessian of the *short geodesic log length sum* to the WP Kähler form

$$\omega_{1/\ell} = \omega_{WP} + i c \sum_{\ell_\gamma < \epsilon} \partial\bar{\partial} \text{Log } \ell_\gamma$$

(for $\text{Log} \approx \min\{\log, 0\}$) to obtain the Kähler form for a modified metric, [McM00]. The constructed metric is *Kähler hyperbolic*: on the universal cover the Kähler form $\omega_{1/\ell}$ is the exterior derivative of a bounded 1-form and the injectivity radius is positive; on the moduli space $\mathcal{M}_{g,n}$ the metric $g_{1/\ell}$ is: complete, finite volume, of bounded sectional curvature. Of primary interest is that the constructed metric $g_{1/\ell}$ is comparable to the Teichmüller metric. The result provides that the Teichmüller metric qualitatively has the properties of a Kähler hyperbolic metric, [McM00]. As applications McMullen establishes a positive lower bound for the Teichmüller Rayleigh-Ritz quotient and a Teichmüller metric isoperimetric inequality for complex manifolds. McMullen also uses the metric to give a simple derivation that the sign of the orbifold Euler characteristic alternates with the parity of the complex dimension. The techniques of [Wol] can be applied to show that in a neighborhood of the compactification divisor $\mathcal{D} = \overline{\mathcal{M}}_g - \mathcal{M}_g \subset \overline{\mathcal{M}}_g$ the

McMullen modification to WP is principally in the directions transverse to the divisor.

Lipman Bers promoted the question of understanding the equivalence of the classical metrics on Teichmüller spaces in 1972, [Ber72]. A collection of authors have now achieved major breakthroughs. Kefeng Liu, Xiaofeng Sun and Shing-Tung Yau have analyzed WP curvature, applied Yau's Schwarz lemma, and considered the Bers embedding to compare and study the Kähler-Einstein metric for Teichmüller space and also the classical metrics, [LSY04]. Sai-Kee Yeung considered the Bers embedding and applied Schwarz type lemmas to compare classical metrics, [Yeu04]. Yeung's approach is based on his analysis of convexity properties of fractional powers of geodesic-length functions, [Yeu03]. Bo-Yong Chen considered McMullen's metric, the WP Kähler potential of Takhtajan-Teo, and L^2 -estimates to establish equivalence of the Bergman and Teichmüller metrics, [BY04].

In earlier work, Cheng-Yau [CY80] and Mok-Yau [MY83] had established existence and completeness for a Kähler-Einstein metric for Teichmüller space. Liu-Sun-Yau study the negative WP Ricci form $-Ricci_{WP}$; the form defines a complete Kähler metric on the Teichmüller space. The authors find that the metric is equivalent to the *asymptotic Poincaré metric* and that asymptotically its holomorphic sectional curvature has negative upper and lower bounds. The authors develop the asymptotics for the curvature of the metric, [LSY04].

To overcome difficulty with controlling interior curvatures Liu-Sun-Yau further introduce the modification

$$g_{LSY} = -Ricci_{WP} + c g_{WP}, \quad c > 0, \quad [LSY04].$$

The authors find that the LSY metric is complete with bounded negative holomorphic sectional curvature, bounded negative Ricci curvature and bounded sectional curvature. The authors apply their techniques and results of others to establish comparability of the seven classical metrics

$$LSY \sim \text{asymptotic Poincaré} \sim \text{Kähler-Einstein} \sim \text{Teichmüller-Kobayashi} \sim \text{McMullen} \sim \text{Bergman} \sim \text{Carathéodory}.$$

Yeung established the comparability of the last five listed classical metrics. Now the classical metrics are expected to have the same qualitative behavior. In combination the metrics have a substantial list of properties. Liu-Sun-Yau use their understanding of the metrics in a neighborhood of the compactification divisor $\mathcal{D} \subset \overline{\mathcal{M}}_g$ to show that the logarithmic cotangent bundle (Bers' bundle of *holomorphic 2-differentials*) of the compactified

moduli space is stable in the sense of Mumford. The authors also show that the bundle is stable with respect to its first Chern class. The authors further find that the Kähler-Einstein metric has bounded geometry in a strong sense.

The comparison of metrics now provides a research opportunity to combine different approaches for studying Teichmüller geometry.

3 WP synthetic geometry.

A collection of authors including Jeffrey Brock, Sumio Yamada and Richard Wentworth have recently studied WP synthetic geometry. A range of new techniques have been developed. Brock established a collection of very interesting results on the large-scale behavior of WP distance. Brock considered the *pants graph* $C_{\mathbf{P}}(F) \subset C(F)$ having vertices the distinct pants decompositions of F and joining edges of unit-length for pants decompositions differing by an elementary move, [Bro03]. He showed that the 0-skeleton of $C_{\mathbf{P}}(F)$ with the edge-metric is quasi-isometric to \mathcal{T} with the WP metric. He further showed for $p, q \in \mathcal{T}$ that the corresponding quasifuchsian hyperbolic three-manifold has convex core volume comparable to $d_{WP}(p, q)$. At large-scale, WP distance and convex core volume are approximately combinatorially determined.

Brock's approach begins with Bers' observation that there is a constant L depending only on the topological type of F , such that a surface R has a pants decomposition with lengths $\ell_{\gamma}(R) < L$. For \mathcal{P} a pants decomposition Brock associates the sublevel set

$$V(\mathcal{P}) = \{R \mid \gamma \in \mathcal{P}, \ell_{\gamma} < L\}.$$

By Bers' theorem the sublevel sets $V(\mathcal{P})$ cover \mathcal{T} , [Ber74]. Brock finds that the configuration of the sublevel sets in terms of WP distance on \mathcal{T} is coarsely approximated by the metric space $C_{\mathbf{P}}(F)$. He finds for $R, S \in \mathcal{T}$ with $R \in V(\mathcal{P}_R), S \in V(\mathcal{P}_S)$ that

$$d_{\mathbf{P}}(\mathcal{P}_R, \mathcal{P}_S) \asymp d_{WP}(R, S)$$

for $d_{\mathbf{P}}$ the $C_{\mathbf{P}}(F)$ edge-metric, [Bro03]. Brock's underlying idea is that $C_{\mathbf{P}}(F)$ is the 1-skeleton for the nerve of the covering $\{V(\mathcal{P})\}$ and that the minimal count of *elementary moves* in $C_{\mathbf{P}}(F)$ approximates WP distance. Paths in the pants graph $C_{\mathbf{P}}(F)$ can be encoded as sequences of elementary moves.

Can sequences of elementary moves be used to study the bi-infinite WP geodesics? Can $C_{\mathbf{P}}(F)$ be used to investigate the large g, n behavior of $\text{diam}_{WP}(\mathcal{M}_{g,n})$?

Brock's model can be used to study and compare the volume growth of $C_{\mathbf{P}}(F)$ and \mathcal{T} . A growth function for $C_{\mathbf{P}}(F)$ is the number of vertices in an edge-metric-ball of a given radius about a base point. A growth function for \mathcal{T} is the WP volume of a ball of a given radius about a base point.

A research question is to determine the growth function for $C_{\mathbf{P}}(F)$ and to effect the comparison to \mathcal{T} .

Questions regarding WP geodesics include behavior in-the-small, as well as in-the-large. Understanding the WP metric tangent cones of $\overline{\mathcal{T}}$, as well as the metric tangent cone bundle over $\overline{\mathcal{T}}$ is a basic matter. Recall the definition of the Alexandrov angle for unit-speed geodesics $\alpha(t)$ and $\beta(t)$ emanating from a common point $\alpha(0) = \beta(0) = o$ of a metric space (X, d) , [BH99, pg. 184]. The Alexandrov angle between α and β is defined by

$$\angle(\alpha, \beta) = \lim_{t \rightarrow 0} \frac{1}{2t} d(\alpha(t), \beta(t)).$$

Basic properties of the angle are provided in the opening and closing sections of [BH99, Chap. II.3]. Angle zero provides an equivalence relation on the space of germs of unit-speed geodesics beginning at o . For the model space $(\mathbb{H}; ds^2 = dr^2 + r^6 d\theta^2, r > 0)$ for the WP metric transverse to a stratum: the geodesics $\{\theta = \text{constant}\}$ are at angle zero at the *origin point* $\{r = 0\}$. The WP metric tangent cone at $p \in \overline{\mathcal{T}}$ is defined as the space of germs of constant-speed geodesics beginning at p modulo the relation of Alexandrov angle zero. Another matter is behavior of WP geodesics in a neighborhood of a stratum. In particular for \mathcal{S} a stratum defined by the vanishing of a geodesic-length function ℓ_γ and θ_γ the corresponding Fenchel-Nielsen angle, is θ_γ bounded along each geodesic ending on \mathcal{S} ? (See the next section for results of Wentworth precluding spiraling for harmonic maps from 2-dimensional domains.)

Research questions on geodesic behavior in-the-small include the following. Define and determine the basic properties of the metric tangent cone bundle of $\overline{\mathcal{T}}$. Determine if WP geodesics can spiral to a stratum. Determine the behavior of the Fenchel-Nielsen angles along finite geodesics.

There are questions regarding in-the-large behavior of WP geodesics. A basic invariant of hyperbolic metrics, the relative systole $sys_{rel}(R)$ of a surface R is the length of the shortest closed geodesic. In [Wol03] we note that the WP injectivity radius inj_{WP} of the moduli space \mathcal{M} is comparable to $(sys_{rel})^{1/2}$. A basic question is to understand the relative systole sys_{rel} along infinite WP geodesics in \mathcal{T} . It would also be interesting to understand the combinatorics of the sequence of short geodesics, and any limits of the (weighted) collections of simple closed geodesics for which the Bers decomposition is realized. Brock, Masur and Minsky have preliminary results on such limits in the space of measured geodesic laminations \mathcal{MGL} . The authors are studying the family of hyperbolic metrics $\{R_t\}$ over a WP geodesic. A geometric model M for a 3-manifold is constructed from the family; the authors seek to show that the model is biLipschitz to the *thick part* of a hyperbolic 3-dimensional structure. To construct M , the authors remove the *thin parts* of the fibers and then use the hyperbolic metric along fibers and rescalings of the WP metric in the parameter to define a 3-dimensional metric.

Two research questions on geodesic behavior in-the-large involve bi-infinite geodesics. Determine the behaviors of sys_{rel} along WP geodesics. Understand the combinatorics along WP geodesics of the sequence of short geodesics (pants decompositions) on surfaces and any limits of the weighted collections of short geodesics.

The divergence of a geodesic and a geodesic/stratum is a measure of the curvature of the join. Although the WP metric on \mathcal{T} has negative sectional curvature there are directions of almost vanishing curvature near the bordification $\overline{\mathcal{T}} - \mathcal{T}$, as well as flats in the bordification. To codify, Zheng Huang has shown that the sectional curvatures on \mathcal{T} are pointwise bounded as $-c'(sys_{rel})^{-1} < sec_{WP} < -c'' sys_{rel}$ for positive constants, [Hua04]. In particular, a complex parameter for the formation of a node determines a tangent 2-plane with large negative curvature realizing the lower curvature bound. The two parameters for a pair of independent formations of nodes determine a tangent 2-plane with almost vanishing curvature. Huang has also found for sys_{rel} bounded away from zero that the sectional curvatures are bounded independent of the genus, [Hua06].

A research matter is to analyze the rate of divergence of pairs consisting of an infinite WP geodesic and either a second infinite WP geodesic or a stratum.

A third matter proposed by Jeffrey Brock is to investigate the dynamics of the WP geodesic flow on the tangent bundle TM . The $CAT(0)$ geometry can be used to show that the *finite and semi-finite* WP geodesics form a subset of smaller Hausdorff dimension, [Wol03, Sec. 6].

Are there WP geodesics with lifts dense in TT ? Are the lifts of axes of (pseudo Anosov) elements of the MCG dense in TT ? What is the growth rate of the minimal translation lengths for the conjugacy classes of the pseudo Anosov elements? Is WP geodesic flow (restricted to the bi-infinite directions) ergodic?

Jason Behrstock-Yair Minsky [BM06] and Ursula Hamenstädt [Ham06] have verified the Brock-Farb conjecture for the maximal dimension of a quasi-flat as $\lfloor \frac{3g+n-2}{2} \rfloor$ (the lower bound is straightforward [BF01, Wol03].) The maximal dimension of a quasi-flat is the rank in the sense of Gromov. The rank is important for understanding the global WP geometry, as well as for understanding mapping class groups. Brock-Farb, Jason Behrstock and Javier Aramayona had already found that certain low dimensional Teichmüller spaces are Gromov-hyperbolic and necessarily rank one, [Ara04, Beh05, BF01].

A research matter is to understand the behavior of quasi-geodesics and especially quasi-flats, the quasi-isometric embeddings of Euclidean space into \mathcal{T} .

4 Harmonic maps to $\overline{\mathcal{T}}$.

A collection of authors including Georgios Daskalopoulos, Richard Wentworth and Sumio Yamada have considered harmonic maps from (finite volume) Riemannian domains Ω into the $CAT(0)$ space $\overline{\mathcal{T}}$, [DKW00, DW03, Wen04, Yam01]. The authors apply the Sobolev theory of Korevaar-Schoen, [KS93], to study maps energy-minimizing for prescribed boundary values. Possible applications are rigidity results for homomorphisms of lattices in Lie groups to mapping class groups and existence results for harmonic representatives of the classifying maps associated to symplectic Lefschetz pencils, [DW03].

The behavior of a harmonic map to a neighborhood of a lower stratum of $\overline{\mathcal{T}}$ is a basic matter. Again this is especially important since $\overline{\mathcal{T}}$ is not locally compact in a neighborhood of a lower stratum; in each neighborhood the corresponding Fenchel-Nielsen angles surject to \mathbb{R} . A harmonic map or even a WP geodesic may *spiral* with unbounded Fenchel-Nielsen angles. With the

non-refraction of WP geodesics at lower stratum [DW03, Yam01, Wol03], a question is to consider analogs of *non-refraction* for harmonic mappings. In particular for the complex of curves $C(F)$ and $\Lambda : \overline{\mathcal{T}} \rightarrow C(F)$ the labeling function and $u : \Omega \rightarrow \overline{\mathcal{T}}$ a harmonic map, what is the behavior of the composition $\Lambda \circ u$? For 2-dimensional domains Wentworth has already provided important results on *non-spiraling* and on $\Lambda \circ u$ being constant on the interior of Ω , [Wen04]. A second general matter is to develop an encompassing approach (including treating regularity and singularity behavior) for approximation of maps harmonic to a neighborhood of a stratum by harmonic maps to model spaces, in particular for: $(\mathbb{H}; ds^2 = dr^2 + r^6 d\theta^2, r > 0)$.

Two research matters on harmonic maps are as follows. Understand the behavior of harmonic maps to neighborhoods of lower strata in $\overline{\mathcal{T}}$. Develop an encompassing approach for approximation of maps harmonic to a neighborhood of a stratum in terms of harmonic maps to model spaces.

5 Characteristic classes and the WP Kähler form.

Andrew McIntyre in joint work with Leon Takhtajan [MT04], Maryam Mirzakhani [Mir04a, Mir04b, Mir], and Lin Weng in part in joint work with Wing-Keung To [Wen01] have been studying the questions in algebraic geometry involving the WP Kähler form. The authors' considerations are guided by the study of the Quillen and Arakelov metrics, and in particular calculations from *string theory*. We sketch aspects of their work and use the opportunity to describe research with Kunio Obitsu.

Andrew McIntyre and Leon Takhtajan have extended the work of Peter Zograf and provided a new *holomorphic factorization formula* for the regularized determinant $\det' \Delta_k$ of the hyperbolic Laplace operator acting on (smooth) symmetric-tensor k -differentials for compact Riemann surfaces, [MT04]. Alexey Kokotov and Dmitry Korotkin have also provided a new *holomorphic factorization formula* for the regularized determinant $\det' \Delta_0$ of the hyperbolic Laplace operator acting on functions, [KK04]. Each factorization involves the exponential of an *action integral*: in the first formula for a Schottky uniformization, and in the second formula for a branched covering of $\mathbb{C}P^1$. The McIntyre-Takhtajan formula provides a holomorphic factorization for the $\partial\bar{\partial}$ *antiderivative* for the celebrated families local index theorem

$$\bar{\partial}\partial \log \frac{\det N_k}{\det' \Delta_k} = \frac{6k^2 - 6k + 1}{6\pi i} \omega_{WP}$$

for N_k the Gram matrix of the natural basis for holomorphic k -differentials relative to the Petersson product, [MT04]. The McIntyre-Takhtajan formula for $\frac{\det N_k}{\det' \Delta_k}$ gives rise to an isometry between the determinant bundle for holomorphic k -differentials with Quillen metric and with a metric defined from the Liouville action. Recall that the quotient $\frac{\det N_k}{\det' \Delta_k}$ is the Quillen norm of the natural frame for $\det N_k$. The formulas represent progress in the ongoing study of the behavior of the Quillen metric, the hyperbolic regularized determinant $\det' \Delta_k$ and positive integral values of the Selberg zeta function, since $\det' \Delta_k = c_{g,k} Z(k)$, $k > 2$, [DP86, Sar87]. The above formulas provide another approach for studying the degeneration of $\det' \Delta_k$; see [Hej90, Wol87a].

Mumford's tautological class κ_1 , the pushdown of the square of the relative dualizing sheaf from the universal curve, is represented by the WP class $\frac{1}{\pi^2} \omega_{WP}$, [Wol90]. The top self-intersection number of κ_1 on $\overline{\mathcal{M}}_{g,n}$ is the WP volume. From effective estimates for intersections of divisors on the moduli space, Georg Schumacher and Stefano Trapani [ST01] have given lower bounds for $vol_{WP}(\overline{\mathcal{M}}_{g,n})$. From Robert Penner's [Pen92] decorated Teichmüller theory and a combinatorial description of the moduli space, Samuel Grushevsky [Gru01] has given upper bounds with the same leading growth order.

In a series of innovative papers Maryam Mirzakhani has presented a collection of new results on hyperbolic geometry and calculations of WP integrals. The asymptotics for the count of the number of simple closed geodesics on a hyperbolic surface of at most a given length is presented in [Mir04a]. She establishes the asymptotic

$$\#\{\gamma \mid \ell_\gamma(R) \leq L\} \sim c_R L^{6g-6+2n}.$$

A recursive method for calculating the WP volumes of moduli spaces of bordered hyperbolic surfaces with prescribed boundary lengths is presented in [Mir04b]. And a proof of the Witten-Kontsevich formula for the tautological classes on $\overline{\mathcal{M}}_{g,n}$ is presented in [Mir].

Central to Mirzakhani's considerations is a recursive scheme for evaluating WP integrals over the moduli space of bordered hyperbolic surfaces with prescribed boundary lengths. Her approach is based on recognizing an integration role for McShane's length sum identity. Greg McShane discovered a remarkable identity for geodesic-lengths of simple closed curves on punctured hyperbolic surfaces, [McS98]. To illustrate the recursive scheme for evaluation of integrals we sketch the consideration for $(g, n) = (1, 1)$.

For the $(1, 1)$ case the identity provides that

$$\sum_{\gamma \text{ scg}} \frac{1}{1 + e^{\ell_\gamma}} = \frac{1}{2}$$

for the sum over simple closed geodesics (scg's). The identity corresponds to a decomposition of a horocycle about the puncture; the identity arises from classifying the behavior of simple complete geodesics emanating from the puncture. Mirzakhani's insight is that the identity can be combined with the $d\tau \wedge d\ell$ formula for ω_{WP} , [Wol85], to give an *unfolding* of the $\mathcal{M}_{1,1}$ volume integral. For a MCG -fundamental domain $\mathcal{F}_{1,1} \subset \mathcal{T}_{1,1}$, she observes that

$$\begin{aligned} \int_{\mathcal{M}_{1,1}} \frac{1}{2} \omega_{WP} &= \int_{\mathcal{F}_{1,1}} \sum_{h \in MCG/Stab_\gamma} \frac{1}{1 + e^{\ell_{h(\gamma)}}} d\tau \wedge d\ell = \\ &= \sum_{h \in MCG/Stab_\gamma} \int_{h^{-1}(\mathcal{F}_{1,1})} \frac{1}{1 + e^\ell} d\tau \wedge d\ell = \int_{\mathcal{T}_{1,1}/Stab_\gamma} \frac{1}{1 + e^\ell} d\tau \wedge d\ell \end{aligned}$$

(using that $\ell_\gamma \circ h^{-1} = \ell_{h(\gamma)}$) with the last integral elementary since

$$\mathcal{T}_{1,1}/Stab_\gamma = \{\ell > 0, 0 < \tau < \ell\}.$$

Mirzakhani established a general identity for bordered hyperbolic surfaces that generalizes McShane's identity, [Mir04b, Sec. 4]. Both identities are based on a sum over configurations of simple closed curves (*subpartitions*) which with a fixed boundary bound a pair of pants. Mirzakhani uses the identity to *unfold* WP integrals to sums of product integrals over lower dimensional moduli spaces for surfaces with boundaries. Her overall approach provides a recursive scheme for determining WP volumes $vol_{WP}(\mathcal{M}_g(b_1, \dots, b_n))$ for the (real analytic) moduli spaces of hyperbolic surfaces with prescribed boundary lengths (b_1, \dots, b_n) . As an instance it is shown that

$$vol_{WP}(\mathcal{M}_1(b)) = \frac{\pi^2}{6} + \frac{b^2}{24}$$

and

$$vol_{WP}(\mathcal{M}_1(b_1, b_2)) = \frac{1}{384}(4\pi^2 + b_1^2 + b_2^2)(12\pi^2 + b_1^2 + b_2^2).$$

In complete generality Mirzakhani found that the WP volume is a polynomial

$$vol_g(b) = \sum_{|\alpha| \leq 3g-3+n} c_\alpha b^{2\alpha}, \quad c_\alpha > 0, \quad c_\alpha \in \pi^{6g-6+2n-2}\mathbb{Q},$$

for b the vector of boundary lengths and α a multi index, [Mir04b]. An easy application is an expansion for the volume of the tube $\mathcal{N}_\epsilon(\mathcal{D}) \subset \overline{\mathcal{M}}_g$ about the compactification divisor. Recently Ser Peow Tan, Yan Loi Wong and Ying Zhang have also generalized McShane's identity for conical hyperbolic surfaces, [TWZ04]. A general identity is obtained by studying gaps formed by simple normal geodesics emanating from a distinguished cone point, cusp or boundary geodesic.

Mirzakhani separately established that the volumes $vol_{WP}(\mathcal{M}_g(b_1, \dots, b_n))$ are determined from the intersection numbers of tautological characteristic classes on $\overline{\mathcal{M}}_{g,n}$. A point of $\overline{\mathcal{M}}_{g,n}$ describes a Riemann surface R possibly with nodes with distinct points x_1, \dots, x_n . The line bundle \mathcal{L}_i on $\overline{\mathcal{M}}_{g,n}$ is the unique line bundle whose fiber over $(R; x_1, \dots, x_n)$ is the cotangent space of R at x_i ; write $\psi_i = c_1(\mathcal{L}_i)$ for the Chern class. Mirzakhani showed using symplectic reduction (for an S^1 quasi-free action following Guillemin-Sternberg) that

$$vol_g(b) = \sum_{|\alpha| \leq N} \frac{b_1^{2\alpha_1} \dots b_n^{2\alpha_n}}{2^{|\alpha|} \alpha! (N - |\alpha|)!} \int_{\overline{\mathcal{M}}_{g,n}} \psi^{\alpha_1} \dots \psi_n^{\alpha_n} \omega_{WP}^{N-|\alpha|}$$

for $N = \dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n}$, [Mir04b]. She then combined the above formula and her recursive integration scheme to find that the collection of integrals

$$\langle \tau_{k_1} \dots \tau_{k_n} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \dots \psi_n^{k_n}, \quad \sum k_i = \dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n},$$

satisfy the recursion for the *string equation* and the *dilaton equation*, [Mir, Sec. 6]. The recursion is the Witten-Kontsevich conjecture. She also found that the volume intergrands satisfy a recursive scheme. Mirzakhani's results represent major progress for the study of volumes and intersection numbers; her results clearly raise the prospect of further insights.

Can Mirzakhani's approach be applied for other integrals? Can the considerations of Grushevsky, Schumacher-Trapani be extended to give further effective intersection estimates?

More than a decade ago Leon Takhtajan and Peter Zograf studied the local index theorem for a family of $\bar{\partial}$ -operators on Riemann surfaces of type (g, n) , [TZ91]. The authors calculated the first Chern form of the determinant line bundle provided with Quillen's metric. For a Riemann surface with punctures there are several candidates for $\det' \Delta$ (a renormalization is necessary since punctures give rise to *continuous spectrum* for Δ). The authors

considered the Selberg zeta function $Z(s)$ and set $\det' \Delta_k = c_{g,n} Z(k)$, $k \geq 2$, [TZ91]. For $\lambda_k = \det N_k$, the determinant line bundle of the bundle of holomorphic k -differentials (with poles allowed at punctures), the authors found for the Chern form

$$c_1(\lambda_k) = \frac{6k^2 - 6k + 1}{12\pi^2} \omega_{WP} - \frac{1}{9} \omega_*$$

with the 2-form ω_* on holomorphic quadratic differentials $\phi, \psi \in Q(R)$

$$\omega_*(\phi, \psi) = \sum_{i=1}^n \Im \int_R \phi \bar{\psi} E_i(z; 2) (ds_{hyp}^2)^{-1}$$

with $E_i(z; 2)$, the Eisenstein-Maass series at $s = 2$ for the cusp x_i . By construction ω_* is a closed $(1, 1)$ form; the associated Hermitian pairing (absent the imaginary part) is a MCG -invariant Kähler metric for \mathcal{T} . The Takhtajan-Zograf (TZ) metric g_{TZ} has been studied in several works, [Obi99, Obi01, Wen01, Wol06]. Kunio Obitsu showed that the metric is not complete [Obi01] and is now studying the degeneration of the metric. His estimates provide a local comparison for the TZ and WP metrics. Although beyond our exposition, we cite the important work of Lin Weng; he has been pursuing an Arakelov theory for punctured Riemann surfaces and has obtained a global comparison for the WP and TZ metrized determinant line bundles $\Delta_{WP}^{\otimes n^2} \leq \Delta_{TZ}^{\otimes ((2g-2+n)^2)}$, [Wen01].

In joint work with Obitsu we are considering the expansion for the *tangential to the compactification divisor* $\mathcal{D} \subset \overline{\mathcal{M}}_g$ component g_{WP}^{tgt} of the WP metric, [OW]. The tangential component is the (orthogonal) complement to Yamada's normal form $dr^2 + r^6 d\theta^2$ for the transversal component. The tangential component expansion is given for a neighborhood of $\mathcal{D} \subset \overline{\mathcal{M}}_g$ for g_{WP} restricted to subspaces parallel to \mathcal{D} . For a family $\{R_\ell\}$ of hyperbolic surfaces given by *pinching* short geodesics all with common length ℓ , we find

$$g_{WP}^{tgt}(\ell) = g_{WP}^{tgt}(0) + \frac{\ell^2}{3} g_{TZ}(0) + O(\ell^3).$$

The formula establishes a direct relationship between the WP and TZ metrics. There is also a relationship with the work of Mirzakhani. The formula is based on an explicit form of the earlier 2-term expansion for the degeneration of hyperbolic metrics, [Wol90, Exp. 4.2].

What is the consequence of the above expansion for the determinant line bundle? Explore the properties of the TZ metric and its relationship to WP geometry.

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