

5.3 – Annuities

Increasing Annuities: You may want to save for retirement by putting some money into a savings account every month (or week, or quarter, etc.). To do this, we use the following formula:

If you make a payment PMT at the end of each compounding period into an account with an initial value PV , paying interest at an *annual* rate r compounded m times per year, then the value after t years is:

$$FV = PV(1 + r/m)^{mt} + PMT \left(\frac{(1+r/m)^{mt} - 1}{r/m} \right)$$

NOTE that usually we'll have $PV = 0$, i.e., we start our savings account from scratch. Also note that the number of payments per year must be the same as the number of compoundings per year.

1. Find the amount you can accumulate after 10 years by saving \$100 per month, at 6.5% annual interest, compounded monthly. (Assume your savings account is empty when you start.)

Answer: $PV = 0$; $r = 0.065$; $m = 12$; $PMT = 100$; $t = 10$

$$FV = 0 + 100 \left(\frac{(1+0.065/12)^{12 \cdot 10} - 1}{0.065/12} \right) = \$16,840.32$$

In your calculator, type:

2. **Who Wants to Be a Millionaire?** Suppose you start your career when you are 21 and you want to save \$1,000,000 by the time you are 60. If you invest your money in an account earning 8% annual interest, compounded weekly, how much do you need to deposit each week?

Answer: $PV = 0$; $FV = 1,000,000$; $m = 52$; $t = 39$; $r = 0.08$

$$1,000,000 = 0 + PMT \left(\frac{(1+0.08/52)^{52 \cdot 39} - 1}{0.08/52} \right)$$

$$1,000,000 = PMT \cdot 14,034.8969 \text{ (keep 4 decimal places)}$$

$$PMT = \$71.25.$$

Note: Type this in your calculator:

Notice that you have only deposited $71.25 \cdot 52 \cdot 39 = \$144,496.96!!!$ The rest ($\$855,503.03$) was all interest!!!

Decreasing Annuity: After you have saved all of this money, you will want to spend it. Maybe after you retire, you will give yourself a weekly (monthly, quarterly, etc.) allowance and you will plan to have spent all of your money by a certain time. We use the **decreasing annuity** formula:

If you invest PV dollars at an *annual* rate r compounded m times per year, and if you receive a payment (or make a withdrawal) of PMT at the end of each compounding period, and if you plan to have FV dollars left after t years, then:

$$PV = FV(1 + r/m)^{-mt} + PMT \left(\frac{1 - (1 + r/m)^{-mt}}{r/m} \right)$$

NOTE that we will usually have $FV = 0$, i.e. you plan to spend ALL of your money in t years. Also notice that the number of withdrawals equals the number of compoundings.

You can distinguish this formula from the increasing annuity formula by the exponent since the decreasing annuity formula has a *negative* exponent.

1. Suppose you have \$75,000 saved for college and it is invested at 9% per year compounded quarterly. You want to make quarterly withdrawals over the next 4 years. How much will you withdraw each quarter in order to spend all of your money after 4 years?

Answer: $FV = 0$; $PMT = ???$; $PV =$; $r =$; $m =$; $t =$

$PMT = \$5633.76$

Notice that you will have spent $PMT \cdot m \cdot t = \$90,140.17$ over the four years. That's \$15,140.17 more than you started with! Where did all of that extra money come from?

Loans and Mortgages can also be viewed as decreasing annuities. The process of paying off a loan (and killing the debt) is called *amortizing* the debt.

1. You bought a house and you need to pay off a \$90,000 loan. It was a 15-year loan (that means your house will be paid off after 15 years) at 7% annual interest per year. What are your monthly mortgage payments?

Answer: $FV = 0$; $PMT = ???$; $PV =$; $r =$; $m =$; $t =$

$PMT = \$808.94$

How much did you actually pay? How much of that was interest?

Retirement Planning: Your pension plan is guaranteed to earn 8% annual interest compounded monthly. You plan to work for 40 years before retiring and then you will withdraw \$6000 per month for 20 years. How much do you need to deposit per month while you are working?

Step 1: Find PV , the amount you will need to have in the account on the day you retire. You will use the decreasing annuity formula to determine how much you need in order to be able to withdraw \$6000 per month. You will assume that you have spent all of your money after 20 years.

$$PMT = 6000; r = \quad ; m = \quad ; t = \quad ; FV =$$

$$PV = 0 + PMT \left(\frac{\quad}{\quad} \right) =$$

Step 2: In step 1, you found that you need to accumulate _____ in an increasing annuity during the 40 years that you will be working. Find the amount you need to deposit each month in order to do this. (Assume you start with no money.)

$$PMT = ???; r = \quad ; m = \quad ; t = \quad ; FV = \quad ; PV = 0$$

$$FV = 0 + PMT \left(\frac{\quad}{\quad} \right) =$$