ORDERED-MINIMAL THEORIES AND OMITTING TYPES

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Abstract

In the 1970s and 80s, L. van den Dries, A. Pillay, C. Steinhorn, J. Knight, and D. Marker pioneered the study of this nice class of linearly ordered structures. Though A. Tarski began this work when he proved quantifier elimination of the real field, the property of being ordered-minimal was not given its current name until Pillay and Steinhorn's paper [3] in 1985, which announced the subject's arrival. Current research in classifying the $\mathcal{O}$-minimal expansions of the real field is ongoing. We investigate and collect some of the 'classical' results on $\mathcal{O}$-minimal structures and theories.

0 Preliminaries

In order to set some standard notation, assume unless otherwise noted that all formulas and types allow parameters from the given structure. We will also assume that all types are complete and nonprincipal, unless otherwise stated.

Definition 0.1. A structure $(\mathcal{M}, <, \dots)$ over language L is *$\mathcal{O}$-minimal* iff for any $D \subseteq \mathcal{M}$ definable by a formula of $L(\mathcal{M})$, D is is a finite union of (possibly infinite or degenerate) intervals.

To motivate the study of $\mathcal{O}$-minimal structures, it is useful to recall the definition of a *minimal structure*, in which every definable subset is either finite or cofinite. It is clear that in the case of a minimal structure or *strongly minimal theory* (a complete theory in which every model is minimal), the collection of definable sets is as simple as possible, since every finite or cofinite set is already definable in the language of equality. The typical example of a strongly minimal theory is that of algebraically closed fields in a given characteristic p, or ACF_p, for any p prime or $p = 0$. In this case, strong minimality is proven through quantifier elimination, and the characterization of definable sets corresponds with the constructible sets familiar to algebraic geometers.

Through a similar quantifier elimination, one can see that the ordered field of reals, and its countable counterpart, the ordered field of real algebraic numbers, are both $\mathcal{O}$-minimal. In a similar sense, the definable sets in an $\mathcal{O}$-minimal structure are as simple as one could hope for, since every interval and hence every finite combination of intervals is definable in the language of order. $\mathcal{O}$-minimality tells you there are no other definable sets.

A look at the title of this paper suggests the following result:

Proposition 0.2. *Let T be complete, and $\mathcal{M} \models T$ be $\mathcal{O}$-minimal. Then every $\mathcal{M}' \models T$ is $\mathcal{O}$-minimal.*

The above proposition, proved in [1], allows us to consider $\mathcal{O}$-minimality as a property of a theory instead of a structure. For instance it is a trivial consequence that the theory of the real ordered field, (which is also complete the theory of any real closed ordered field), $RCOF$, is $\mathcal{O}$-minimal.

The results cited above can in fact be improved. Just as the only strongly minimal extension of the theory of fields of characteristic 0 in the language of fields is ACF_0, it can be shown that the only $\mathcal{O}$-minimal extension of the theory of rings in the language of rings is $RCOF$. To illustrate the principles involved in such a result, we state and prove a less complicated but similar proposition.

Proposition 0.3. *Let $L = \{+, -, <, 0, 1\}$, and let $OG \subseteq T$ (the theory of nontrivial ordered groups). Then T is $\mathcal{O}$-minimal if and only if $T = DOAG$, the theory of divisible ordered abelian groups.*

Proof: First note that $DOAG$ is complete (this is done by quantifier elimination). The backward direction, as in the above cases, is also gotten through quantifier elimination. To prove the forward direction, we need a lemma.

Lemma 0.4. *Let G be an $\mathcal{O}$-minimal group, and let $H \leq G$ be a definable subgroup. Then $H = \{0\}$ or $H = G$.*

Proof of Lemma: Suppose that $H \neq \{0\}$. Then H is a finite union of intervals, and hence contains a largest interval J. By properties of an ordered group, we may assume J symmetric about 0.

Case 1. $J = [-h, h]$ for some $h \in G$. Then if there is $h' \in H$ such that $h < h' < 2h$, then $\{2h'' - h' : h \in [-h, h]\}$ is a nontrivial extension of J, a contradiction. And if $H \cap (h, 2h) = \emptyset$, then there is $h' \in G \setminus H$ such that $h - h' \in J \subseteq H$ and thus $h - (h - h') = h - h + h' = h' \in H$ by closure under group operations, a contradiction to $h' \in G \setminus H$.

Case 2. $J = (-h, h)$. Then there is some $0 < g < h$ such that $h - g \in J$. Thus $h - g + g = h \in H$, so $J = [-h, h]$, contradicting the assumption that $h \notin J$. ⊣

Using the Lemma, we can easily complete the proof. Suppose that G were an $\mathcal{O}$-minimal ordered group. Then for any $g \in G$, it is clear that $C(g) = \{h \in G : h+g = g+h\}$ is a nontrivial subgroup group and hence $C(g) = G$, and thus G is abelian. Similarly, for any $n \in \omega$, the set $\{ng : g \in G\}$ is a nontrivial subgroup and thus all of G, so G is divisible. ⊣

1 Classical Results

$\mathcal{O}$-minimality of a structure indicates that the algebraic and order relations behave nicely with respect to one another. The first result of this section crystallizes the advantage that this behavior affords us. The idea is that once we have established a structure is $\mathcal{O}$-minimal, then when dealing with types, we can ignore all of the relations except for the order.

Proposition 1.1. *Let $\gamma(x) \in S_1^{\mathcal{M}}(A)$ be given, with $\mathcal{M}$ an $\mathcal{O}$-minimal structure. Then $\gamma(x)$ is generated (uniquely) by $\{x < a : a \in cl(A)$ and $\gamma(x) \vdash x < a\} \cup \{x > a : a \in cl(A)$ and $\gamma(x) \vdash x > a\} \cup \{x = a : a \in cl(A)$ and $\gamma(x) \vdash x = a\}$*

Following the correspondence with strongly minimal theories, it turns out that $\mathcal{O}$-minimal theories enjoy very nice definability properties.

Definition 1.2. For any model $\mathcal{M} \models T$ and $A \subseteq \mathcal{M}$, we define $cl(A)$, the *definable closure of A in $\mathcal{M}$*, $=\{b \in \mathcal{M} : b$ is definable by a formula with parameters in $A\}$. Note that in an $\mathcal{O}$-minimal structure, the definable closure is the same as the algebraic closure, since each element of a finite set can be distinguished by its place in the order.

The following result is the basis for most of the arguments in the remainder of this paper. For the rest of the paper, let us fix an $\mathcal{O}$-minimal theory T and a model $\mathcal{M} \models T$.

Theorem 1.3. *(Monotonicity) Let $\mathcal{M} \models T$, and $f : \mathcal{M} \to \mathcal{M}$ be an $\mathcal{M}$-definable partial function. Then the domain of f can be written as a finite union of disjoint intervals such that on each interval, f is continuous and either strictly monotone or constant on each interval.*

The proof can be found in [3]. As an immediate consequence of Monotonicity, we can see that $\mathcal{O}$-minimal theories allow for a notion of dimension via the following:

Corollary 1.4. *(Symmetry) If $a \in cl(A \cup \{b\})$, then either $a \in cl(A)$ or $b \in cl(A \cup \{a\})$.*

Proof: Suppose that $a \in cl(A \cup \{b\})$, and $a \notin cl(A)$. Then there is an A-definable $f : \mathcal{M} \to \mathcal{M}$ such that $f(b) = a$, and by monotonicity an interval I, definable over $cl(A)$, containing b such that f is either monotone or constant. If f is constant on I, then a is definable over A by $f(I)$. Thus f is monotone on I, and suppose without loss of generality f is increasing. Then there is some interval J containing a, also definable over $cl(A)$, such that f is a bijection between I and J. Thus there is a definable inverse f^{-1}, and b is definable over $A \cup \{a\}$ as $f^{-1}(a)$. ⊣

While $\mathcal{O}$-minimal theories are not stable, the above corollary, together with the below result from [3], form a strong parallel with stable theories:

Theorem 1.5. *(Existence and uniqueness of prime models over sets) For any $\mathcal{M} \models T$ and any $A \subseteq \mathcal{M}$, there is a model, denoted $Pr(A)$, such that for any $A \subseteq \mathcal{N} \models T$, there is $f : Pr(A) \cong\prec \mathcal{N}$, and $Pr(A)$ is unique up to A-isomorphism.*

2 Types and The Rudin-Keisler Order

We will see that there are two main kinds of complete types in an $\mathcal{O}$-minimal structure. There are many different formulations of the below definitions, all of them equivalent.

Definition 2.1. Let $\mathcal{M} \models T$ be a linearly ordered structure. A type $\sigma(x) \in S_1(\mathcal{M})$ is a *cut* over $\mathcal{M}$ iff there are a and b such that $a < x < b \in \sigma$, and for any $a \in \mathcal{M}$, if $a < x \in \sigma$, then there is $b \in \mathcal{M}$ such that $a < b$ and $b < x \in \sigma$. Similarly, for any $a \in \mathcal{M}$, if $a > x \in \sigma$, then there is $b \in \mathcal{M}$ such that $a > b$ and $b > x \in \sigma$.

Any complete nonprincipal type which is not a cut will be called a *noncut*. Note that by Proposition 1.1, every cut uniquely determines a complete type over $\mathcal{M}$, thus for the rest of the paper, we will confuse cuts and noncuts with their associated types without loss of generality.

Example 2.2. Denote by $\mathbb{Q}^*$ the real closure of $\mathbb{Q}$. Then $tp_{\mathbb{Q}^*}(\pi)$ is a cut, while the type of an element larger than all the rational numbers is a noncut.

If we further stipulate that all types be nonprincipal (and hence not realized in the original model), then any noncut is of one of the following forms (which we will refer to later by their roman numeral):

i. For some $q \in \mathcal{M}$, $\{m < x < q : m \in \mathcal{M}$ and $m < q\}$ (a *left infinitesimal*)
ii. For some $q \in \mathcal{M}$, $\{m > x > q : m \in \mathcal{M}$ and $m > q\}$ (a *right infinitesimal*)
iii. $\{x < m : m \in M\}$ (a *left infinite* element)
iv. $\{m < x : m \in M\}$ (a *right infinite* element)

The next result says that realizing a noncut over a model does not force us to realize a cut.

Proposition 2.3. *Let $\gamma(x)$ be a cut, and $\sigma(x)$ be a noncut. Fix b, a realization of σ. Then γ is omitted in $Pr(\mathcal{M} \cup \{b\})$.*

Proof: Let $\mathcal{M}' = Pr(\mathcal{M} \cup \{b\})$, and suppose c realizes γ. Since $c \in \mathcal{M}'$, then c is atomic over $cl(\mathcal{M} \cup \{b\})$. We may assume the atom for c is of the form (β_0, β_1), where $\beta_0, \beta_1 \in cl(\mathcal{M} \cup \{b\})$, since the endpoints of a closed interval are definable. But in this case, by manipulation of the $<$ relation with β_i and the formulas in γ, both β_0 and β_1 are contained in the cut γ. Thus without loss of generality, we may assume $c \in cl(\mathcal{M} \cup \{b\})$. Now let f be an $\mathcal{M}$-definable function such that $f(b) = c$. There are four cases, all of which are similar, and thus we will prove just one.

Suppose the noncut σ is of form *i*, an infinitesimal left of $q \in \mathcal{M}$. Then by monotonicity there is some $a \in \mathcal{M}$ such that f is monotone and continuous on $(a, q) \subseteq \mathcal{M}$. Since $\mathcal{M} \prec \mathcal{M}'$, we also know f is monotone and continuous on (a, q), considered as a subset of $\mathcal{M}'$. Suppose that f is strictly increasing; the other case is similar. Now let $X = \{x \in \mathcal{M} :$ such that there is $y \in (a, q)$ with $f(y) = x\}$.

Subcase 1: X is unbounded. Then there is some $d \in \mathcal{M}$ such that $q > d > a$ and $f(d) > c$. But since $\mathcal{M}, \mathcal{M}' \models \forall x > d(x < q \rightarrow f(x) > f(d))$, then $\mathcal{M} \models f(b) > f(d) > c$, a contradiction since $f(b) = c$.

Subcase 2: X is bounded. Then $X = (\beta, \delta)$, for some $\beta, \delta \in \mathcal{M}$. Then clearly the cut γ is entirely contained in (β, δ). Now since γ is a cut, there is $e \in \mathcal{M}$ such that $e < \delta$ and $x < e \in \gamma$.

Thus there is $d \in \mathcal{M} \cap (a, q)$ such that $f(d) = e$. But then $d < b$ since $d \in \mathcal{M}$. Thus $f(d) < f(b)$, so $e < c$, a contradiction to c realizing γ. $\dashv$.

Definition 2.4. (the *Rudin-Keisler Order*). Let $\gamma, \tau \in S_1(\mathcal{M})$. We say $\gamma \triangle_{\mathcal{M}} \tau$ (alternatively $\gamma \triangleleft_{\mathcal{M}} \tau$) iff for any c realizing τ, γ is realized in $Pr(\mathcal{M} \cup \{c\})$.

Example 2.5. $tp_{\mathbb{Q}^*}(\pi^{17}) \triangle_{\mathcal{M}} tp_{\mathbb{Q}^*}(\sqrt{\pi})$. This is because $Pr(\mathbb{Q}^* \cup \{\sqrt{\pi}\})$ is the real closure of $\mathbb{Q}^* \cup \{\sqrt{\pi}\}$. It is also the case that $tp_{\mathbb{Q}^*}(\sqrt{\pi}) \triangle_{\mathcal{M}} tp_{\mathbb{Q}^*}(\pi^{17})$, since the real closure of $\mathbb{Q}^* \cup \{\pi^{17}\}$ must have roots for $x^{34} - \pi^{17} = 0$. This is no accident, and in fact is a further illustration of the good behavior between order and algebra in $\mathcal{O}$-minimal theories.

Proposition 2.6. *If $\mathcal{M}$ is $\mathcal{O}$-minimal, then $\triangle_{\mathcal{M}}$ is an equivalence relation on $S_1(\mathcal{M})$.*

Proof: The transitivity and reflexivity are trivial. Symmetry depends on the symmetry from Corollary 1.4. Suppose there is c realizing γ and b realizing τ, and that $\gamma \triangle_{\mathcal{M}} \tau$. Then there is $c' \in cl(\mathcal{M} \cup \{b\})$ realizing γ. Since $c' \notin \mathcal{M}$, we know that by Symmetry, $b \in cl(\mathcal{M} \cup \{c'\})$. Thus there is an $\mathcal{M}$-definable f such that $f(c') = b$. But since by assumption $tp_{\mathcal{M}}(c) = tp_{\mathcal{M}}(c')$ and f was definable over $\mathcal{M}$, then $tp_{\mathcal{M}}(f(c)) = tp_{\mathcal{M}}(f(c')) = \tau$. Hence any model realizing γ must also realize τ, so $\tau \triangle_{\mathcal{M}} \gamma$. $\dashv$

Corollary 2.7. *Let $\gamma(x)$ be a noncut, and $\sigma(x)$ be a cut. Fix b, a realization of σ. Then γ is omitted in $Pr(\mathcal{M} \cup \{b\})$.*

The proof is immediate from 2.3 and 2.6. $\dashv$

Corollary 2.8. *The maximum size of any $\triangle_{\mathcal{M}}$-equivalence class is $|M|$.*

Proof: To see this, suppose that c realizes a type γ over $\mathcal{M}$. Then $Pr(\mathcal{M} \cup \{c\})$ must realize every element of the class to which γ belongs. Hence the class can be no larger than $|Pr(\mathcal{M} \cup \{c\})| = |\mathcal{M}|$.

3 Omitting Types in Large Models

The results in the final section stand by themselves, but for additional motivation, we start with a definition.

Definition 3.1. The *Hanf number* (sometimes called the *Morley number*) of a logic L is the smallest infinite cardinal κ such that if any sentence φ of L has a model $\mathcal{M} \models \varphi$ such that $|\mathcal{M}| \geq \kappa$, then φ has a model larger than η for every cardinal η.

It is easily seen (by Löwenheim-Skolem) that the usual $L_{\omega\omega}$ has Hanf number ω. Since the introduction of the above calculation, model theorists have taken up the following variant on the standard Hanf number.

Definition 3.2. The *Hanf number for omitting types over* T, written $H(T)$, is the smallest infinite cardinal κ such that for all L-types γ, if T has a model in every infinite cardinal $\eta < \kappa$, then T has a model omitting γ in all infinite cardinals. The *Hanf number for omitting complete types over* T, written $H^C(T)$ is defined similarly, with 'complete types' substituted in for 'it types'.

At the end of this section, though we will have in fact seen more, we will be able to calculate an upper bound on $H^C(T)$ for *any* $\mathcal{O}$-minimal T. To start, we develop methods for extending types from smaller models to elementary extensions. Note that all of the technical lemmas in this section are proved in [2].

Definition 3.3. Suppose τ is a noncut over $\mathcal{M}$ and $\mathcal{N} \succ \mathcal{M}$. We define $\tau_{\mathcal{N}}$ in the following way:

if $\tau(x) = \{m < x < q : m \in \mathcal{M}$ and $m < q\}$ then $\tau_{\mathcal{N}}(x) = \{n < x < q : n \in \mathcal{N}$ and $n < q\}$

if $\tau(x) = \{x < m : m \in M\}$ then define $\tau_{\mathcal{N}}(x) = \{x < n : n \in N\}$

and similarly for the other two cases.

It is clear that such type extensions always exist over a noncut, and that $\tau_{\mathcal{N}}$ will still be a noncut. But further than that, one can also use a Monotonicity argument to prove the following:

Lemma 3.4. *Suppose γ and τ are $\triangle_{\mathcal{M}}$-inequivalent, and τ is a noncut. Then for any c realizing τ, if $\mathcal{M}' = Pr(\mathcal{M} \cup \{c\})$, then γ and $\tau_{\mathcal{M}'}$ are $\triangle_{\mathcal{M}'}$-inequivalent.*

Given the above argument, we would like to find a way to canonically extend a *cut*, but unfortunately we cannot. An illustration of this is the typical example of π over $\mathbb{Q}^*$, in which it is clear that any nonprincipal extension of $tp_{\mathbb{Q}^*}(\pi)$ over $Pr(\mathbb{Q} * \cup\{\pi\})$ is a noncut. It turns out that the trouble in this case is caused by the fact that $tp_{\mathbb{Q}^*}(\pi)$ is realized by precisely one element in $Pr(\mathbb{Q} * \cup\{\pi\})$.

Definition 3.5. A cut γ is *uniquely realizable* over $\mathcal{M}$ iff for any c realizing γ, c is the unique realization of γ in $Pr(\mathcal{M} \cup \{c\})$.

It is the case that extending *nonuniquely realizable cuts* always gives us a cut in the larger model. To see this, suppose that γ is a nonuniquely realizable cut over $\mathcal{M}$ and suppose c realizes γ. Then there is some $c' \neq c$ in $cl(\mathcal{M} \cup \{c\})$ which also realizes γ. Thus there is an $\mathcal{M}$-definable function f such that $f(c) = c'$, and an interval I in $\mathcal{M}$ containing c on which f is monotone. Since the interval must contain the entire cut, f must be continuous and monotonic on the entire cut. Thus there is no least or greatest element $Pr(\mathcal{M} \cup \{c\})$ realizing γ. Given the above, we can canonically extend γ:

Definition 3.6. Let $\gamma(x)$ be a nonuniquely realizable cut over $\mathcal{M}$. Fix c a realization of γ, and let $\mathcal{M}' = Pr(\mathcal{M} \cup \{c\})$. Define $\gamma_{\mathcal{M}'}(x) = \{x > m : m \in \mathcal{M}'$ and '$m < \gamma(x)'\} \cup \{x < m : m \in \mathcal{M}'$ and realizing $\gamma\}$, where '$m < \gamma(x)'$ means m is smaller than every element realizing γ in $\mathcal{M}$.

From this definition, we can get the following results, parallel to Lemma 3.4.

Lemma 3.7. *Suppose $\gamma, \tau \in S_1(\mathcal{M})$, with γ a nonuniquely realizable cut, γ and τ are $\triangle_{\mathcal{M}}$-inequivalent, and c realizes γ. Then if $\mathcal{M}' = Pr(\mathcal{M} \cup \{c\})$, it is also true that $\gamma_{\mathcal{M}'}$ and τ are $\triangle_{\mathcal{M}'}$-inequivalent.*

Lemma 3.8. *Suppose $\gamma, \tau \in S_1(\mathcal{M})$, with γ a nonuniquely realizable cut and τ a uniquely realizable cut. Then γ and τ are $\triangle_{\mathcal{M}}$-inequivalent.*

Given an $\mathcal{O}$-minimal model and a type γ, we would now like to develop a canonical stretching process that will create a large model which omits γ. The standard case that we can adjust for the other cases is as follows.

Let $\tau(x)$ be a noncut and $\gamma(x)$ a type, both over $\mathcal{M}$, such that τ and γ are $\triangle_{\mathcal{M}}$-inequivalent. Let δ be any ordinal. We build an elementary chain $\{\mathcal{M}_\alpha : \alpha \leq \delta\}$ as follows:

i. $\mathcal{M}_0 = \mathcal{M}$

ii. $\mathcal{M}_{\alpha+1} = Pr(\mathcal{M}_\alpha \cup \{c_\alpha\})$, where c_α realizes $\tau_{\mathcal{M}_\alpha}$

iii. $\mathcal{M}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{M}_\alpha$ if λ is a limit

The proof of the following is a straightforward transfinite induction argument.

Lemma 3.9. *M_δ omits γ.*

By an analogous construction, if $\tau(x)$ is a cut and $\gamma(x)$ a noncut or a nonuniquely realizable cut, we can stretch over τ to form an elementary chain whose limit omits γ.

We can encapsulate these results as follows:

I. If given a noncut $\tau(x)$, we may stretch a nonuniquely realizable cut or an inequivalent noncut to get an arbitrarily large model omitting τ.

II. If given a cut $\gamma(x)$, we may stretch over a noncut or an inequivalent nonuniquely realizable cut to get an arbitrarily large model omitting γ.

In case I., the converse is also true:

Lemma 3.10. *Let τ be a noncut over $\mathcal{M}$ and suppose that for each $\mathcal{M}' \succ \mathcal{M}$ which omits τ, it must be that for any noncut σ over $\mathcal{M}'$, $\tau \triangle_{\mathcal{M}'} \sigma$. Let $\mathcal{N} \succ \mathcal{M}$ be such that $|\mathcal{N}| > |S_1(\mathcal{M})|$. Then $\mathcal{N}$ realizes τ.*

To further examine statement II., we have the following definition.

Definition 3.11. A theory T *admits noncuts* iff for any $\mathcal{M} \models T$, we there is a noncut over $\mathcal{M}$.

Example 3.12. Most $\mathcal{O}$-minimal theories admit noncuts, including the usual example of $\mathbb{Q}^*$. A nonexample is $\omega + \omega^*$ in the language containing just $\{<\}$, where ω^* is the reverse ordering on ω. This structure is $\mathcal{O}$-minimal, but infinite and infinitesimal noncuts are inconsistent with $Th((\omega + \omega^*, <))$. However in this case, it is clear that *every* cut is nonuniquely realizable, since any cut consists of elements infinitely far from the outer endpoints of $\omega + \omega^*$. This turns out to be true for all theories which do not admit noncuts.

Lemma 3.13. *If T does not admit noncuts, then every cut γ over T is nonuniquely realizable.*

The proof of Lemma 3.13 follows from the fact that *every* $\mathcal{O}$-minimal theory which does not admit noncuts is order-isomorphic to $\omega + \omega^*$. We now have enough information to compute $H^C(T)$.

Theorem 3.14. *Let T be $\mathcal{O}$-minimal in a countable language. Then $H^C(T) \leq (2^{|T|})^+$.*

Proof: Let T be an $\mathcal{O}$-minimal theory, $\gamma(x) \in S_1(\emptyset)$, and suppose there is a model $\mathcal{N} \models T$ of size $(2^{|T|})^+$ which omits γ. Let $\kappa > (2^{|T|})^+$ be an infinite cardinal. Let $\mathcal{M}_0$ be the prime model of T. We will use the stretching techniques outlined above to construct a model $\mathcal{N}' \succ \mathcal{N}$ such that $|\mathcal{N}'| \geq \kappa$ and $\mathcal{N}'$ omits γ.
Case 1. γ is a cut. If T admits noncuts, stretch a noncut over $\mathcal{N}$. If not, and if there is a nonuniquely realizable cut inequivalent to γ over $\mathcal{M}_0$, then stretch this cut over $\mathcal{M}_0$. If T does not admit noncuts and there is no nonuniquely realizable cut inequivalent to γ, then every cut is equivalent to γ. Thus any model of size at least $|T|^+$ must realize a cut and hence must realize γ, contradicting the assumption that there is a model of size $(2^{|T|})^+$ omitting γ.
Case 2. γ is a noncut, in which case there is either an inequivalent noncut over $\mathcal{N}$ or a nonuniquely realizable cut over $\mathcal{N}$ which must be inequivalent. We can then stretch over one of these to give the large model as required. $\dashv$

References

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