Law of Large Numbers for Increasing Subsequences of Random Permutations

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Let the random variable $Z_{n,k}$ denote the number of increasing subsequences of length $k$ in a random permutation from $S_n$, the symmetric group of permutations of $\{1, \ldots, n\}$. We show that the weak law of large numbers holds for $Z_{n,k_n}$ if $k_n = o(n^{\frac{2}{5}})$; that is,

$$\lim_{n \to \infty} \frac{Z_{n,k_n}}{nEZ_{n,k_n}} = 1 \text{ in probability.}$$

The proof uses the second moment method and demonstrates that this method cannot work if the condition $k_n = o(n^{\frac{2}{5}})$ does not hold. It follows from results concerning the longest increasing subsequence of a random permutation that the law of large numbers cannot hold for $Z_{n,k_n}$ if $k_n \geq cn^{\frac{1}{2}}$, with $c > 2$. Presumably there is a critical exponent $l_0$ such that the law of large numbers holds if $k_n = O(n^l)$, with $l < l_0$, and does not hold if $\limsup_{n \to \infty} \frac{k_n}{n^l} > 0$, for some $l > l_0$. Several phase transitions concerning increasing subsequences occur at $l = \frac{1}{2}$, and these would suggest that $l_0 = \frac{1}{2}$. However, we show that the law of large numbers fails for $Z_{n,k_n}$ if $\limsup_{n \to \infty} \frac{k_n}{n^\frac{2}{9}} = \infty$. Thus the critical exponent, if it exists, must satisfy $l_0 \in [\frac{2}{5}, \frac{4}{9}]$. To show that the law of large numbers fails, we use a celebrated result by Baik, Deift and Johansson (1999) concerning the length of the longest increasing subsequence in a random permutation, as well as the following result, which is of independent interest: Place $n$ cards numbered from 1 to $n$ on a line in increasing order from left to right. Randomly designate $k_n$ of the cards. Pick up the $n - k_n$ other cards and randomly reinsert them between the $k_n$ designated cards that remained on the line. Denote the resulting measure on $S_n$ by $\mu_{n,k_n}$, and denote the uniform measure on $S_n$ by $U_n$. Then the law of large numbers holds for $Z_{n,k_n}$ if and only if the distance between $U_n$ and $\mu_{n,k_n}$ converges to 0 in the total variation norm as $n \to \infty$. 