Review Problems for Test 2

(1). Let

\[ A = \begin{pmatrix} 3 & 0 & 1 & 4 \\ -5 & 4 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 2 & 1 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

(a) Calculate \( \det(A) \).  
(b) What does the value in (a) tell you about the four-dimensional volume spanned by \( \{ \sum_{i=1}^4 t_i(\mathbf{A}\mathbf{b}_i) : 0 \leq t_i \leq 1, \ i = 1, \ldots, 4 \} \), where the vectors \( \mathbf{b}_i \) are the columns of \( B \)?
(c) Calculate the \((4,1)\) element of \( A^{-1} \) by determinants.

(2). Suppose that \( A \) is a \( 5 \times 6 \) matrix with columns \( \mathbf{a}_1, \ldots, \mathbf{a}_6 \in \mathbb{R}^5 \), and denote the columns of \( A^T \) by \( \mathbf{b}_1, \ldots, \mathbf{b}_5 \), where

\[ \text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & -1 \\ 0 & 0 & 1 & 6 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{rref}(A^T) = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]

Let \( S \) denote the linear transformation from \( \mathbb{R}^6 \) to \( \mathbb{R}^5 \) with standard matrix representation \( A \). Give bases for the \( \ker(S) = \text{null-space of } S \), for \( \text{range}(S) \), for \( \text{Nul}(A^T) \), and for \( \text{Col}(A^T) \).
Suppose that $S : \mathbb{R}^{10} \to \mathbb{R}^4$ and $T : \mathbb{R}^4 \to \mathbb{R}^6$ are linear transformations.

(a). What is the size of the standard matrix representation of the transformation $T \circ S$ which sends $x \in \mathbb{R}^{10}$ to $T(S(x))$?

(b). If the matrix representing $S$ has exactly 4 linearly independent rows, then what is $\text{range}(S)$?

(c). If the condition of (b) holds, and also the only vector which $T$ sends to $0 \in \mathbb{R}^6$ is the $0$-vector in $\mathbb{R}^4$, then what is $\dim(\text{range}(T))$?

Consider the two bases $\mathcal{B}_1 = \{1, t, t^2, t^3\}$ and $\mathcal{B}_2 = \{1, 2t, t^2 - 1, 3t^3 - t^2 + t\}$ of $\mathcal{P}_3$. (a) Give the matrix which transforms each vector $[v]_{\mathcal{B}_1}$ of coordinates for a polynomial $v \in \mathcal{P}_3$ with respect to $\mathcal{B}_2$ into the vector of coordinates $[v]_{\mathcal{B}_2}$ of coordinates with respect to $\mathcal{B}_2$. (b) Give the matrix which transforms $[v]_{\mathcal{B}_2}$ to $[v]_{\mathcal{B}_1}$ and use it to write down the general expression for the standard basis representation of the polynomial $a(1) + b(2t) + c(t^2 - 1) + d(3t^3 - t^2 + t)$. 

(3).

(4).
(5). A Markov chain with three states has one-step transition probabilities defined as follows:

from state 1, prob’s of going to 1, 2, 3 are: 0.6, 0.1, 0.3
from state 2, prob’s of going to 1, 2, 3 are: 0.1, 0.4, 0.5
from state 3, prob’s of going to 1, 2, 3 are: 0.3, 0.5, 0.2

(a) Starting in state 3, what is the probability of being in state 1 exactly 2 steps later?

(b) What is the steady-state probability of being in state 2?