Actuarial Mathematics and Life-Table Statistics

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Contents

0.1 Preface

This book is a course of lectures on the mathematics of actuarial science. The idea behind the lectures is as far as possible to deduce interesting material on contingent present values and life tables directly from calculus and common-sense notions, illustrated through word problems. Both the Interest Theory and Probability related to life tables are treated as wonderful concrete applications of the calculus. The lectures require no background beyond a third semester of calculus, but the prerequisite calculus courses must have been solidly understood. It is a truism of pre-actuarial advising that students who have not done really well in and digested the calculus ought not to consider actuarial studies.

It is not assumed that the student has seen a formal introduction to probability. Notions of relative frequency and average are introduced first with reference to the ensemble of a cohort life-table, the underlying formal random experiment being random selection from the cohort life-table population (or, in the context of probabilities and expectations for 'lives aged x', from the subset of l_x members of the population who survive to age x). The calculation of expectations of functions of a time-to-death random variables is rooted on the one hand in the concrete notion of life-table average, which is then approximated by suitable idealized failure densities and integrals. Later, in discussing Binomial random variables and the Law of Large Numbers, the combinatorial and probabilistic interpretation of binomial coefficients are derived from the Binomial Theorem, which the student the is assumed to know as a topic in calculus (Taylor series identification of coefficients of a polynomial.) The general notions of expectation and probability are introduced, but for example the Law of Large Numbers for binomial variables is treated (rigorously) as a topic involving calculus inequalities and summation of finite series. This approach allows introduction of the numerically and conceptually useful large-deviation inequalities for binomial random variables to explain just how unlikely it is for binomial (e.g., life-table) counts to deviate much percentage-wise from expectations when the underlying population of trials is large.

The reader is also not assumed to have worked previously with the Theory of Interest. These lectures present Theory of Interest as a mathematical problem-topic, which is rather unlike what is done in typical finance courses.

Getting the typical Interest problems — such as the exercises on mortgage refinancing and present values of various payoff schemes — into correct format for numerical answers is often not easy even for good mathematics students.

The main goal of these lectures is to reach — by a conceptual route — mathematical topics in Life Contingencies, Premium Calculation and Demography not usually seen until rather late in the trajectory of quantitative Actuarial Examinations. Such an approach can allow undergraduates with solid preparation in calculus (not necessarily mathematics or statistics majors) to explore their possible interests in business and actuarial science. It also allows the majority of such students — who will choose some other avenue, from economics to operations research to statistics, for the exercise of their quantitative talents — to know something concrete and mathematically coherent about the topics and ideas actually useful in Insurance.

A secondary goal of the lectures has been to introduce varied topics of applied mathematics as part of a reasoned development of ideas related to survival data. As a result, material is included on statistics of biomedical studies and on reliability which would not ordinarily find its way into an actuarial course. A further result is that mathematical topics, from differential equations to maximum likelihood estimators based on complex life-table data, which seldom fit coherently into undergraduate programs of study, are 'vertically integrated' into a single course.

While the material in these lectures is presented systematically, it is not separated by chapters into unified topics such as Interest Theory, Probability Theory, Premium Calculation, etc. Instead the introductory material from probability and interest theory are interleaved, and later, various mathematical ideas are introduced as needed to advance the discussion. No book at this level can claim to be fully self-contained, but every attempt has been made to develop the mathematics to fit the actuarial applications as they arise logically.

The coverage of the main body of each chapter is primarily 'theoretical'. At the end of each chapter is an Exercise Set and a short section of Worked Examples to illustrate the kinds of word problems which can be solved by the techniques of the chapter. The Worked Examples sections show how the ideas and formulas work smoothly together, and they highlight the most important and frequently used formulas.

Chapter 1

Basics of Probability and the Theory of Interest

The first lectures supply some background on elementary Probability Theory and basic Theory of Interest. The reader who has not previously studied these subjects may get a brief overview here, but will likely want to supplement this Chapter with reading in any of a number of calculus-based introductions to probability and statistics, such as Larson (1982), Larsen and Marx (1985), or Hogg and Tanis (1997) and the basics of the Theory of Interest as covered in the text of Kellison (1970) or Chapter 1 of Gerber (1997).

1.1 Probability, Lifetimes, and Expectation

In the *cohort life-table model*, imagine a number l_0 of individuals born simultaneously and followed until death, resulting in data d_x, l_x for each integer age $x = 0, 1, 2, \ldots,$ where

$$l_x = \text{number of lives aged } x \quad \text{(i.e. alive at birthday } x \text{)}$$

and

$$d_x \;=\; l_x - l_{x+1} \quad = \text{number dying between ages } x,\, x+1$$

Now, allow the age-variable to be denoted by t and to take all real values, not just whole numbers x, and treat $S_0(t) = l_t/l_0$ as the fraction of individuals in a life table surviving to exact age t. This nonincreasing function S_0 would be called the empirical "survivor" or "survival" function. Although it takes on only rational values with denominator l_0, it can be approximated by a survivor function $S(t)$ which is continuously differentiable (or piecewise continuously differentiable with just a few break-points) and takes values exactly $= l_x/l_0$ at integer ages x. Then for all positive real t, $S(u) - S(u+t)$ is the fraction of the initial cohort which fails between time u and $u+t$, and for integers x, t,

$$\frac{S(x) - S(x+t)}{S(x)} = \frac{l_x - l_{x+t}}{l_x}$$

denotes the fraction of those alive at exact age x who fail before $x+t$.

Question: what do probabilities have to do with the life table and survival function ?

To answer this, we first introduce probability as simply a relative frequency, using numbers from a cohort life-table like that of the accompanying Illustrative Life Table. In response to a probability question, we supply the fraction of the relevant life-table population, to obtain identities like

$$Pr(\text{life aged 29 dies between exact ages 35 and 41 or between 52 and 60 })$$

$$= S(35) - S(41) + S(52) - S(60) = \Big\{(l_{35} - l_{41}) + (l_{52} - l_{60})\Big\}\Big/ l_{29}$$

where our convention is that a *life aged 29* is one of the cohort surviving to the 29th birthday.

The idea here is that all of the lifetimes covered by the life table are understood to be governed by an identical "mechanism" of failure, and that any probability question about a single lifetime is really a question concerning the fraction of those lives about which the question is asked (e.g., those alive at age x) whose lifetimes will satisfy the stated property (e.g., die either between 35 and 41 or between 52 and 60). This "frequentist" notion of probability of an event as the relative frequency with which the event occurs in a large population of (independent) identical units is associated with the phrase "law of large numbers", which will be discussed later. For now, remark only that the life table population should be large for the ideas presented so far to make good sense. See Table 1.1 for an illustration of a cohort life-table with realistic numbers.

Table 1.1: Illustrative Life-Table, simulated to resemble realistic US (Male) life-table. For details of simulation, see Section 3.4 below.

Age x	l_x	d_x	x	l_x	d_x
0	100000	2629	40	92315	295
1	97371	141	41	92020	332
2	97230	107	42	91688	408
3	97123	63	43	91280	414
4	97060	63	44	90866	464
5	96997	69	45	90402	532
6	96928	69	46	89870	587
7	96859	52	47	89283	680
8	96807	54	48	88603	702
9	96753	51	49	87901	782
10	96702	33	50	87119	841
11	96669	40	51	86278	885
12	96629	47	52	85393	974
13	96582	61	53	84419	1082
14	96521	86	54	83337	1088
15	96435	105	55	82249	1213
16	96330	83	56	81036	1344
17	96247	125	57	79692	1423
18	96122	133	58	78269	1476
19	95989	149	59	76793	1572
20	95840	154	60	75221	1696
21	95686	138	61	73525	1784
22	95548	163	62	71741	1933
23	95385	168	63	69808	2022
24	95217	166	64	67786	2186
25	95051	151	65	65600	2261
26	94900	149	66	63339	2371
27	94751	166	67	60968	2426
28	94585	157	68	58542	2356
29	94428	133	69	56186	2702
30	94295	160	70	53484	2548
31	94135	149	71	50936	2677
32	93986	152	72	48259	2811
33	93834	160	73	45448	2763
34	93674	199	74	42685	2710
35	93475	187	75	39975	2848
36	93288	212	76	37127	2832
37	93076	228	77	34295	2835
38	92848	272	78	31460	2803
39	92576	261			

Note: see any basic probability textbook, such as Larson (1982), Larsen and Marx (1985), or Hogg and Tanis (1997) for formal definitions of the notions of sample space, event, probability, and conditional probability.

The main ideas which are necessary to understand the discussion so far are really matters of common sense when applied to relative frequency but require formal axioms when used more generally:

- Probabilities are numbers between 0 and 1 assigned to subsets of the entire range of possible outcomes. In the examples, the subsets which are assigned probabilities include sub-intervals of the interval of possible human lifetimes measured in years, and also disjoint unions of such subintervals. These sets in the real line are viewed as possible *events* summarizing ages at death of newborns in the cohort population. At this point, we are regarding each set A of ages as determining the subset of the cohort population whose ages at death fall in A.

- The probability $P(A \cup B)$ of the union $A \cup B$ of disjoint (i.e., nonoverlapping) sets A and B is necessarily equal to the sum of the separate probabilities $P(A)$ and $P(B)$.

- When probabilities are requested with reference to a smaller universe of possible outcomes, such as $B =$ *lives aged 29*, rather than all members of a cohort population, the resulting *conditional probabilities* of events A are written $P(A \,|\, B)$ and calculated as $P(A \cap B)/P(B)$, where $A \cap B$ denotes the *intersection* or *overlap* of the two events A, B. *The phrase "lives aged 29" defines an event which in terms of ages at death says simply "age at death is 29 or larger" or, in relation to the cohort population, specifies the subset of the population which survives to exact age 29 (i.e., to the 29th birthday).*

- Two events A, B are defined to be *independent* when $P(A \cap B) = P(A) \cdot P(B)$ or — equivalently, as long as $P(B) > 0$ — the conditional probability $P(A|B)$ expressing the probability of A if B were known to have occurred, is the same as the (unconditional) probability $P(A)$.

The life-table data, and the mechanism by which members of the population die, are summarized first through the survivor function $S(x)$ which at integer values of x agrees with the ratios l_x/l_0. Note that $S(x)$ has values between 0 and 1, and can be interpreted as the probability for a single individual to survive at least x time units. Since fewer people are alive at larger ages, $S(t)$ is a decreasing function of the continuous age-variable t, and in applications $S(t)$ should be piecewise continuously differentiable (largely for convenience, and because any analytical expression which would be chosen for $S(t)$ in practice *will* be piecewise smooth). In addition, by definition, $S(0) = 1$. Another way of summarizing the probabilities of survival given by this function is to define the **density** function

$$f(x) = -\frac{dS}{dx}(x) = -S'(x)$$

as the (absolute) rate of decrease of the function S. Then, by the fundamental theorem of calculus, for any ages $a < b$,

$$P(\textit{life aged 0 dies between ages a and b})$$

$$= S(a) - S(b) = \int_a^b (-S'(x))\,dx = \int_a^b f(x)\,dx \qquad (1.1)$$

which has the very helpful geometric interpretation that the probability of dying within the interval $[a, b]$ is equal to the area under the curve $y = f(x)$ over the x-interval $[a, b]$. Note also that the 'probability' rule which assigns the integral $\int_A f(x)\,dx$ to the set A (which may be an interval, a union of intervals, or a still more complicated set) obviously satisfies the first two of the bulleted axioms displayed above.

The **terminal age** ω of a life table is an integer value large enough that $S(\omega)$ is negligibly small, but no value $S(t)$ for $t < \omega$ is zero. For practical purposes, no individual lives to the ω birthday. While ω is finite in real life-tables and in some analytical survival models, most theoretical forms for $S(x)$ have no finite age ω at which $S(\omega) = 0$, and in those forms $\omega = \infty$ by convention.

Now we are ready to define some terms and motivate the notion of expectation. Think of the age T at which a specified newly born member of the population will die as a **random variable**, which for present purposes means a variable which takes various values t with probabilities governed (at integer ages) by the life table data l_x and the survivor function $S(t)$ or density function $f(t)$ in a formula like the one just given in equation (1.1). Suppose there is a contractual amount Y which must be paid (say, to the heirs of that individual) at the time T of death of the individual, and suppose that the contract provides a specific function $Y = g(T)$ according to which this payment depends on (the whole-number part of) the age T at which death occurs. What is the average value of such a payment over all individuals whose lifetimes are reflected in the life-table ? Since $d_x = l_x - l_{x+1}$ individuals (out of the original l_0) die at ages between x and $x+1$, thereby generating a payment $g(x)$, the total payment to all individuals in the life-table can be written as

$$\sum_x (l_x - l_{x+1})\, g(x)$$

Thus the average payment, at least under the assumption that $Y = g(T)$ depends only on the largest whole number $[T]$ less than or equal to T, is

$$\begin{aligned} \textstyle\sum_x (l_x - l_{x+1})\, g(x)\, /\, l_0 \;&=\; \textstyle\sum_x (S(x) - S(x+1)) g(x) \\ =\; \textstyle\sum_x \int_x^{x+1} f(t)\, g(t)\, dt &= \textstyle\int_0^\infty f(t)\, g(t)\, dt \end{aligned} \tag{1.2}$$

This quantity, the total contingent payment over the whole cohort divided by the number in the cohort, is called the **expectation** of the random payment $Y = g(T)$ in this special case, and can be interpreted as the weighted average of all of the different payments $g(x)$ actually received, where the weights are just the relative frequency in the life table with which those payments are received. More generally, if the restriction that $g(t)$ depends only on the integer part $[t]$ of t were dropped , then the expectation of $Y = g(T)$ would be given by the same formula

$$E(Y) = E(g(T)) = \int_0^\infty f(t)\, g(t)\, dt$$

The last displayed integral, like all expectation formulas, can be understood as a weighted average of values $g(T)$ obtained over a population, with weights equal to the probabilities of obtaining those values. Recall from the Riemann-integral construction in Calculus that the integral $\int f(t) g(t) dt$ can be regarded approximately as the sum over very small time-intervals $[t,\, t+\Delta]$ of the quantities $f(t)g(t)\Delta$, quantities which are interpreted as the base Δ of a rectangle multiplied by its height $f(t)g(t)$, and the rectangle closely covers the area under the graph of the function $f\,g$ over the interval $[t,\, t+\Delta]$. The term $f(t)g(t)\Delta$ can alternatively be interpreted as the product of the value $g(t)$ — essentially equal to any of the values $g(T)$ which can be realized when T falls within the interval $[t,\, t+\Delta]$ — multiplied by $f(t)\,\Delta$. The latter quantity is, by the Fundamental Theorem of the Calculus, approximately equal for small Δ to the area under the function f over the interval $[t,\, t+\Delta]$, and is by definition equal to the probability with which $T \in [t,\, t+\Delta]$. In summary, $E(Y) = \int_0^\infty g(t) f(t) dt$ is the average of values $g(T)$ obtained for lifetimes T within small intervals $[t,\, t+\Delta]$ weighted by the probabilities of approximately $f(t)\Delta$ with which those T and $g(T)$ values are obtained. The expectation is a weighted average because the weights $f(t)\Delta$ sum to the integral $\int_0^\infty f(t)dt = 1$.

The same idea and formula can be applied to the restricted population of lives aged x. The resulting quantity is then called the **conditional expected value of** $g(T)$ given that $T \geq x$. The formula will be different in two ways: first, the range of integration is from x to ∞, because of the resitriction to individuals in the life-table who have survived to exact age x; second, the density $f(t)$ must be replaced by $f(t)/S(x)$, the so-called **conditional density given** $T \geq x$, which is found as follows. From the definition of conditional probability, for $t \geq x$,

$$P(t \leq T \leq t+\Delta \,|\, T \geq x) \;=\; \frac{P(\,\{t \leq T \leq t+\Delta\} \cap \{T \geq x\})}{P(T \geq x)}$$

$$=\; \frac{P(t \leq T \leq t+\Delta)}{P(T \geq x)} \;=\; \frac{S(t) - S(t+\Delta)}{S(x)}$$

Thus the density which can be used to calculate conditional probabilities $P(a \leq T \leq b \,|\, T \geq x)$ for $x < a < b$ is

$$\lim_{\Delta \to 0} \frac{1}{\Delta} P(t \leq T \leq t+\Delta \,|\, T \geq x) \;=\; \lim_{\Delta \to 0} \frac{S(t) - S(t+\Delta)}{S(x)\,\Delta} \;=\; -\frac{S'(t)}{S(x)} \;=\; \frac{f(t)}{S(x)}$$

The result of all of this discussion of conditional expected values is the formula, with associated weighted-average interpretation:

$$E(g(T) \,|\, T \geq x) \;=\; \frac{1}{S(x)} \int_x^\infty g(t)\, f(t)\, dt \qquad (1.3)$$

1.2 Theory of Interest

Since payments based upon unpredictable occurrences or *contingencies* for insured lives can occur at different times, we study next the Theory of Interest, which is concerned with valuing streams of payments made over time. The general model in the case of constant interest, to which we restrict in the current sub-section, is as follows. Money is deposited in an account like a bank-account and grows according to a schedule determined by both the interest rate and the occasions when interest amounts are *compounded*, that is, deemed to be added to the account. The compounding rules are important because they determine when new interest interest begins to be earned on previously earned interest amounts.

Compounding at time-intervals $h = 1/m$, with *nominal* interest rate $i^{(m)}$, means that a unit amount accumulates to $(1 + i^{(m)}/m)$ after a time $h = 1/m$. The principal or account value $1 + i^{(m)}/m$ at time $1/m$ accumulates over the time-interval from $1/m$ until $2/m$, to $(1+i^{(m)}/m)\cdot(1+i^{(m)}/m) = (1+i^{(m)}/m)^2$. Similarly, by induction, a unit amount accumulates to $(1 + i^{(m)}/m)^n = (1 + i^{(m)}/m)^{Tm}$ after the time $T = nh$ which is a multiple of n whole units of h. In the limit of continuous compounding (i.e., $m \to \infty$), the unit amount compounds to $e^{\delta T}$ after time T, where the instantaneous annualized nominal interest rate $\delta = \lim_m i^{(m)}$ (also called the *force of interest*) will be shown to exist. In either case of compounding, the actual *Annual Percentage Rate* or **APR** or *effective annual interest rate* is defined as the amount (minus 1, and multiplied by 100 if it is to be expressed as a percentage) to which a unit compounds after a single year, i.e., respectively as

$$i_{\text{APR}} = \Big(1 + \frac{i^{(m)}}{m}\Big)^m - 1 \qquad \text{or} \qquad e^{\delta} - 1$$

The amount to which a unit invested at time 0 accumulates at the effective interest rate i_{APR} over a time-duration T (still assumed to be a multiple of $1/m$) is therefore

$$\Big(1 + i_{\text{APR}}\Big)^T = \Big(1 + \frac{i^{(m)}}{m}\Big)^{mT} = e^{\delta T}$$

This amount is called the *accumulation factor* operating over the interval of duration T at the fixed interest rate. Moreover, the first and third expressions of the displayed equation also make perfect sense when the duration T is any positive real number, not necessarily a multiple of $1/m$.

All the nominal interest rates $i^{(m)}$ for different periods of compounding are related by the formulas

$$(1+i^{(m)}/m)^m \;=\; 1+i \;=\; 1+i_{\text{APR}} \quad , \quad i^{(m)} \;=\; m\,\left\{(1+i)^{1/m} \,-\, 1\right\} \tag{1.4}$$

Similarly, interest can be said to be governed by the *discount rates* for various compounding periods, defined by

$$1 \,-\, d^{(m)}/m \;=\; (1+i^{(m)}/m)^{-1}$$

Solving the last equation for $d^{(m)}$ gives

$$d^{(m)} \;=\; i^{(m)}/(1+i^{(m)}/m) \tag{1.5}$$

The idea of discount rates is that if \$1 is loaned out at interest, then the amount $d^{(m)}/m$ is the correct amount to be repaid at the *beginning* rather than the end of each fraction $1/m$ of the year, with repayment of the principal of \$1 at the end of the year, in order to amount to the same effective interest rate. The reason is that, according to the definition, the amount $1-d^{(m)}/m$ accumulates at nominal interest $i^{(m)}$ (compounded m times yearly) to $(1-d^{(m)}/m)\cdot(1+i^{(m)}/m)=1$ after a time-period of $1/m$.

The quantities $i^{(m)}$, $d^{(m)}$ are naturally introduced as the interest payments which must be made respectively at the ends and the beginnings of successive time-periods $1/m$ in order that the principal owed at each time j/m on an amount \$ 1 borrowed at time 0 will always be \$ 1. To define these terms and justify this assertion, consider first the simplest case, $m=1$. If \$ 1 is to be borrowed at time 0, then the single payment at time 1 which fully compensates the lender, if that lender could alternatively have earned interest rate i, is \$ $(1+i)$, which we view as a payment of \$ 1 *principal* (the face amount of the loan) and \$ i interest. In exactly the same way, if \$ 1 is borrowed at time 0 for a time-period $1/m$, then the repayment at time $1/m$ takes the form of \$ 1 principal and \$ $i^{(m)}/m$ interest. Thus, if \$ 1 was borrowed at time 0, an interest payment of \$ $i^{(m)}/m$ at time $1/m$ leaves an amount \$ 1 still owed, which can be viewed as an amount borrowed on the time-interval $(1/m, 2/m]$. Then a payment of \$ $i^{(m)}/m$ at time $2/m$ still leaves an amount \$ 1 owed at

$2/m$, which is deemed borrowed until time $3/m$, and so forth, until the loan of \$ 1 on the final time-interval $((m-1)/m,\, 1]$ is paid off at time 1 with a final interest payment of \$ $i^{(m)}/m$ together with the principal repayment of \$ 1. The overall result which we have just proved intuitively is:

> \$ 1 at time 0 is equivalent to the stream of m payments of \$ $i^{(m)}/m$ at times $1/m,\, 2/m,\, \ldots,\, 1$ plus the payment of \$ 1 at time 1.

Similarly, if interest is to be paid at the *beginning* of the period of the loan instead of the end, the interest paid at time 0 for a loan of \$ 1 would be $d = i/(1+i)$, with the only other payment a repayment of principal at time 1. To see that this is correct, note that since interest d is paid at the same instant as receiving the loan of \$ 1 , the net amount actually received is $1 - d = (1+i)^{-1}$, which accumulates in value to $(1-d)(1+i) =$ \$ 1 at time 1. Similarly, if interest payments are to be made at the beginnings of each of the intervals $(j/m,\, (j+1)/m]$ for $j = 0,\, 1,\, \ldots,\, m-1$, with a final principal repayment of \$ 1 at time 1, then the interest payments should be $d^{(m)}/m$. This follows because the amount effectively borrowed (after the immediate interest payment) over each interval $(j/m,\, (j+1)/m]$ is \$ $(1 - d^{(m)}/m)$, which accumulates in value over the interval of length $1/m$ to an amount $(1 - d^{(m)}/m)(1 + i^{(m)}/m) = 1$. So throughout the year-long life of the loan, the principal owed at (or just before) each time $(j+1)/m$ is exactly \$ 1. The net result is

> \$ 1 at time 0 is equivalent to the stream of m payments of \$ $d^{(m)}/m$ at times $0,\, 1/m,\, 2/m,\, \ldots,\, (m-1)/m$ plus the payment of \$ 1 at time 1.

A useful algebraic exercise to confirm the displayed assertions is:

Exercise. Verify that the present values at time 0 of the payment streams with m interest payments in the displayed assertions are respectively

$$\sum_{j=1}^{m} \frac{i^{(m)}}{m}\,(1+i)^{-j/m} + (1+i)^{-1} \qquad \text{and} \qquad \sum_{j=0}^{m-1} \frac{d^{(m)}}{m}\,(1+i)^{-j/m} + (1+i)^{-1}$$

and that both are equal to 1. These identities are valid for all $i > 0$.

1.2.1 Variable Interest Rates

Now we formulate the generalization of these ideas to the case of non-constant instantaneously varying, but known or observed, nominal interest rates $\delta(t)$, for which the old-fashioned name would be *time-varying force of interest*. Here, if there is a compounding-interval $[kh, (k+1)h)$ of length $h = 1/m$, one would first use the instantaneous continuously-compounding interest-rate $\delta(kh)$ available at the beginning of the interval to calculate an equivalent annualized nominal interest-rate over the interval, i.e., to find a number $r_m(kh)$ such that

$$\left(1 + \frac{r_m(kh)}{m}\right) \;=\; \left(e^{\delta(kh)}\right)^{1/m} \;=\; \exp\left(\frac{\delta(kh)}{m}\right)$$

In the limit of large m, there is an essentially constant principal amount over each interval of length $1/m$, so that over the interval $[b, b+t)$, with instantaneous compounding, the unit principal amount accumulates to

$$\lim_{m\to\infty} e^{\delta(b)/m} e^{\delta(b+h)/m} \cdots e^{\delta(b+[mt]h)/m}$$

$$= \exp\left(\lim_{m} \frac{1}{m} \sum_{k=0}^{[mt]-1} \delta(b + k/m)\right) \;=\; \exp\left(\int_0^t \delta(b+s)\, ds\right)$$

The last step in this chain of equalities relates the concept of continuous compounding to that of the Riemann integral. To specify continuous-time varying interest rates in terms of effective or APR rates, instead of the instantaneous nominal rates $\delta(t)$, would require the simple conversion

$$r_{\text{APR}}(t) \;=\; e^{\delta(t)} - 1 \quad , \qquad \delta(t) \;=\; \ln\Big(1 \,+\, r_{\text{APR}}(t)\Big)$$

Next consider the case of deposits $s_0, s_1, \ldots, s_k, \ldots, s_n$ made at times $0, h, \ldots, kh, \ldots, nh$, where $h = 1/m$ is the given compounding-period, and where nominal annualized instantaneous interest-rates $\delta(kh)$ (with compounding-period h) apply to the accrual of interest on the interval $[kh, (k+1)h)$. If the accumulated bank balance just after time kh is denoted by B_k , then how can the accumulated bank balance be expressed in terms of s_j and $\delta(jh)$? Clearly

$$B_{k+1} \;=\; B_k \cdot \Big(1 \,+\, \frac{i^{(m)}(kh)}{m}\Big) \,+\, s_{k+1} \quad , \qquad B_0 \;=\; s_0$$

The preceding *difference equation* can be solved in terms of successive summation and product operations acting on the sequences s_j and $\delta(jh)$, as follows. First define a function A_k to denote the accumulated bank balance at time kh for a unit invested at time 0 and earning interest with instantaneous nominal interest rates $\delta(jh)$ (or equivalently, nominal rates $r_m(jh)$ for compounding at multiples of $h = 1/m$) applying respectively over the whole compounding-intervals $[jh, (j+1)h)$, $j = 0, \ldots, k-1$. Then by definition, A_k satisfies a *homogeneous equation* analogous to the previous one, which together with its solution is given by

$$A_{k+1} \;=\; A_k \cdot \Big(1 \,+\, \frac{r_m(kh)}{m}\Big), \qquad A_0 = 1, \qquad A_k \;=\; \prod_{j=0}^{k-1} \Big(1 + \frac{r_m(jh)}{m}\Big)$$

The next idea is the second basic one in the theory of interest, namely the idea of *equivalent investments* leading to the definition of *present value* of an income stream/investment. Suppose that a stream of deposits s_j accruing interest with annualized nominal rates $r_m(jh)$ with respect to compounding-periods $[jh, (j+1)h)$ for $j = 0, \ldots, n$ is such that a single deposit D at time 0 would accumulate by compound interest to give exactly the same final balance F_n at time $T = nh$. Then the present cash amount D in hand is said to be **equivalent** to the value of a contract to receive s_j at time jh, $j = 0, 1, \ldots, n$. In other words, the **present value** of the contract is precisely D. We have just calculated that an amount 1 at time 0 compounds to an accumulated amount A_n at time $T = nh$. Therefore, an amount a at time 0 accumulates to $a \cdot A_n$ at time T, and in particular $1/A_n$ at time 0 accumulates to 1 at time T. Thus the present value of 1 at time $T = nh$ is $1/A_n$. Now define G_k to be the present value of the stream of payments s_j at time jh for $j = 0, 1, \ldots, k$. Since B_k was the accumulated value just after time kh of the same stream of payments, and since the present value at 0 of an amount B_k at time kh is just B_k/A_k, we conclude

$$G_{k+1} \;=\; \frac{B_{k+1}}{A_{k+1}} \;=\; \frac{B_k\,(1 + r_m(kh)/m)}{A_k\,(1 + r_m(kh)/m)} + \frac{s_{k+1}}{A_{k+1}} \;, \quad k \geq 1 \;, \qquad G_0 = s_0$$

Thus $G_{k+1} - G_k = s_{k+1}/A_{k+1}$, and

$$G_{k+1} = s_0 + \sum_{i=0}^{k} \frac{s_{i+1}}{A_{i+1}} = \sum_{j=0}^{k+1} \frac{s_j}{A_j}$$

In summary, we have simultaneously found the solution for the accumulated balance B_k just after time kh and for the present value G_k at time 0 :

$$G_k = \sum_{i=0}^{k} \frac{s_i}{A_i} , \qquad B_k = A_k \cdot G_k , \qquad k = 0, \ldots, n$$

The intuitive interpretation of the formulas just derived relies on the following simple observations and reasoning:

(a) The present value at fixed interest rate i of a payment of \$1 exactly t years in the future, must be equal to the amount which must be put in the bank at time 0 to accumulate at interest to an amount 1 exactly t years later. Since $(1+i)^t$ is the factor by which today's deposit increases in exactly t years, the present value of a payment of \$1 delayed t years is $(1+i)^{-t}$. Here t may be an integer or positive real number.

(b) Present values superpose additively: that is, if I am to receive a payment stream C which is the sum of payment streams A and B, then the present value of C is simply the sum of the present value of payment stream A and the present value of payment stream B.

(c) As a consequence of (a) and (b), the present value for constant interest rate i at time 0 of a payment stream consisting of payments s_j at future times t_j, $j = 0, \ldots, n$ must be the summation

$$\sum_{j=0}^{n} s_j \, (1+i)^{-t_j}$$

(d) Finally, to combine present values on distinct time intervals, at possibly different interest rates, remark that if fixed interest-rate i applies to the time-interval $[0, s]$ and the fixed interest rate i' applies to the time-interval $[s, t+s]$, then the present value at time s of a future payment of a at time $t+s$ is $b = a(1+i')^{-t}$, and the present value at time 0 of a payment b at time s is $b\,(1+i)^{-s}$. The idea of present value is that these three payments, a at time $s+t$, $b = a(1+i')^{-t}$ at time s, and $b(1+i)^{-s} = a\,(1+i')^{-t}\,(1+i)^{-s}$ at time 0 , are all equivalent.

(e) Applying the idea of paragraph (d) repeatedly over successive intervals of length $h = 1/m$ each, we find that the present value of a payment of $1 at time t (assumed to be an integer multiple of h), where $r(kh)$ is the applicable *effective* interest rate on time-interval $[kh,\,(k+1)h]$, is

$$1/A(t) = \prod_{j=1}^{mt} (1 + r(jh))^{-h}$$

where $A(t) = A_k$ is the amount previously derived as the accumulation-factor for the time-interval $[0, t]$.

The formulas just developed can be used to give the *internal rate of return* r over the time-interval $[0, T]$ of a unit investment which pays amount s_k at times t_k, $k = 0, \ldots, n$, $0 \le t_k \le T$. This constant (effective) interest rate r is the one such that

$$\sum_{k=0}^{n} s_k \left(1 + r\right)^{-t_k} = 1$$

With respect to the APR r , the present value of a payment s_k at a time t_k time-units in the future is $s_k \cdot (1+r)^{-t_k}$. Therefore the stream of payments s_k at times t_k, $(k = 0, 1, \ldots, n)$ becomes equivalent, for the uniquely defined interest rate r, to an immediate (time-0) payment of 1.

Example 1 *As an illustration of the notion of effective interest rate, or internal rate of return, suppose that you are offered an investment option under which a* \$ 10,000 *investment made now is expected to pay* \$ 300 *yearly for 5 years (beginning 1 year from the date of the investment), and then* \$ 800 *yearly for the following five years, with the principal of* \$ 10,000 *returned to you (if all goes well) exactly 10 years from the date of the investment (at the same time as the last of the* \$ 800 *payments. If the investment goes as planned, what is the effective interest rate you will be earning on your investment ?*

As in all calculations of effective interest rate, the present value of the payment-stream, at the unknown interest rate $r = i_{APR}$, must be balanced with the value (here \$ 10,000) which is invested. (That is because the indicated payment stream is being regarded as equivalent to bank interest at rate r.) The balance equation in the Example is obviously

$$10,000 \;=\; 300 \sum_{j=1}^{5} (1+r)^{-j} \;+\; 800 \sum_{j=6}^{10} (1+r)^{-j} \;+\; 10,000\,(1+r)^{-10}$$

The right-hand side can be simplified somewhat, in terms of the notation $x = (1+r)^{-5}$, to

$$\frac{300}{1+r}\left(\frac{1-x}{1-(1+r)^{-1}}\right) + \frac{800x}{(1+r)}\left(\frac{1-x}{1-(1+r)^{-1}}\right) + 10000\,x^2$$

$$= \;\frac{1-x}{r}\,(300+800x) \;+\; 10000x^2 \tag{1.6}$$

Setting this simplified expression equal to the left-hand side of $10,000$ does not lead to a closed-form solution, since both $x = (1+r)^{-5}$ and r involve the unknown r. Nevertheless, we can solve the equation roughly by 'tabulating' the values of the simplified right-hand side as a function of r ranging in increments of 0.005 from 0.035 through 0.075. (We can guess that the correct answer lies between the minimum and maximum payments expressed as a fraction of the principal.) This tabulation yields:

r	.035	.040	.045	.050	.055	.060	.065	.070	.075
(1.6)	11485	11018	10574	10152	9749	9366	9000	8562	8320

From these values, we can see that the right-hand side is equal to \$ $10,000$ for a value of r falling between 0.05 and 0.055. Interpolating linearly to approximate the answer yields $r = 0.050 + 0.005 * (10000 - 10152)/(9749 - 10152) = 0.05189$, while an accurate equation-solver (the one in the **Splus** or **R** function *uniroot*) finds $r = 0.05186$.

1.2.2 Continuous-time Payment Streams

There is a completely analogous development for continuous-time deposit streams with continuous compounding. Suppose $D(t)$ to be the **rate** per unit time at which savings deposits are made, so that if we take m to go to

∞ in the previous discussion, we have $D(t) = \lim_{m\to\infty} m s_{[mt]}$, where $[\cdot]$ again denotes greatest-integer. Taking $\delta(t)$ to be the time-varying nominal interest rate with continuous compounding, and $B(t)$ to be the accumulated balance as of time t (analogous to the quantity $B_{[mt]} = B_k$ from before, when $t = k/m$), we replace the previous difference-equation by

$$B(t+h) \;=\; B(t)\,(1 + h\,\delta(t)) \;+\; h\,D(t) \;+\; o(h)$$

where $o(h)$ denotes a remainder such that $o(h)/h \to 0$ as $h \to 0$. Subtracting $B(t)$ from both sides of the last equation, dividing by h, and letting h decrease to 0, yields a *differential equation* at times $t > 0$:

$$B'(t) \;=\; B(t)\,\delta(t) + D(t)\;, \qquad A(0) = s_0 \tag{1.7}$$

The method of solution of (1.7), which is the standard one from differential equations theory of multiplying through by an *integrating factor*, again has a natural interpretation in terms of present values. The integrating factor $1/A(t) = \exp(-\int_0^t \delta(s)\,ds)$ is the present value at time 0 of a payment of 1 at time t, and the quantity $B(t)/A(t) \;=\; G(t)$ is then the present value of the deposit stream of s_0 at time 0 followed by continuous deposits at rate $D(t)$. The ratio-rule of differentiation yields

$$G'(t) \;=\; \frac{B'(t)}{A(t)} - \frac{B(t)\,A'(t)}{A^2(t)} \;=\; \frac{B'(t)\;-\;B(t)\,\delta(t)}{A(t)} \;=\; \frac{D(t)}{A(t)}$$

where the substitutuion $A'(t)/A(t) \;\equiv\; \delta(t)$ has been made in the third expression. Since $G(0) = B(0) = s_0$, the solution to the differential equation (1.7) becomes

$$G(t) = s_0 + \int_0^t \frac{D(s)}{A(s)}\,ds\;, \qquad B(t) = A(t)\,G(t)$$

Finally, the formula can be specialized to the case of a constant unit-rate payment stream ($D(x) \;=\; 1,\; \delta(x) \;=\; \delta \;=\; \ln(1+i),\; 0 \le x \le T$) with no initial deposit (i.e., $s_0 \;=\; 0$). By the preceding formulas, $A(t) = \exp(t\ln(1+i)) = (1+i)^t$, and the present value of such a payment stream is

$$\int_0^T 1\cdot\exp(-t\ln(1+i))\,dt = \frac{1}{\delta}\left(1-(1+i)^{-T}\right)$$

Recall that the *force of interest* $\delta = \ln(1+i)$ is the limiting value obtained from the nominal interest rate $i^{(m)}$ using the difference-quotient representation:

$$\lim_{m\to\infty} i^{(m)} = \lim_{m\to\infty} \frac{\exp((1/m)\ln(1+i)) - 1}{1/m} = \ln(1+i)$$

The present value of a payment at time T in the future is, as expected,

$$\left(1 + \frac{i^{(m)}}{m}\right)^{-mT} = (1+i)^{-T} = \exp(-\delta T)$$

1.3 *Exercise Set 1*

The first homework set covers the basic definitions in two areas: (i) probability as it relates to events defined from *cohort life-tables*, including the theoretical machinery of population and conditional survival, distribution, and density functions and the definition of expectation; (ii) the theory of interest and present values, with special reference to the idea of income streams of equal value at a fixed rate of interest.

(1). For how long a time should \$100 be left to accumulate at 5% interest so that it will amount to twice the accumulated value (over the same time period) of another \$100 deposited at 3% ?

(2). Use a calculator to answer the following numerically:

(a) Suppose you sell for \$6,000 the right to receive for 10 years the amount of \$1,000 per year payable quarterly (beginning at the end of the first quarter). What effective rate of interest makes this a fair sale price ? (You will have to solve numerically or graphically, or interpolate a tabulation, to find it.)

(b) \$100 deposited 20 years ago has grown at interest to \$235. The interest was compounded twice a year. What were the nominal and effective interest rates ?

(c) How much should be set aside (the same amount each year) at the beginning of each year for 10 years to amount to \$1000 at the end of the 10th year at the interest rate of part (b) ?

In the following problems, $S(x)$ denotes the probability for a newborn in a designated population to survive to exact age x . If a *cohort life table* is under discussion, then the probability distribution relates to a randomly chosen member of the newborn cohort.

(3). Assume that a population's survival probability function is given by $S(x) = 0.1(100 - x)^{1/2}$, for $0 \le x \le 100$.

(a) Find the probability that a life aged 0 will die between exact ages 19 and 36.

(b) Find the probability that a life aged 36 will die before exact age 51.

(4). (a) Find the expected age at death of a member of the population in problem (3).

(b) Find the expected age at death of a life aged 20 in the population of problem (3).

(5). Use the Illustrative Life-table (Table 1.1) to calculate the following probabilities. (In each case, assume that the indicated span of years runs from birthday to birthday.) Find the probability

(a) that a life aged 26 will live at least 30 more years;

(b) that a life aged 22 will die between ages 45 and 55;

(c) that a life aged 25 will die either before age 50 or after the 70'th birthday.

(6). In a certain population, you are given the following facts:

(i) The probability that two independent lives, respectively aged 25 and 45, *both* survive 20 years is 0.7.

(ii) The probability that a life aged 25 will survive 10 years is 0.9.

Then find the probability that a life aged 35 will survive to age 65.

(7). Suppose that you borrowed \$1000 at 6% APR, to be repaid in 5 years in a lump sum, and that after holding the money idle for 1 year you invested the money to earn 8% APR for the remaining four years. What is the effective interest rate you have earned (ignoring interest costs) over 5 years on the \$1000 which you borrowed ? Taking interest costs into account, what is the present value of your profit over the 5 years of the loan ? Also re-do the problem if instead of repaying all principal and interest at the end of 5 years, you must make a payment of accrued interest at the end of 3 years, with the additional interest and principal due in a single lump-sum at the end of 5 years.

(8). Find the total present value at 5% APR of payments of $1 at the end of 1, 3, 5, 7, and 9 years and payments of $2 at the end of 2, 4, 6, 8, and 10 years.

1.4 Worked Examples

Example 1. How many years does it take for money to triple in value at interest rate i ?

The equation to solve is $3 = (1+i)^t$, so the answer is $\ln(3)/\ln(1+i)$, with numerical answer given by

$$t = \begin{cases} 22.52 & \text{for} \quad i = 0.05 \\ 16.24 & \text{for} \quad i = 0.07 \\ 11.53 & \text{for} \quad i = 0.10 \end{cases}$$

Example 2. Suppose that a sum of \$1000 is borrowed for 5 years at 5%, with interest deducted immediately in a lump sum from the amount borrowed, and principal due in a lump sum at the end of the 5 years. Suppose further that the amount received is invested and earns 7%. What is the value of the net profit at the end of the 5 years ? What is its present value (at 5%) as of time 0 ?

First, the amount received is $1000\,(1-d)^5 = 1000/(1.05)^5 = 783.53$, where $d = .05/1.05$, since the amount received should compound to precisely the principal of \$1000 at 5% interest in 5 years. Next, the compounded value of 783.53 for 5 years at 7% is $783.53\,(1.07)^5 = 1098.94$, so the net profit at the end of 5 years, after paying off the principal of 1000, is \$98.94. The present value of the profit ought to be calculated with respect to the 'going rate of interest', which in this problem is presumably the rate of 5% at which the money is borrowed, so is $98.94/(1.05)^5 = 77.52$.

Example 3. For the following small cohort life-table (first 3 columns) with 5 age-categories, find the probabilities for all values of $[T]$, both unconditionally and conditionally for lives aged 2, and find the expectation of both $[T]$ and $(1.05)^{-[T]-1}$.

The basic information in the table is the first column l_x of numbers surviving. Then $d_x = l_x - l_{x+1}$ for $x = 0, 1, \ldots, 4$. The random variable T is the life-length for a randomly selected individual from the age=0 cohort, and therefore $P([T] = x) = P(x \le T < x+1) = d_x/l_0$. The conditional probabilities given survivorship to age-category 2 are simply the ratios with numerator d_x for $x \ge 2$, and with denominator $l_2 = 65$.

x	l_x	d_x	$P([T]=x)$	$P([T]=x \mid T \geq 2)$	1.05^{-x-1}
0	100	20	0.20	0	0.95238
1	80	15	0.15	0	0.90703
2	65	10	0.10	0.15385	0.86384
3	55	15	0.15	0.23077	0.82770
4	40	40	0.40	0.61538	0.78353
5	0	0	0	0	0.74622

In terms of the columns of this table, we evaluate from the definitions and formula (1.2)

$$E([T]) \; = \; 0 \cdot (0.20) + 1 \cdot (0.15) + 2 \cdot (0.10) + 3 \cdot (0.15) + 4 \cdot (0.40) \; = \; 2.4$$

$$E([T] \,|\, T \geq 2) \; = \; 2 \cdot (0.15385) + 3 \cdot (0.23077) + 4 \cdot (0.61538) \; = \; 3.4615$$

$$E(1.05^{-[T]-1}) \; = \; 0.95238 \cdot 0.20 \; + \; 0.90703 \cdot 0.15 \; + \; 0.86384 \cdot 0.10 \, +$$

$$+ \, 0.8277 \cdot 0.15 \; + \; 0.78353 \cdot 0.40 \; = \; 0.8497$$

The expectation of $[T]$ is interpreted as the average per person in the cohort life-table of the number of completed whole years before death. The quantity $(1.05)^{-[T]-1}$ can be interpreted as the present value at birth of a payment of \$1 to be made at the end of the year of death, and the final expectation calculated above is the average of that present-value over all the individuals in the cohort life-table, if the going rate of interest is 5%.

Example 4. Suppose that the death-rates $q_x = d_x/l_x$ for integer ages x in a cohort life-table follow the functional form

$$q_x \; = \; \begin{cases} 4 \cdot 10^{-4} & \text{for } 5 \leq x < 30 \\ 8 \cdot 10^{-4} & \text{for } 30 \leq x \leq 55 \end{cases}$$

between the ages x of 5 and 55 inclusive. Find analytical expressions for $S(x)$, l_x, d_x at these ages if $l_0 = 10^5$, $S(5) = .96$.

The key formula expressing survival probabilities in terms of death-rates q_x is:

$$\frac{S(x+1)}{S(x)} \; = \; \frac{l_{x+1}}{l_x} \; = \; 1 - q_x$$

or

$$l_x \; = \; l_0 \; \cdot \; S(x) \; = \; (1 - q_0)(1 - q_1) \cdots (1 - q_{x-1})$$

So it follows that for $x = 5, \ldots, 30$,

$$\frac{S(x)}{S(5)} = (1 - .0004)^{x-5} \, , \qquad l_x = 96000 \cdot (0.9996)^{x-5}$$

so that $S(30) = .940446$, and for $x = 31, \ldots, 55$,

$$S(x) = S(30) \cdot (.9992)^{x-30} = .940446 \, (.9992)^{x-30}$$

The death-counts d_x are expressed most simply through the preceding expressions together with the formula $d_x = q_x \, l_x$.

1.5 Useful Formulas from Chapter 1

$$S(x) \;=\; \frac{l_x}{l_0} \quad , \qquad d_x = l_x - l_{x+1}$$

p. 1

$$P(x \le T \le x+k) \;=\; \frac{S(x) - S(x+k)}{S(x)} \;=\; \frac{l_x - l_{x+k}}{l_x}$$

p. 2

$$f(x) \;=\; -S'(x) \quad , \qquad S(x) - S(x+k) \;=\; \int_x^{x+t} f(t)\,dt$$

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$$E\left(g(T) \,\middle|\, T \ge x \right) \;=\; \frac{1}{S(x)} \int_x^{\infty} g(t)\,f(t)\,dt$$

p. 7

$$1 + i_{\text{APR}} \;=\; \left(1 + \frac{i^{(m)}}{m}\right)^m \;=\; \left(1 - \frac{d^{(m)}}{m}\right)^{-m} \;=\; e^{\delta}$$

p. 9

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