Actuarial Mathematics and Life-Table Statistics

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Chapter 6

Commutation Functions, Reserves & Select Mortality

In this Chapter, we consider first the historically important topic of *Commutation Functions* for actuarial calculations, and indicate why they lose their computational usefulness as soon as the insurer entertains the possibility (as demographers often do) that life-table survival probabilities display some slow secular trend with respect to year of birth. We continue our treatment of premiums and insurance contract valuation by treating briefly the idea of insurance reserves and policy cash values as the life-contingent analogue of mortgage amortization and refinancing. The Chapter concludes with a brief section on Select Mortality, showing how models for select-population mortality can be used to calculate whether modified premium and deferral options are sufficient protections for insurers to insure such populations.

6.1 Idea of Commutation Functions

The Commutation Functions are a computational device to ensure that net single premiums for life annuities, endowments, and insurances from the same life table and figured at the same interest rate, for lives of differing ages and for policies of differing durations, can all be obtained from a single table look-up. Historically, this idea has been very important in saving calculational labor when arriving at premium quotes. Even now, assuming that a governing life table and interest rate are chosen provisionally, company employees without quantitative training could calculate premiums in a spreadsheet format with the aid of a life table.

To fix the idea, consider first the contract with the simplest net-single-premium formula, namely the pure n-year endowment. The expected present value of $1 one year in the future *if the policyholder aged* x *is alive at that time* is denoted in older books as ${}_nE_x$ and is called the **actuarial present value** of a life-contingent n-year future payment of 1:

$$A_{x:\overline{n}|}^{\;\;1} \;=\; {}_nE_x \;=\; v^n \, {}_np_x$$

Even such a simple life-table and interest-related function would seem to require a table in the *two* integer parameters x, n, but the following expression immediately shows that it can be recovered simply from a single tabulated column:

$$A_{x:\overline{n}|}^{\;\;1} \;=\; \frac{v^{n+x}\, l_{n+x}}{v^x\, l_x} \;=\; \frac{D_{x+n}}{D_x} \;, \qquad D_y \equiv v^y\, l_y \tag{6.1}$$

In other words, at least for integer ages and durations we would simply augment the insurance-company life-table by the column D_x. The addition of just a few more columns allows the other main life-annuity and insurance quantities to be recovered with no more than simple arithmetic. Thus, if we begin by considering whole life insurances (with only one possible payment at the end of the year of death), then the net single premium is re-written

$$A_x \;=\; A_{x:\overline{\infty}|}^{1} \;=\; \sum_{k=0}^{\infty} v^{k+1}\, {}_kp_x \,\cdot\, q_{x+k} \;=\; \sum_{k=0}^{\infty} \frac{v^{x+k+1}\,(l_{x+k} - l_{x+k+1})}{v^x\, l_x}$$

$$=\; \sum_{y=x}^{\infty} v^{y+1}\, \frac{d_y}{D_x} \;=\; \frac{M_x}{D_x} \;, \qquad M_x \;\equiv\; \sum_{y=x}^{\infty} v^{y+1}\, d_y$$

The insurance of finite duration also has a simple expression in terms of the same *commutation* columns M, D :

$$A_{x:\overline{n}|}^{1} \;=\; \sum_{k=0}^{n-1} v^{k+1}\, \frac{d_{k+x}}{D_x} \;=\; \frac{M_x - M_{x+n}}{D_x} \tag{6.2}$$

Next let us pass to to life annuities. Again we begin with the life annuity-due of infinite duration:

$$\ddot{a}_x \;=\; \ddot{a}_{\overline{x:\infty}|} \;=\; \sum_{k=0}^{\infty} v^{k+x} \frac{l_{k+x}}{D_x} \;=\; \frac{N_x}{D_x} \;, \qquad N_x \;=\; \sum_{y=x}^{\infty} v^y \, l_y \tag{6.3}$$

The commutation column N_x turns is the reverse cumulative sum of the D_x column:

$$N_x \quad = \quad \sum_{y=x}^{\infty} \; D_y$$

The expected present value for the finite-duration life-annuity due is obtained as a simple difference

$$\ddot{a}_{\overline{x:n}|} \;=\; \sum_{k=0}^{n-1} v^{k+x} \; \frac{l_{x+k}}{D_x} \;=\; \frac{N_x - N_{x+n}}{D_x}$$

There is no real need for a separate commutation column M_x since, as we have seen, there is an identity relating net single premiums for whole life insurances and annuities:

$$A_x \;=\; 1 \,-\, d\,\ddot{a}_x$$

Writing this identity with A_x and $\ddot{a}_x$ replaced by their respective commutation-function formulas, and then multupliying through by D_x, immediately yields

$$M_x \;=\; D_x \,-\, d\,N_x \tag{6.4}$$

Based on these tabulated **commutation columns** D, M, N, a quantitatively unskilled person could use simple formulas and tables to provide on-the-spot insurance premium quotes, a useful and desirable outcome even in these days of accessible high-powered computing. Using the case-(i) interpolation assumption, the m-payment-per-year net single premiums $A^{(m)}{}^{\,1}_{\overline{x:n}|}$ and $\ddot{a}^{(m)}_{\overline{x:n}|}$ would be related to their single-payment counterparts (whose commutation-function formulas have just been provided) through the standard formulas

$$A^{(m)}{}^{\,1}_{\overline{x:n}|} \;=\; \frac{i}{i^{(m)}}\, A^{\,1}_{\overline{x:n}|} \;, \qquad \ddot{a}^{(m)}_{\overline{x:n}|} \;=\; \alpha(m)\,\ddot{a}_{\overline{x:n}|} \,-\, \beta(m)\left(1 -\, A_{\overline{x:n}|}^{\;\;1}\right)$$

Table 6.1: Commutation Columns for the simulated US Male Illustrative Life-Table, Table 1.1, with APR interest-rate 6%.

Age x	l_x	D_x	N_x	M_x
0	100000	100000.00	1664794.68	94233.66
5	96997	72481.80	1227973.94	69507.96
10	96702	53997.89	904739.10	51211.65
15	96435	40238.96	663822.79	37574.87
20	95840	29883.37	484519.81	27425.65
25	95051	22146.75	351486.75	19895.48
30	94295	16417.71	252900.70	14315.13
35	93475	12161.59	179821.07	10178.55
40	92315	8975.07	125748.60	7117.85
45	90402	6567.71	85951.37	4865.17
50	87119	4729.55	56988.31	3225.75
55	82249	3336.63	36282.55	2053.73
60	75221	2280.27	21833.77	1235.87
65	65600	1486.01	12110.79	685.52
70	53484	905.34	5920.45	335.12
75	39975	505.65	2256.41	127.72

That is,

$$A^{(m)1}_{\ x:\overline{n}|} \;=\; \frac{i}{i^{(m)}} \cdot \frac{M_x - M_{x+n}}{D_x}$$

$$\ddot{a}^{(m)}_{x:\overline{n}|} \;=\; \alpha(m)\,\frac{N_x - N_{x+n}}{D_x} \;-\; \beta(m)\,(1 - \frac{D_{x+n}}{D_x})$$

To illustrate the realistic sizes of commutation-column numbers, we reproduce as Table 6.1 the main commutation-columns for 6% APR interest, in 5-year intervals, for the illustrative simulated life table given on page 3.

6.1.1 Variable-benefit Commutation Formulas

The only additional formulas which might be commonly needed in insurance sales are the variable-benefit term insurances with linearly increasing or decreasing benefits, and we content ourselves with showing how an additional commutation-column could serve here. First consider the infinite-duration policy with linearly increasing benefit

$$IA_x \;=\; \sum_{k=0}^{\infty} (k+1)\, v^{k+1}\, {}_kp_x \,\cdot\, q_{x+k}$$

This net single premium can be written in terms of the commutation functions already given together with

$$R_x \;=\; \sum_{k=0}^{\infty} (x+k+1)\, v^{x+k+1}\, d_{x+k}$$

Clearly, the summation defining IA_x can be written as

$$\sum_{k=0}^{\infty} (x+k+1)\, v^{k+1}\, {}_kp_x \,\cdot\, q_{x+k} \;-\; x \sum_{k=0}^{\infty} v^{k+1}\, {}_kp_x \,\cdot\, q_{x+k} \;=\; \frac{R_x}{D_x} \;-\; x\,\frac{M_x}{D_x}$$

Then, as we have discussed earlier, the finite-duration linearly-increasing-benefit insurance has the expression

$$IA^{1}_{x:\overline{n}|} \;=\; IA_x - \sum_{k=n}^{\infty} (k+1)\, v^{k+x+1}\, \frac{d_{x+k}}{D_x} \;=\; \frac{R_x - xM_x}{D_x} \;-\; \frac{R_{x+n} - xM_{x+n}}{D_x}$$

and the net single premium for the linearly-decreasing-benefit insurance, which pays benefit $n-k$ if death occurs between exact policy ages k and $k+1$ for $k = 0, \ldots, n-1$, can be obtained from the increasing-benefit insurance through the identity

$$DA^{1}_{x:\overline{n}|} \;=\; (n+1)A^{1}_{x:\overline{n}|} - IA^{1}_{x:\overline{n}|}$$

Throughout *all* of our discussions of premium calculation — not just the present consideration of formulas in terms of commutation functions — we have assumed that for ages of prospective policyholders, the same interest rate and life table would apply. In a future Chapter, we shall consider the problem of premium calculation and reserving under variable and stochastic interest-rate assumptions, but for the present we continue to fix the interest rate i. Here we consider briefly what would happen to premium calculation and the commutation formalism if the key assumption that the same life table applies to all insureds were to be replaced by an assumption involving interpolation between (the death rates defined by) two separate life tables applying to different birth cohorts. This is a particular case of a topic which we shall also take up in a future chapter, namely how extra (*'covariate'*) information about a prospective policyholder might change the survival probabilities which should be used to calculate premiums *for that policyholder*.

6.1.2 Secular Trends in Mortality

Demographers recognize that there are secular shifts over time in life-table age-specific death-rates. The reasons for this are primarily related to public health (e.g., through the eradication or successful treatment of certain disease conditions), sanitation, diet, regulation of hours and conditions of work, etc. As we have discussed previously in introducing the concept of force of mortality, the modelling of shifts in mortality patterns with respect to likely causes of death at different ages suggests that it is most natural to express shifts in mortality in terms of force-of-mortality and death rates rather than in terms of probability density or population-wide relative numbers of deaths in various age-intervals. One of the simplest models of this type, used for projections over limited periods of time by demographers (*cf.* the text *Introduction to Demography* by M. Spiegelman), is to view age-specific death-rates q_x as locally linear functions of calendar time t. Mathematically, it may be slightly more natural to make this assumption of linearity directly about the force of mortality. Suppose therefore that in calendar year t, the force of mortality $\mu_x^{(t)}$ at all ages x is assumed to have the form

$$\mu_x^{(t)} \;=\; \mu_x^{(0)} \,+\, b_x\, t \tag{6.5}$$

where $\mu_x^{(0)}$ is the force-of-mortality associated with some standard life table as of some arbitrary but fixed calendar-time origin $t = 0$. The age-dependent slope b_x will generally be extremely small. Then, placing superscripts $^{(t)}$ over all life-table entries and ratios to designate calendar time, we calculate

$${}_kp_x^{(t)} \;=\; \exp\Big(- \int_0^k \mu_{x+u}^{(t)}\, du \Big) \;=\; {}_kp_x^{(t)} \,\cdot\, \exp(-t \sum_{j=0}^{k-1} b_{x+j})$$

If we denote

$$B_x \;=\; \sum_{y=0}^{x} b_y$$

and assume that the life-table radix l_0 is not taken to vary with calendar time, then the commutation-function $D_x = D_x^{(t)}$ takes the form

$$D_x^{(t)} \;=\; v^x \, l_0 \, {}_xp_0^{(t)} \;=\; D_x^{(0)} \, e^{-tB_x} \tag{6.6}$$

Thus the commutation-columns $D_x^{(0)}$ (from the standard life-table) and B_x are enough to reproduce the time-dependent commutation column D, but now the calculation is not quite so simple, and the time-dependent commutation columns M, N become

$$M_x^{(t)} \;=\; \sum_{y=x}^{\infty} v^y \, (l_y^{(t)} - l_{y+1}^{(t)}) \;=\; \sum_{y=x}^{\infty} D_y^{(0)} \, e^{-tB_y} \left(1 - e^{-t\, b_{y+1}} \, q_y^{(0)}\right) \tag{6.7}$$

$$N_x^{(t)} \;=\; \sum_{y=x}^{\infty} D_y^{(0)} \, e^{-tB_y} \tag{6.8}$$

For simplicity, one might replace equation (6.7) by the approximation

$$M_x^{(t)} \;=\; \sum_{y=x}^{\infty} D_y^{(0)} \left(p_y^{(0)} + t\, b_{y+1} \, q_y^{(0)}\right) e^{-tB_y}$$

None of these formulas would be too difficult to calculate with, for example on a hand-calculator; moreover, since the calendar year t would be fixed for the printed tables which an insurance salesperson would carry around, the virtues of commutation functions in providing quick premium-quotes would not be lost if the life tables used were to vary systematically from one calendar year to the next.

6.2 Reserve & Cash Value of a Single Policy

In long-term insurance policies paid for by level premiums, it is clear that since risks of death rise very rapidly with age beyond middle-age, the early premium payments must to some extent exceed the early insurance costs.

Our calculation of risk premiums ensures by definition that for each insurance and/or endowment policy, the expected total present value of premiums paid in will equal the expected present value of claims to be paid out. However, it is generally *not* true *within each year* of the policy that the expected present value of amounts paid in are equal to the expected present value of amounts paid out. In the early policy years, the difference paid in versus out is generally in the insurer's favor, and the surplus must be set aside as a **reserve** against expected claims in later policy-years. It is the purpose of the present section to make all of these assertions mathematically precise, and to show how the reserve amounts are to be calculated. Note once and for all that loading plays no role in the calculation of reserves: throughout this Section, 'premiums' refer only to pure-risk premiums. The loading portion of actual premium payments is considered either as reimbursement of administrative costs or as profit of the insurer, but in any case does not represent buildup of value for the insured.

Suppose that a policyholder aged x purchased an endowment or insurance (of duration at least t) t years ago, paid for with level premiums, and has survived to the present. Define the **net (level) premium reserve** as of time t to be the excess ${}_tV$ of the expected present value of the amount to be paid out under the contract in future years over the expected present value of further *pure risk* premiums to be paid in (including the premium paid immediately, in case policy age t coincides with a time of premium-payment). Just as the notations $P^1_{x:\overline{n}|}$, $P_{x:\overline{n}|}^{\ \ 1}$, etc., are respectively used to denote the level annual premium amounts for a term insurance, an endowment, etc., we use the same system of sub- and superscripts with the symbol ${}_tV$ to describe the reserves on these different types of policies.

By definition, the net premium reserve of any of the possible types of contract as of policy-age $t = 0$ is 0 : this simply expresses the balance between expected present value of amounts to be paid into and out of the insurer under the policy. On the other hand, the *terminal reserve* under the policy, which is to say the reserve ${}_nV$ just before termination, will differ from one type of policy to another. The main possibilities are the pure term insurance, with reserves denoted ${}_tV^1_{x:\overline{n}|}$ and terminal reserve ${}_nV^1_{x:\overline{n}|} = 0$, and the endowment insurance, with reserves denoted ${}_tV_{x:\overline{n}|}^{\ \ 1}$ and terminal reserve ${}_nV_{x:\overline{n}|}^{\ \ 1} = 1$. In each of these two examples, just before policy termination there are no further premiums to be received or insurance benefits to be paid, so that the terminal reserve coincides with the terminal (i.e., endowment) benefit to be paid at policy termination. Note that for simplicity, we do not consider here the insurances or endowment insurances with $m > 1$ possible payments per year. The reserves for such policies can be obtained as previously discussed from the one-payment-per-year case via interpolation formulas.

The definition of net premium reserves is by nature *prospective*, referring to future payment streams and their expected present values. From the definition, the formulas for reserves under the term and endowment insurances are respectively given by:

$$ {}_tV^1_{x:\overline{n}|} \;=\; A^1_{x+t:\overline{n-t}|} \;-\; P^1_{x:\overline{n}|} \cdot \ddot{a}_{x+t:\overline{n-t}|} \tag{6.9} $$

$$ {}_tV_{x:\overline{n}|} \;=\; A_{x+t:\overline{n-t}|} \;-\; P_{x:\overline{n}|} \cdot \ddot{a}_{x+t:\overline{n-t}|} \tag{6.10} $$

One identity from our previous discussions of net single premiums immediately simplifies the description of reserves for endowment insurances. Recall the identity

$$ \ddot{a}_{x:\overline{n}|} \;=\; \frac{1 - A_{x:\overline{n}|}}{d} \qquad \Longrightarrow \qquad A_{x:\overline{n}|} \;=\; 1 \;-\; d\,\ddot{a}_{x:\overline{n}|} $$

Dividing the second form of this identity through by the net single premium for the life annuity-due, we obtain

$$ P_{x:\overline{n}|} \;=\; \frac{1}{\ddot{a}_{x:\overline{n}|}} \;-\; d \tag{6.11} $$

after which we immediately derive from the definition the reserve-identity

$$ {}_tV_{x:\overline{n}|} \;=\; \ddot{a}_{x+t:\overline{n-t}|}\Big(P_{x+t:\overline{n-t}|} \;-\; P_{x:\overline{n}|}\Big) \;=\; 1 \;-\; \frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}} \tag{6.12} $$

6.2.1 Retrospective Formulas & Identities

The notion of *reserve* discussed so far expresses a difference between expected present value of amounts to be paid out and to be paid in to the insurer dating from t years following policy initiation, *conditionally given the survival of the insured life aged* x *for the additional* t *years to reach age* $x + t$. It stands to reason that, assuming this reserve amount to be positive, there must have been up to time t an excess of premiums collected over insurance acquired up to duration t. This latter amount is called the **cash value** of the policy when accumulated actuarially to time t, and represents a cash amount to which the policyholder is entitled (less administrative expenses) if he wishes to discontinue the insurance. (Of course, since the insured has lived to age $x+t$, in one sense the insurance has not been 'used' at all because it did not generate a claim, but an insurance up to policy age t was in any case the only part of the purchased benefit which could have brought a payment from the insurer up to policy age t.) The insurance bought had the time-0 present value $A^{1}_{x:\overline{t}|}$, and the premiums paid prior to time t had time-0 present value $\ddot{a}_{x:\overline{t}|} \cdot P$, where P denotes the level annualized premium for the duration-n contract actually purchased.

To understand clearly why there is a close connection between retrospective and prospective differences between expected (present value of) amounts paid in and paid out under an insurance/endowment contract, we state a general and slightly abstract proposition.

> Suppose that a life-contingent payment stream (of duration n at least t) can be described in two stages, as conferring a benefit of expected time-0 present value U_t on the policy-age-interval $[0,t)$ (up to but not including policy age t), and **also** conferring a benefit *if the policyholder is alive as of policy age* t with expected present value *as of time* t which is equal to F_t. **Then the total expected time-0 present value of the contractual payment stream is**
>
> $$U_t + v^t \, {}_tp_x \, F_t$$

Before applying this idea to the balancing of prospective and retrospective reserves, we obtain without further effort three useful identities by recognizing that a term life-insurance, insurance endowment, and life annuity-due (all of duration n) can each be divided into a before- and after- t component along the lines of the displayed Proposition. (Assume for the following discussion that the intermediate duration t is also an integer.) For the insurance, the benefit U_t up to t is $A^{1}_{x:\overline{t}|}$, and the contingent after-t benefit F_t is $A^{1}_{x+t:\overline{n-t}|}$. For the endowment insurance, $U_t = A^{1}_{x:\overline{t}|}$ and $F_t = A_{x+t:\overline{n-t}|}$.

Finally, for the life annuity-due, $U_t = \ddot{a}_{\overline{x:t}|}$ and $F_t = \ddot{a}_{\overline{x+t:n-t}|}$. Assembling these facts using the displayed Proposition, we have our three identities:

$$A^{1}_{\overline{x:n}|} \;=\; A^{1}_{\overline{x:t}|} + v^t\, {}_tp_x\, A^{1}_{\overline{x+t:n-t}|} \tag{6.13}$$
$$A_{\overline{x:n}|} \;=\; A^{1}_{\overline{x:t}|} + v^t\, {}_tp_x\, A_{\overline{x+t:n-t}|} \tag{6.14}$$
$$\ddot{a}_{\overline{x:n}|} \;=\; \ddot{a}_{\overline{x:t}|} + v^t\, {}_tp_x\, \ddot{a}_{\overline{x+t:n-t}|} \tag{6.15}$$

The factor $v^t\, {}_tp_x$, which discounts present values from time t to the present values at time 0 *contingent on survival to* t, has been called above the **actuarial present value**. It coincides with ${}_tE_x = A_{\overline{x:t}|}^{\ \ 1}$, the expected present value at 0 of a payment of 1 to be made at time t if the life aged x survives to age $x+t$. This is the factor which connects the excess of insurance over premiums on $[0, t)$ with the reserves ${}_tV$ on the insurance/endowment contracts which refer prospectively to the period $[t, n]$. Indeed, substituting the identities (6.13), (6.14), and (6.15) into the identities

$$A^{1}_{\overline{x:n}|} = P^{1}_{\overline{x:n}|}\, \ddot{a}_{\overline{x:n}|} \quad , \qquad A_{\overline{x:n}|} = P_{\overline{x:n}|}\, \ddot{a}_{\overline{x:n}|}$$

yields

$$v^t\, {}_tp_x\, {}_tV^{1}_{\overline{x:n}|} \;=\; -\Big[\, A^{1}_{\overline{x:t}|} - P^{1}_{\overline{x:n}|}\, \ddot{a}_{\overline{x:n}|} \,\Big] \tag{6.16}$$
$$v^t\, {}_tp_x\, {}_tV_{\overline{x:n}|} \;=\; -\Big[\, A_{\overline{x:t}|} - P_{\overline{x:t}|}\, \ddot{a}_{\overline{x:n}|} \,\Big] \tag{6.17}$$

The interpretation in words of these last two equations is that the actuarial present value of the net (level) premium reserve at time t (of either the term insurance or endowment insurance contract) is equal to the negative of the expected present value of the difference between the contract proceeds and the premiums paid on $[0, t)$.

Figure 6.1 provides next an illustrative set of calculated net level premium reserves, based upon 6% interest, for a 40-year term life insurance with face-amount $1,000$ for a life aged 40. The mortality laws used in this calculation, chosen from the same families of laws used to illustrate force-of-mortality curves in Figures 2.3 and 2.4 in Chapter 2, are the same plausible mortality laws whose survival functions are pictured in Figure 2.5. These mortality laws are realistic in the sense that they closely mirror the US male mortality in 1959 (*cf.* plotted points in Figure 2.5.) The cash-reserve curves ${}_tV_{\overline{40:40}|}$ as functions of $t = 0, \ldots, 40$ are pictured graphically in Figure 6.1. Note that these reserves can be very substantial: at policy age 30, the reserves on the \$1000 term insurance are respectively \$459.17, \$379.79, \$439.06, and \$316.43.

6.2.2 Relating Insurance & Endowment Reserves

The simplest formulas for net level premium reserves in a general contract arise in the case of endowment insurance ${}_t\mathrm{V}^{1}_{\overline{x:n}|}$, as we have seen in formula (6.12). In fact, for endowment and/or insurance contracts in which the period of payment of level premiums coincides with the (finite or infinite) policy duration, the reserves for term-insurance and pure-endowment can also be expressed rather simply in terms of ${}_t\mathrm{V}_{\overline{x:n}|}$. Indeed, reasoning from first principles with the prospective formula, we find for the pure endowment

$$ {}_t\mathrm{V}_{\overline{x:n}|}^{\;\;1} \;=\; v^{n-t}\;{}_{n-t}p_{x+t} \;-\; \frac{v^n\;{}_np_x}{\ddot{a}_{\overline{x:n}|}}\,\ddot{a}_{\overline{x+t:n-t}|} $$

from which, by substitution of formula (6.12), we obtain

$$ \mathrm{V}_{\overline{x:n}|}^{\;\;1} \;=\; v^{n-t}\;{}_{n-t}p_{x+t} \;-\; v^n\;{}_np_x\,\left(1 \;-\; {}_t\mathrm{V}_{\overline{x:n}|}\right) \qquad (6.18) $$

Then, for net level reserves or cash value on a term insurance, we conclude

$$ \mathrm{V}^{1}_{\overline{x:n}|} \;=\; (1 - v^n\;{}_np_x)\,{}_t\mathrm{V}_{\overline{x:n}|} \;+\; v^n\;{}_np_x \;-\; v^{n-t}\;{}_{n-t}p_{x+t} \qquad (6.19) $$

6.2.3 Reserves under Constant Force of Mortality

We have indicated above that the phenomenon of positive reserves relates in some way to the aging of human populations, as reflected in the increasing force of mortality associated with life-table survival probabilities. A simple benchmark example helps us here: we show that when life-table survival is governed at all ages x and greater by a constant force of mortality μ, the reserves for term insurances are identically 0. In other words, the expected present value of premiums paid in *within each policy year* exactly compensates, under constant force of mortality, for the expected present value of the amount to be paid out *for that same policy year*.

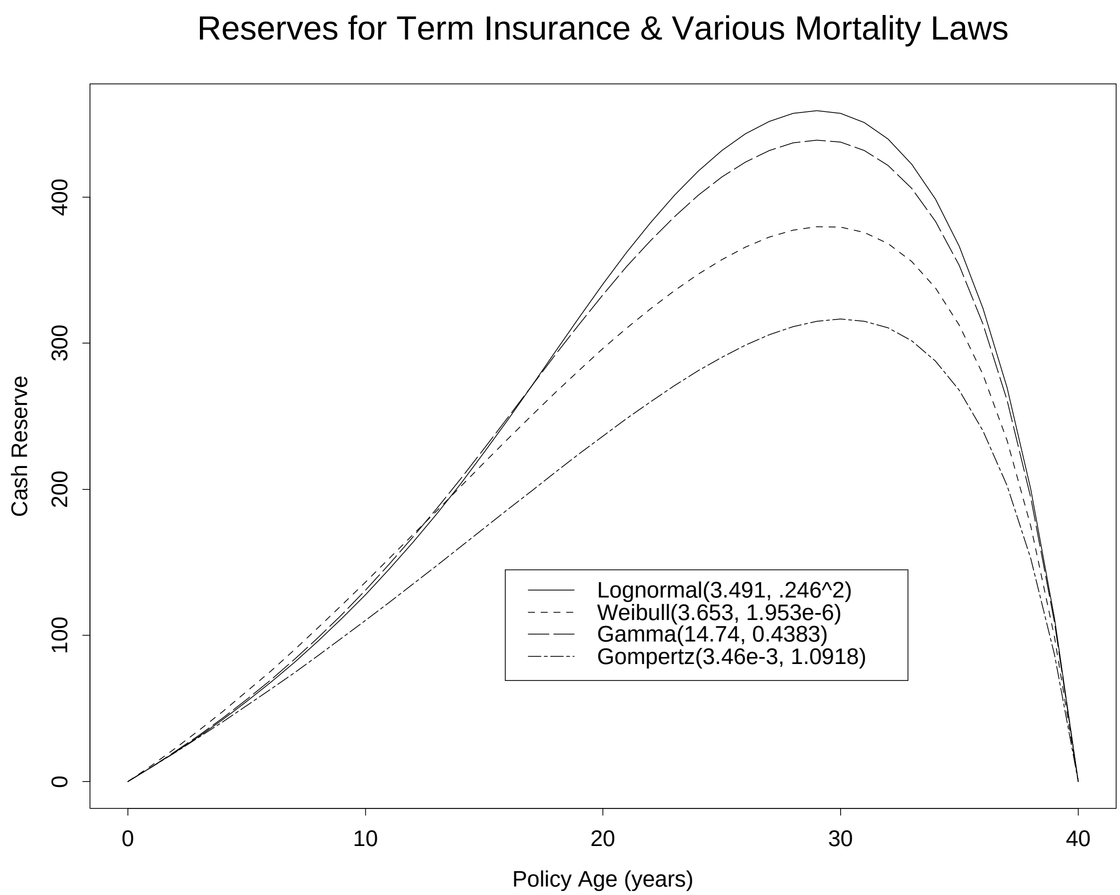

Figure 6.1: Net level premium reserves as a function of policy age for a 40-year term insurance to a life aged 40 with level benefit 1000, at 6%, under the same mortality laws pictured in Figures 2.5, with median age 72 at death. For definitions of the mortality laws see the Examples of analytical force-of-mortality functions in Chapter 2.

The calculations here are very simple. Reasoning from first principles, under constant force of mortality

$$A^{1}_{x:\overline{n}|} \;=\; \sum_{k=0}^{n-1} v^{k+1}\, e^{-\mu\, k}\, (1-e^{-\mu}) \;=\; (e^{\mu}-1)\,(ve^{-\mu})\, \frac{1\,-\,(ve^{-\mu})^n}{1\,-\,ve^{-\mu}}$$

while

$$\ddot{a}_{x:\overline{n}|} \;=\; \sum_{k=0}^{n-1} v^{k}\, e^{-\mu\, k} \;=\; \frac{1\,-\,(ve^{-\mu})^n}{1\,-\,ve^{-\mu}}$$

It follows from these last two equations that

$$P^{1}_{x:\overline{n}|} \;=\; v\,(1\,-\,e^{-\mu})$$

which does not depend upon either age x (within the age-range where constant force of mortality holds) or n. The immediate result is that for $0 < t < n$

$${}_tV^{1}_{x:\overline{n}|} \;=\; \ddot{a}_{x+t:\overline{n-t}|}\left(P^{1}_{x+t:\overline{n-t}|} - P^{1}_{x:\overline{n}|}\right) \;=\; 0$$

By contrast, since the terminal reserve of an endowment insurance must be 1 even under constant force of mortality, the intermediate net level premium reserves for endowment insurance must be positive and growing. Indeed, we have the formula deriving from equation (6.12)

$${}_tV_{x:\overline{n}|} \;=\; 1\,-\,\frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}} \;=\; 1\,-\,\frac{1-(ve^{-\mu})^{n-t}}{1-(ve^{-\mu})^{n}}$$

6.2.4 Reserves under Increasing Force of Mortality

Intuitively, we should expect that a force-of-mortality function which is everywhere increasing for ages greater than or equal to x will result in intermediate reserves for term insurance which are positive at times between $0, n$ and in reserves for endowment insurance which increase from 0 at $t = 0$ to 1 at $t = n$. In this subsection, we prove those assertions, using the simple observation that when $\mu(x+y)$ is an increasing function of positive y,

$${}_kp_{x+y} \;=\; \exp(-\int_0^k \mu_(x+y+z)\,dz) \quad \searrow \quad \text{in } y \tag{6.20}$$

First suppose that $0 \le s < t \le n$, and calculate using the displayed fact that ${}_kp_{x+y}$ decreases in y,

$$\ddot{a}_{\overline{x+t:n-t}|} = \sum_{k=0}^{n-t-1} v^{k+1}\,{}_kp_{x+t} \le \sum_{k=0}^{n-t-1} v^{k+1}\,{}_kp_{x+s} < \ddot{a}_{\overline{x+s:n-s}|}$$

Therefore $\ddot{a}_{\overline{x+t:n-t}|}$ is a decreasing function of t, and by formula (6.12), ${}_tV_{\overline{x:n}|}$ is an increasing function of t, ranging from 0 at $t = 0$ to 1 at $t = 1$.

It remains to show that for force of mortality which is increasing in age, the net level premium reserves ${}_tV^1_{\overline{x:n}|}$ for term insurances are positive for $t = 1, \ldots, n-1$. By equation (6.9) above, this assertion is equivalent to the claim that

$$A^1_{\overline{x+t:n-t}|}/\ddot{a}_{\overline{x+t:n-t}|} > A^1_{\overline{x:n}|}/\ddot{a}_{\overline{x:n}|}$$

To see why this is true, it is instructive to remark that each of the displayed ratios is in fact a multiple v times a weighted average of death-rates: for $0 \le s < n$,

$$\frac{A^1_{\overline{x+s:n-s}|}}{\ddot{a}_{\overline{x+s:n-s}|}} = v\left\{\frac{\sum_{k=0}^{n-s-1} v^k\,{}_kp_{x+s}\,q_{x+s+k}}{\sum_{k=0}^{n-s-1} v^k\,{}_kp_{x+s}}\right\}$$

Now fix the age x arbitrarily, together with $t \in \{1, \ldots, n-1\}$, and define

$$\overline{q} = \frac{\sum_{j=t}^{n-1} v^j\,{}_jp_x\,q_{x+j}}{\sum_{j=t}^{n-1} v^j\,{}_jp_x}$$

Since μ_{x+y} is assumed increasing for all y, we have from formula (6.20) that $q_{x+j} = 1 - p_{x+j}$ is an increasing function of j, so that for $k < t \le j$, $q_{x+k} < q_{x+j}$ and

$$\text{for all} \quad k \in \{0, \ldots, t-1\}\,, \qquad q_{x+k} < \overline{q} \tag{6.21}$$

Moreover, dividing numerator and denominator of the ratio defining $\overline{q}$ by $v^t\,{}_tp_x$ gives the alternative expression

$$v\,\overline{q} = \frac{\sum_{a=0}^{n-t-1} v^a\,{}_ap_{x+t}\,q_{x+t+a}}{\sum_{a=0}^{n-t-1} v^a\,{}_ap_{x+t}} = P^1_{\overline{x+t:n-t}|}$$

Finally, using equation (6.21) and the definition of $\overline{q}$ once more, we calculate

$$\frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} \;=\; v\Big(\frac{\sum_{k=0}^{t-1} v^k\, {}_kp_x\, q_{x+k} \;+\; \sum_{j=t}^{n-t-1} v^j\, {}_jp_x\, q_{x+j}}{\sum_{k=0}^{t-1} v^k\, {}_kp_x \;+\; \sum_{j=t}^{n-t-1} v^j\, {}_jp_x}\Big)$$

$$=\; v\Big(\frac{\sum_{k=0}^{t-1} v^k\, {}_kp_x\, q_{x+k} \;+\; \overline{q}\, \sum_{j=t}^{n-t-1} v^j\, {}_jp_x}{\sum_{k=0}^{t-1} v^k\, {}_kp_x \;+\; \sum_{j=t}^{n-t-1} v^j\, {}_jp_x}\Big) \;<\; v\,\overline{q} \;=\; P^1_{x+t:\overline{n-t}|}$$

as was to be shown. The conclusion is that under the assumption of increasing force of mortality for all ages x and larger, ${}_tV^1_{x:\overline{n}|} > 0$ for $t = 1, \ldots, n-1$.

6.2.5 Recursive Calculation of Reserves

The calculation of reserves can be made very simple and mechanical in numerical examples with small illustrative life tables, due to the identities (6.13) and (6.14) together with the easily proved recursive identities (for integers t)

$$A^1_{x:\overline{t+1}|} \;=\; A^1_{x:\overline{t}|} \;+\; v^{t+1}\, {}_tp_x\, q_{x+t}$$

$$A_{x:\overline{t+1}|} \;=\; A_{x:\overline{t}|} \;-\; v^t\, {}_tp_x \;+\; v^{t+1}\, {}_tp_x \;=\; A_{x:\overline{t}|} \;-\; d\, v^t\, {}_tp_x$$

$$\ddot{a}_{x:\overline{t+1}|} \;=\; \ddot{a}_{x:\overline{t}|} \;+\; v^t\, {}_tp_x$$

Combining the first and third of these recursions with (6.13), we find

$${}_tV^1_{x:\overline{n}|} \;=\; \frac{-1}{v^t\, {}_tp_x}\Big(A^1_{x:\overline{t}|} \;-\; P^1_{x:\overline{n}|}\, \ddot{a}_{x:\overline{t}|}\Big)$$

$$=\; \frac{-1}{v^t\, {}_tp_x}\Big(A^1_{x:\overline{t+1}|} \;-\; v^{t+1}\, {}_tp_x\, q_{x+t} \;-\; P^1_{x:\overline{n}|}\Big[\ddot{a}_{x:\overline{t+1}|} \;-\; v^t\, {}_tp_x\Big]\Big)$$

$$=\; v\, p_{x+t}\;\; {}_{t+1}V^1_{x:\overline{n}|} \;+\; v\, q_{x+t} \;-\; P^1_{x:\overline{n}|}$$

The result for term-insurance reserves is the single recursion

$${}_tV^1_{x:\overline{n}|} \;=\; v\, p_{x+t}\;\; {}_{t+1}V^1_{x:\overline{n}|} \;+\; v\, q_{x+t} \;-\; P^1_{x:\overline{n}|} \qquad (6.22)$$

which we can interpret in words as follows. The reserves at integer policy-age t are equal to the sum of the one-year-ahead actuarial present value of the reserves at time $t+1$ and the excess of the present value of this year's expected insurance payout $(v\, q_{x+t})$ over this year's received premium $(P^1_{x:\overline{n}|})$.

A completely similar algebraic proof, combining the one-year recursions above for endowment insurance and life annuity-due with identity (6.14), yields a recursive formula for endowment-insurance reserves (when $t < n$) :

$$_tV_{x:\overline{n}|} \; = \; v\, p_{x+t} \;\; _{t+1}V_{x:\overline{n}|} \; + \; v\, q_{x+t} \; - \; P^1_{x:\overline{n}|} \tag{6.23}$$

The verbal intepretation is as before: the future reserve is discounted by the one-year actuarial present value and added to the expected present value of the one-year term insurance minus the one-year cash (risk) premium.

6.2.6 Paid-Up Insurance

An insured may want to be aware of the cash value (equal to the reserve) of an insurance or endowment either in order to discontinue the contract and receive the cash or to continue the contract in its current form and borrow with the reserve as collateral. However, it may also happen for various reasons that an insured may want to continue insurance coverage but is no longer able or willing to par further premiums. In that case, for an administrative fee the insurer can convert the premium reserve to a single premium for a new insurance (either with the same term, or whole-life) with lesser benefit amount. This is really not a new topic, but a combination of the previous formulas for reserves and net single premiums. In this sub-section, we give the simplified formula for the case where the cash reserve is used as a single premium to purchase a new whole-life policy. Two illustrative worked examples on this topic are given in Section 6.6 below.

The general formula for reserves, now specialized for whole-life insurances, is

$$_tV_x \; = \; \mathrm{A}_{x+t} \; - \; \frac{\mathrm{A}_x}{\ddot{\mathrm{a}}_x} \cdot \ddot{\mathrm{a}}_{x+t} \; = \; 1 \; - \; \frac{\ddot{\mathrm{a}}_{x+t}}{\ddot{\mathrm{a}}_x}$$

This formula, which applies to a unit face amount, would be multiplied through by the level benefit amount B. Note that loading is disregarded in this calculation. The idea is that any loading which may have applied has been collected as part of the level premiums; but in practice, the insurer might apply some further (possibly lesser) loading to cover future administrative costs. Now if the cash or reserve value $_tV_x$ is to serve as net single premium for a new insurance, the new face-amount F is determined as of the t policy-anniversary by the balance equation

$$B \cdot {}_t\mathrm{V}_x \;=\; F \cdot \mathrm{A}_{x+t}$$

which implies that the equivalent **face amount of paid-up insurance** as of policy-age t is

$$F \;=\; \frac{B\,{}_t\mathrm{V}_x}{\mathrm{A}_{x+t}} \;=\; B\left(1 - \frac{\ddot{\mathrm{a}}_{x+t}}{\ddot{\mathrm{a}}_x}\right) \Big/ \left(1 - d\,\ddot{\mathrm{a}}_{x+t}\right) \tag{6.24}$$

6.3 Select Mortality Tables & Insurance

Insurers are often asked to provide life insurance converage to groups and/or individuals who belong to special populations with mortality significantly worse than that of the general population. Yet such **select populations** may not be large enough, or have a sufficiently long data-history, within the insurer's portfolio for survival probabilities to be well-estimated in-house. In such cases, insurers may provide coverage under special premiums and terms. The most usual example is that such select risks may be issued insurance with restricted or no benefits for a specified period of time, e.g. 5 years. The stated rationale is that after such a period of deferral, the select group's mortality will be sufficiently like the mortality of the general population in order that the insurer will be adequately protected if the premium in increased by some standard multiple. In this Section, a slightly artifical calculation along these lines illustrates the principle and gives some numerical examples.

Assume for simplicity that the general population has constant force of mortality μ, and that the select group has larger force of mortality μ^*. If the interest rate is i, and $v = 1/(1+i)$, then the level yearly risk premium for a k-year deferred whole-life insurance (of unit amount, payable at the end of the year of death) to a randomly selected life (of any specified age x) from the general population is easily calculated to be

$$\text{Level Risk Premium} \;=\; v^k\,{}_kp_x\,A_{x+k}/\ddot{\mathrm{a}}_x \;\;=\;\; (1-e^{-\mu})\,v^{k+1}\,e^{-\mu k} \tag{6.25}$$

If this premium is multiplied by a factor $\kappa > 1$ and applied as the risk premium for a k-year deferred whole-life policy to a member of the select

Table 6.2: Excess payout as given by formula (6.26) under a k-year deferred whole life insurance with benefit \$1000, issued to a select population with constant force of mortality μ^* for a level yearly premium which is a multiple κ times the pure risk premium for the standard population which has force-of-mortality μ. Interest rate is $i = 0.06$ APR throughout.

k	μ	μ^*	κ	Excess Payout
0	0.02	0.03	1	109
0	0.02	0.03	2	-112
5	0.02	0.03	1	63
5	0.02	0.03	1.42	0
0	0.05	0.10	1	299
0	0.05	0.10	3	-330
5	0.05	0.10	1	95
5	0.05	0.10	1.52	0
3	0.05	0.10	1	95
3	0.05	0.10	1.68	0

population, then the expected excess (per unit of benefit) of the amount paid out under this select policy over the risk premiums collected, is

$$\text{Excess Payout} \;=\; v^k \, {}_kp_x^* \, A_{x+k}^* \;-\; \kappa\,(1-e^{-\mu})\, v^{k+1}\, e^{-\mu k}\, \ddot{a}_x^*$$

$$= \;\frac{v^{k+1}}{1 - ve^{-mu^*}}\,\Big\{(1-e^{-\mu})e^{-\mu k} \;-\; (1-e^{-\mu^*})e^{-\mu^* k}\Big\} \qquad (6.26)$$

where the probability, insurance, and annuity notations with superscripts * are calculated using the select mortality distribution with force of mortality μ^*. Because of the constancy of forces of mortality both in the general and the select populations, the premiums and excess payouts do not depend on the age of the insured. Table 6.3 shows the values of some of these excess payouts, for $i = 0.06$, under several combinations of k, μ, μ^*, and κ. Note that in these examples, select mortality with force of mortality multiplied by 1.5 or 2 is offset, with sufficient protection to the insurer, by an increase of 40–60% in premium on whole-life policies deferring benefits by 3 or 5 years.

Additional material will be added to this Section later. A calculation *along the same lines as the foregoing Table, but using the Gompertz*$(3.46e-3, 1.0918)$ *mortality law previously found to approximate well the realistic Life Table data in Table 1.1, will be included for completeness.*

6.4 *Exercise Set 6*

For the first problem, use the Simulated Illustrative Life Table with commutator columns given as Table 6.1 on page 152, using 6% APR as the going rate of interest. (Also assume, wherever necessary, that the distribution of deaths within whole years of age is uniform.)

(1). (a) Find the level premium for a 20-year term insurance of \$5000 for an individual aged 45, which pays at the end of the half-year of death, where the payments are to be made semi-annually.

(b) Find the level annual premium for a whole-life insurance for an individual aged 35, which pays \$30,000 at the end of year of death if death occurs before exact age 55 and pays \$60,000 at the instant (i.e., day) of death at any later age.

(2). You are given the following information, indirectly relating to the fixed rate of interest i and life-table survival probabilities ${}_kp_x$.

(i) For a one-payment-per-year level-premium 30-year endowment insurance of 1 on a life aged x, the amount of reduced paid-up endowment insurance at the end of 10 years is 0.5.

(ii) For a one-payment-per-year level-premium 20-year endowment insurance of 1 on a life aged $x+10$, the amount of reduced paid-up insurance at the end of 5 years is 0.3.

Assuming that cash values are equal to net level premium reserves and reduced paid-up insurances are calculated using the equivalence principle, so that the cash value is equated to the net single premium of an endowment insurance with reduced face value, calculate the amount of reduced paid-up insuranceat the end of 15 years for the 30-year endowment insuranceof 1 on a life aged x. See the following Worked Examples for some relevant formulas.

(3). Give a formula for A_{45} in terms of the following quantities alone:

$$ {}_{25}p_{20} \ , \quad \ddot{a}_{\overline{20:25}|} \ , \quad \mathrm{P}_{\overline{20:25}|} \ , \quad {}_{25}P_{20} \ , \quad v^{25} $$

where

$$ \mathrm{P}_{\overline{x:n}|} \;=\; \mathrm{A}_{\overline{x:n}|} \Big/ \ddot{a}_{\overline{x:n}|} \qquad \text{and} \qquad {}_tP_x \;=\; A_x \Big/ \ddot{a}_{\overline{x:t}|} $$

(4). A life aged 32 purchases a life annuity of 3000 per year. From tables, we find commutation function values

$$ N_{32} \;=\; 2210 \quad , \qquad N_{34} \;=\; 1988 \quad , \qquad D_{33} = 105 $$

Find the net single premium for the annuity purchased if the first yearly payment is to be received (a) immediately, (b) in 1 year, and (c) in 2 years.

(5). Henry, who has just reached his 70^{th} birthday, is retiring immediately with a monthly pension income of 2500 for life, beginning in 1 month. Using the uniform-failure assumption between birthdays and the commutation function values $M_{70} = 26.2$ and $D_{70} = 71$ associated with the interest rate $i = 0.05$, find the expected present value of Henry's retirement annuity.

(6). Find the cash value of a whole-life insurance for $\$100,000$ on a life aged 45 with yearly premium-payments (which began at issuance of the policy) after 25 years, assuming interest at 5% and constant force-of-mortality $\mu_{40+t} \;=\; 0.07$ for all $t > 0$.

(7). Suppose that 25 years ago, a life then aged 40 bought a whole-life insurance policy with quarterly premium payments and benefit payable at the end of the quarter of death, with loading-factor 4%. Assume that the interest rate used to figure present values and premiums on the policy was 6% and that the life-table survival probabilities used were ${}_tp_{40} \;=\; (60-t)/60$. If the insured is now alive at age 65, then find the face amount of paid-up insurance which he is entitled to — with no further premiums paid and no further loading applied — on a whole-life policy with benefits payable at the end of quarter-year of death.

(7). Verify formulas (6.25) and (6.26).

6.5 Illustration of Commutation Columns

Consider the following articial life-table fragment, which we imagine to be available together with data also for all older ages,on a population of potential insureds:

x	l_x	d_x
45	75000	750
46	74250	760
47	73490	770
48	72720	780
49	71940	790
50	71150	

Let us imagine that the going rate of interest is 5% APR, and that we are interested in calculating various life insurance and annuity risk-premiums for level-benefit contracts and payments only on policy anniversaries ($m = 1$), on lives aged 45 to 50. One way of understanding what commutation columns do is to remark that *all* whole-life net single premiums of this type are calculable directly from the table-fragment given along with the single additional number $A_{50} = 0.450426$. The point is that all of the commutation columns $D_x,\ N_x,\ M_x$ for ages 45 to 49 can now be filled in. First, we use the identity (6.4) to obtain

$$D_{50} = 1.05^{-50}\, 71150 = 6204.54, \qquad M_{50} = D_{50}\, 0.450426 = 2794.69$$

$$N_{50} = D_{50}\, \ddot{a}_{50} = \frac{D_{50}}{d}\,(1 - A_{50}) = \frac{1.05}{0.05}\,(D_{50} - M_{50}) = 71606.99$$

Next we fill in the rest of the columns for ages 45 to 49 by the definition of D_x as $1.05^x\, l_x$ and the simple recursive calculations

$$N_x = N_{x+1} + D_x \quad , \qquad M_x = M_{x+1} + v^{x+1}\, d_x$$

Filling these values in for $x = 49,\, 48,\, \ldots,\, 45$ gives the completed fragment

x	l_x	d_x	D_x	N_x	M_x
45	75000	750	8347.24	106679.11	3267.81
46	74250	760	7870.25	98808.85	3165.07
47	73490	770	7418.76	91390.10	3066.85
48	72720	780	6991.45	84398.64	2972.47
49	71940	790	6587.11	77811.53	2891.80
50	71150		6204.54	71606.99	2794.69

From this table fragment we can deduce, for example, that a whole-life annuity-due of \$2000 per year to a life aged 47 has expected present value $2000\, N_{47}/D_{47} \;=\; 24637.57$, or that a five-year term insurance of $100,000$ to a life aged 45 has net single premium $100000 \cdot (M_{45} - M_{50})/D_{50} \;=\; 5661.66$, or that a whole-life insurance of $200,000$ with level premiums payable for 5 years to a life aged 45 has level pure-risk premiums of $200,000 \cdot M_{45}/(N_{45} - N_{50}) \;=\; 18631.78$.

6.6 Examples on Paid-up Insurance

Example 1. Suppose that a life aged 50 has purchased a whole life insurance for \$100,000 *with level annual premiums and has paid premiums for* 15 *years. Now, being aged 65, the insured wants to stop paying premiums and convert the cash value of the insurance into (the net single premium for) a fully paid-up whole-life insurance. Using the APR interest rate of* 6% *and commutator columns given in Table 6.1 and disregarding premium loading, calculate the face amount of the new, paid-up insurance policy.*

Solution. Applying formula (6.24) to the Example, with $x = 50$, $t = 15$, $B = 100,000$, and using Table 6.1,gives

$$F \;=\; 100,000 \left(D_{65} - \frac{N_{65}\, D_{50}}{N_{50}} \right) \Big/ \left(D_{65} \,-\, d\, N_{65} \right)$$

$$= \; 100,000 \; \frac{1486.01 - \frac{4729.55}{56988.31}\, 12110.79}{1486.01 \;-\; \frac{0.06}{1.06}\, 12110.79} \;\; = \; 60077.48$$

If the new insurance premium were to be figured with a loading such as $L' = 0.02$, then the final amount figured using pure-risk premiums would be *divided* by 1.02, because the cash value would then be regarded as a single risk premium which *when inflated by the factor* $1+L'$ purchases the contract of expected present value $F \cdot \mathrm{A}_{x+t}$.

The same ideas can be applied to the re-figuring of the amount of other insurance contracts, such as an endowment, based upon an incomplete stream of premium payments.

Example 2. Suppose that a 20*-year pure endowment for* 50,000 *on a newborn governed by the life-table and commutator columns in Table 6.1, with seminannual premiums, but that after* 15 *years the purchaser can afford no more premium payments. Disregarding loading, and assuming uniform distribution of death-times within single years of age, what is the benefit amount which the child is entitled to receive at age* 20 *if then alive ?*

Solution. Now the prospective formula for cash value or reserve based on initial benefit amount B is

$$B \left({}_5p_{15}\, v^5 \; - \; {}_{20}p_0\, v^{20} \, \frac{\ddot{a}^{(2)}_{\overline{15:5}|}}{\ddot{a}^{(2)}_{\overline{0:20}|}} \right)$$

which will be used to balance the endowment $F\, A_{\overline{15:5}|}^{\;\;1}$. Therefore, substituting the approximate formula (5.6), we obtain

$$F \;=\; B \cdot \left({}_5p_{15}\, v^5 \; - \; {}_{20}p_0\, v^{20} \, \frac{\alpha(2)\, \ddot{a}^{(2)}_{\overline{15:5}|} - \beta(2)(1 - \, {}_5p_{15}\, v^5)}{\alpha(2)\, \ddot{a}^{(2)}_{\overline{0:20}|} - \beta(2)(1 - \, {}_{20}p_0\, v^{20})} \right) \Big/ \, (\, {}_5p_{15}\, v^5)$$

In the particular example, where $i = 0.06$, $\alpha(2) = 1.000212$, and $\beta(2) = 0.257391$, we find

$$F \;=\; 50000 \cdot \left(1 \; - \; \frac{1.000212\, (N_{15} - N_{20}) \; - \; 0.257391\, (D_{15} - D_{20})}{1.000212\, (N_0 - N_{20}) \; - \; 0.257391\, (D_0 - D_{20})} \right)$$

and using Table 6.1 it follows that $F \;=\; 42400.91$.

6.7 Useful formulas from Chapter 6

$$\text{Commutation Columns} \quad D_y \;=\; v^y\, l_y \quad , \qquad M_x \;=\; \sum_{y=x}^{\infty} v^{y+1}\, d_y$$

p. 150

$${}_nE_x \;=\; \frac{D_{x+n}}{D_x} \quad , \qquad A_x \;=\; \frac{M_x}{D_x} \quad , \qquad A^{1}_{x:\overline{n|}} \;=\; \frac{M_x - M_{x+n}}{D_x}$$

p. 150

$$N_x \;=\; \sum_{y=x}^{\infty} v^y\, l_y \;=\; \sum_{y=x}^{\infty} \; D_y \quad , \qquad \ddot{a}_x \;=\; \frac{N_x}{D_x}$$

p. 151

$$\ddot{a}_{x:\overline{n|}} \;=\; \sum_{k=0}^{n-1} v^{k+x}\, \frac{l_{x+k}}{D_x} \;=\; \frac{N_x - N_{x+n}}{D_x}$$

p. 150

$$M_x \;=\; D_x \,-\, d\, N_x$$

p. 151

$$A^{(m)1}_{x:\overline{n|}} \;=\; \frac{i}{i^{(m)}} \,\cdot\, \frac{M_x - M_{x+n}}{D_x}$$

p. 152

$$\ddot{a}^{(m)}_{x:\overline{n|}} \;=\; \alpha(m)\, \frac{N_x - N_{x+n}}{D_x} \,-\, \beta(m)\,(1 - \frac{D_{x+n}}{D_x})$$

p. 152

$${}_tV^{1}_{x:\overline{n|}} \;=\; A^{1}_{x+t:\overline{n-t|}} \,-\, P^{1}_{x:\overline{n|}} \,\cdot\, \ddot{a}_{x+t:\overline{n-t|}}$$

p. 157

$$ {}_tV_{\overline{x:n}|} \;=\; A_{\overline{x+t:n-t}|} \;-\; P_{\overline{x:n}|} \cdot \ddot{a}_{\overline{x+t:n-t}|} $$

p. 157

$$ P_{\overline{x:n}|} \;=\; \frac{1}{\ddot{a}_{\overline{x:n}|}} \;-\; d $$

p. 157

$$ {}_tV_{\overline{x:n}|} \;=\; \ddot{a}_{\overline{x+t:n-t}|}\Big(P_{\overline{x+t:n-t}|} \;-\; P_{\overline{x:n}|}\Big) \;=\; 1 \;-\; \frac{\ddot{a}_{\overline{x+t:n-t}|}}{\ddot{a}_{\overline{x:n}|}} $$

p. 157

$$ A^{1}_{\overline{x:n}|} \;=\; A^{1}_{\overline{x:t}|} \;+\; v^t\,{}_tp_x\; A^{1}_{\overline{x+t:n-t}|} $$

p. 159

$$ A_{\overline{x:n}|} \;=\; A^{1}_{\overline{x:t}|} \;+\; v^t\,{}_tp_x\; A_{\overline{x+t:n-t}|} $$

p. 159

$$ \ddot{a}_{\overline{x:n}|} \;=\; \ddot{a}_{\overline{x:t}|} \;+\; v^t\,{}_tp_x\; \ddot{a}_{\overline{x+t:n-t}|} $$

p. 159

$$ v^t\,{}_tp_x\;{}_tV^{1}_{\overline{x:n}|} \;=\; -\Big[\,A^{1}_{\overline{x:t}|} \;-\; P^{1}_{\overline{x:n}|}\;\ddot{a}_{\overline{x:n}|}\,\Big] $$

p. 159

$$ v^t\,{}_tp_x\;{}_tV_{\overline{x:n}|} \;=\; -\Big[\,A_{\overline{x:t}|} \;-\; P_{\overline{x:t}|}\;\ddot{a}_{\overline{x:n}|}\,\Big] $$

p. 159

$$ \mathrm{V}^{\;1}_{\overline{x:n}|} \;=\; v^{n-t}\;{}_{n-t}p_{x+t} \;-\; v^n\,{}_np_x\;\big(1 \;-\; {}_t\mathrm{V}_{\overline{x:n}|}\big) $$

p. 160

$$\mathrm{V}^{1}_{\overline{x:n}|} \;=\; (1 - \,v^n\,{}_np_x)\,{}_t\mathrm{V}_{\overline{x:n}|} \;+\; v^n\,{}_np_x \;-\; v^{n-t}\;{}_{n-t}p_{x+t}$$

p. 160

$${}_tV^{1}_{\overline{x:n}|} \;=\; v\,p_{x+t}\;\; {}_{t+1}V^{1}_{\overline{x:n}|} \;+\; v\,q_{x+t} \;-\; P^{1}_{\overline{x:n}|}$$

p. 164

$$\text{Paid-up insurance Amt} \quad = \; B\left(1 \,-\, \frac{\ddot{a}_{x+t}}{\ddot{a}_x}\right) \Big/ \;(1 \,-\, d\,\ddot{a}_{x+t})$$

p. 166

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