Non-Markovian Example

As indicated in class, this is an example of a lumped-state random sequence constructed from a Homogeneous Markov Chain, and we supply calculations to show the lumped-state chain is non-Markovian.

Let \( \{X_k : k \geq 0\} \) be a homogeneous Markov Chain with state-space \( S = \{0, 1, 2\} \) defined by \( X_0 = 1 \) and transition probability matrix

\[
P = \left( P(X_{k+1} = j | X_k = i) \right)_{i,j = 0,1,2} = \begin{pmatrix}
.8 & .1 & .1 \\
.2 & .7 & .1 \\
0 & .1 & .9
\end{pmatrix}
\]

Define a binary-valued random sequence

\[
Y_k = I_{[X_k \geq 1]} = \begin{cases}
1 & \text{if } X_k = 1, 2 \\
0 & \text{if } X_k = 0
\end{cases}
\]

Then one easily calculates \( P(Y_1 = 1) = P(X_1 \geq 1) = .8 \) and

\[
P(Y_2 = 1, Y_1 = 1) = P(Y_2 = 1, Y_1 = 1, X_1 = 1) + P(Y_2 = 1, Y_1 = 1, X_1 = 2)
= (.7)(.8) + (.1)(1) = .66
\]

Next, it is easy to see that

\[
P(Y_2 = 1, Y_1 = 0) = P(X_1 = 0, X_2 \geq 1) = (.2)(.2) = .04
\]

so that \( P(Y_2 = 1) = P(Y_2 = 1, Y_1 = 1) + P(Y_2 = 1, Y_1 = 0) = .7 \). Now

\[
P(Y_3 = 1, Y_2 = 1, Y_1 = 0) = P(X_1 = 0, X_2 = 1, X_3 \geq 1) +
+ P(X_1 = 0, X_2 = 2, X_3 \geq 1) = (.2)(.1)(.8) + (.2)(.1)(1) = .036
\]

and

\[
P(Y_3 = 1, Y_2 = 1, Y_1 = 1) = P(X_1 = 1, X_2 = 1, Y_3 = 1)
+ P(X_1 = 1, X_2 = 2, Y_3 = 1) + P(X_1 = 2, X_2 = 1, Y_3 = 1) + P(X_1 = 2, X_2 = 2, Y_3 = 1)
= (.7)(.7)(.8) + (.7)(.1)(1) + (.1)(.1)(.8) + (.1)(.9)(1) = .56
\]

Finally, pulling these results together we have

\[
P(Y_3 = 1, Y_2 = 1) = .56 + .036 = .596
\]

so that

\[
P(Y_3 = 1 | Y_2 = 1) = \frac{.596}{.66} = .8914 \neq P(Y_3 = 1 | Y_2 = 1, Y_1 = 1) = \frac{.56}{.66} = .8485
\]

which shows that \( \{Y_k : k \geq 0\} \) is not Markovian.