Take-Home Stat 650 Midterm Problems

Instructions. As indicated on the course web-page, you can on a voluntary basis, for Monday 4/10/06, do these problems to modify your score on the in-class midterm test last Monday. Whatever fraction of these problems you get right, I will multiply by \( \frac{3}{4} \) of the difference between 100 and your raw test-score and add to your raw score. The objective of this re-test is to make sure you can do certain types of important, basic problems in the course. All four problems count equally.

(1). A Markov chain on the state-space \( S = \{1, 2, \ldots, 7\} \) has transition matrix

\[
P = \begin{pmatrix}
0 & .2 & 0 & .8 & 0 & 0 & 0 \\
.5 & 0 & .5 & 0 & 0 & 0 & 0 \\
0 & .6 & 0 & .4 & 0 & 0 & 0 \\
.4 & 0 & .6 & 0 & 0 & 0 & 0 \\
0 & .5 & 0 & 0 & 0 & .2 & .3 \\
0 & 0 & 0 & 0 & 0 & .8 & .2 \\
0 & 0 & 0 & 0 & 0 & .2 & .8 \\
\end{pmatrix}
\]

Find \( \lim_{k \to \infty} P^{2k} \). Does the limit of \( P^n \) as \( n \to \infty \) exist?

(2). Let \( N \geq 4 \) be any fixed integer, and consider the Markov chain \( \{X_k\}_{k \geq 0} \) on state-space \( \{0, 1, 2, \ldots, N+1\} \) whose transitions have the property that the states 0 and \( N + 1 \) are absorbing, and that given \( X_k \), the increment \( X_{k+1} - X_k \) is chosen with equal probability from \( \{-1, 1\} \) when \( X_k \in \{1, N\} \) and from \( \{-2, -1, 1, 2\} \) when \( 2 \leq X_k \leq N - 1 \).

(a) Show that \( X_k \) is a martingale, and find \( P_x(X_k \text{ hits 0 before } N+1) \), where \( x \neq 0, N + 1 \).

(b). Let \( Y_0 = 0 \), and for positive \( k \), \( Y_k = \sum_{i=0}^{k-1} I_{[X_i=1 \text{ or } N]} \). In addition, define the stopping time \( \tau \), the first hitting time for the absorbing states \( \{0, N + 1\} \), i.e.,

\[
\tau = \inf \{ k \geq 1 : X_k = 0 \text{ or } X_k = N + 1 \}
\]

and \( M_k \equiv X_k^2 - \frac{1}{2} k + \frac{3}{2} Y_k \), and assume \( X_0 \neq 0, N + 1 \). Then calculate \( E(X_k^2 | X_{k-1}) \) in order to show that \( M_{\min(\tau, k)} \) is also a martingale with respect to \( X_0, X_1, \ldots, X_k \). (See this definition on p.106 of Durrett.)

(3). Suppose that we flip a biased coin, with Heads probability \( \frac{1}{3} \), independently and repeatedly. Let the variable \( W_k = 1 \) if the \( k^{th} \) flip is Heads,
and \( W_k = 0 \) otherwise, for \( k \geq 1 \), and define \( W_0 = 0 \). Define \( Z_0 = 0 \), and for all \( k \geq 0 \), define \( Z_k = 0 \) whenever \( W_k = 0 \), and otherwise

\[
Z_k = \max \left\{ j : 0 \leq j \leq k, \ W_k = W_{k-1} = \cdots = W_{k-j+1} = 1 \right\}
\]

That is, \( Z_k \) is the number of Heads since the last Tail, as of the \( k^{th} \) flip. Show that \( \{Z_k\} \) is a positively recurrent irreducible Markov chain with states equal to the nonnegative integers, and find the expectation of the time \( \tau = V_3 = \min\{k \geq 1 : Z_k = 3\} \).

Hint: in #3, first establish positive recurrence, finding the stationary probability for the state 0; and then use the form of the first few transition probabilities to relate the stationary probabilities \( \pi_0, \pi_1, \pi_2, \pi_3 \).

(4). Suppose that \( T_i \) for \( 1 \leq i \leq 4 \) are iid Exponential(\( \lambda \)) random variables. By using a conditioning argument, find

\[
P(T_4 > \max(T_1, T_2 + T_3))
\]