

Stat 700 In-Class Test, Fall 2009

Instructions. This test is closed-book, but you may bring up to two written or typed 8.5×11 notebook sides of formulas, definitions, etc. as memory aids. You may use a calculator, although you need not simplify numerical-answer problems to decimal answers. (Numerical expressions which can be evaluated on a calculator will be good enough for full credit.) In every problem where you are asked to ‘find’ or ‘calculate’ something, unless the method is completely standard you must explain or justify briefly that your method finds the requested information.

(1). Suppose that the data-observation $Y \in \{0, 1, 2, 3, 4\}$ (the number of successes in 4 tosses of a coin with heads-probability ϑ) is distributed given the parameter ϑ as $\text{Binom}(4, \vartheta)$, and that $\vartheta \in [0, 1]$ follows the prior density $\text{Unif}(0, 1)$.

(a) Find the Bayes optimal action $a \in \mathcal{A} \equiv [0, 1]$ subject to the loss function $L(\vartheta, a) \equiv (\vartheta - a)^2$ when the observed data value is $Y = 3$.

(b) Note that our usual theorems (from Sec. 1.4) about calculating the Bayes rule hold only when \mathcal{A} is convex. If in part (a) the allowed actions consist only of $\mathcal{A} \equiv \{1/4, 1/2, 3/4\}$, then find the Bayes optimal action based on $Y = 3$.

(2). The random 3-vector $W = (W_1, W_2, W_3)$ is assumed to have the multivariate normal distribution

$$\begin{aligned} W &\sim \mathcal{N}\left(\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{pmatrix}\right) \\ &= \mathcal{N}\left(\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}^{\otimes 2} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}^{\otimes 2} + \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}^{\otimes 2}\right) \end{aligned}$$

where recall that for a column vector \mathbf{v} , the notation $\mathbf{v}^{\otimes 2}$ is defined by $\mathbf{v}^{\otimes 2} = \mathbf{v} \mathbf{v}^{tr}$.

(a). Find the distribution of $(W_1 - W_2, W_1 + W_2 - 2W_3)$ either by giving the joint density or by identifying the distributional type and specifying the parameter values.

(b). Express W in the form $\underline{\mu} + B\mathbf{Z}$ for a random 3-vector \mathbf{Z} with *iid* standard normal components, where B is a symmetric 3×3 matrix.

(3). Suppose that *iid* random variables X_i have the density

$$f_X(x) \equiv f_X(x, \lambda, \mu) = \frac{1}{2} \lambda e^{-\lambda x} I_{[x>0]} + \frac{1}{2} \mu e^{\mu x} I_{[x<0]}$$

where $\vartheta \equiv (\lambda, \mu) \in (0, \infty)^2$.

(a). Show that $\vartheta = (\lambda, \mu) \in \Theta_1 = (0, \infty)^2$ is identifiable if the observed data are (X_1, X_2, \dots, X_n) .

(b). Show that $\vartheta = (\lambda, \mu) \in \Theta_1 = (0, \infty)^2$ is **not** identifiable if the observed data are $(|X_1|, |X_2|, \dots, |X_n|)$.

(c). Specify as large a subset $\Theta_2 \subset (0, \infty)^2$ as you can so that $\vartheta \in \Theta_2$ is identifiable from data $(|X_1|, |X_2|, \dots, |X_n|)$.

Hint for (b) and (c). To do these parts, find the density of $|X_1|$, e.g. by calculating $\lim_{\epsilon \rightarrow 0} \epsilon^{-1} P(t < |X_1| < t + \epsilon)$ for $t > 0$.

(4). A sample of two-dimensional observations (X_i, Y_i) for $i = 1, 2, \dots, n$ are drawn from the density

$$f(x, y; \vartheta) \equiv f_{X,Y}(x, y; \vartheta) = C (x^2 + y^2)^\alpha e^{-(x^2+y^2)\beta} \quad , \quad x, y \in \mathbf{R}^2$$

where $\vartheta = (\alpha, \beta)$, $\alpha, \beta > 0$, and $C = C(\alpha, \beta)$ is an appropriately defined constant.

(a). Show that with $Z_i = X_i^2 + Y_i^2$, the vector statistic (Z_1, Z_2, \dots, Z_n) is sufficient for ϑ .

(b). Find the distribution of (Z_1, Z_2, \dots, Z_n) .

(c). Find a two-dimensional sufficient statistic for ϑ based on (X_i, Y_i) . Is it minimal?