First 3 Problems of Stat 701 HW3, Due Fri. 3/7.

(1). Each of the variables W_i in the data-sample $W_i,\ i = 1, \ldots, n$, has density

$$f(w, \vartheta) \;=\; \frac{1}{2}\,(1 + \vartheta w)\, I_{(-1,1)}(w)$$

with unknown parameter assumed to lie in the interval $(-1, 1)$.

(a) Find a general lower bound for the variance of all unbiased estimators of ϑ based on the W-sample.

(b) Find the Method of Moments Estimator of ϑ based on the W-sample of size n.

(c). Give the maximum likelihood estimating equation for ϑ based on the sample. Do not try to solve it, but show that its solution is unique.

(2). Let $X_1, \ldots, X_n$ be a sample from a population with probability density

$$2\,|x - \vartheta|^3 \,\exp(-(x - \vartheta)^4), \qquad -\infty < x < \infty$$

where $\vartheta \in \mathbf{R}$ i an unknown parameter.

(a) Find the method of moments estimator of ϑ and, after an appropriate normalization, its nondegenerate limiting distribution. (Hint: Recall the identities $\Gamma(y+1) = y\Gamma(y)$ for $y > 0$, and $\Gamma(1/2) = \sqrt{\pi}$).

(b) Calculate the Fisher information on ϑ and the asymptotic efficiency of the method of moments estimator.

(3). Consider a sample $X_1, \ldots, X_n$ from the population with density

$$f(x) = \frac{1 + \lambda(x - \mu)^2}{1 + \lambda}\,\phi(x - \mu)$$

where $\phi(\cdot)$ is the standard normal density and $(\mu, \lambda) \in \mathbf{R} \times [0, \infty)$ is unknown. *Take as given that for a standard normal r.v.* Z, $E(Z^2) = 1$, $E(Z^4) = 3$, *and* $E(Z^6) = 15$.

(a). Find the method of moments estimator of (μ, λ), and show that it is consistent.

(b). For the particular case $\mu = 0, \lambda = 1$, find the limiting nondegenerate (joint) distribution, as $n \to \infty$, of a suitably centered and scaled version of your estimator in part (a).

(c). Compare the asymptotic variance of your estimator in (b) to the corresponding asymptotic variance of the MLE. (Requires numerical integration.)