STAT 770 Sep. 14 Lecture Part B
Loose Ends from Previous Lecture Segments

Reading for this lecture:

Section 1.6 in Agresti for Bayes, Sections 1.2.5 and 1.5 for Multinomial and Poisson likelihoods.

We consolidate topics mentioned in previous lectures:

(i). Bayes Estimates & Intervals for Binomial/Multinomial data.

(ii). Connection between Multinomial & Poisson likelihoods from different data-collection methods for 2×2 contingency tables.

(iii). LRT in Multinomial Setting.

Bayes Multinomial Examples

Let $\underline{\mathbf{Y}} = \{X_a\}_{a=1}^n$ with $P(X_a = j) = \beta_j$ if $1 \le j \le d$, and $P(X_a = d+1) = 1 - \beta_1 - \cdots - \beta_d$, $\quad N_j \equiv \sum_{a=1}^n I_{[Y_a = j]}$

Dirichlet($\underline{\mu}$) **prior** $\propto \left[\prod_{j=1}^d \beta_j^{\mu_j - 1}\right] \cdot (1 - \sum_{j=1}^d \beta_j)^{\mu_{d+1}-1}$

where $\beta_j > 0$, $j = 1, \ldots, d$, such that $\beta_1 + \cdots + \beta_d < 1$

Posterior $f(\beta \,|\, \underline{\mathbf{Y}}) \sim$ Dirichlet$(\mu_1 + N_1, \ldots, \mu_{d+1} + N_{d+1})$

$\beta_j \sim$ Beta$\Big(\mu_j + N_j, \sum_{k \le d+1:\, k \ne j} (\mu_k + N_k)\Big)$ given $\underline{\mathbf{Y}}$

$$E(\beta_j \,|\, \underline{\mathbf{N}}) = (\mu_j + N_j) \Big/ \sum_{k=1}^{d+1} (\mu_k + N_k)$$

Bayes Data Examples

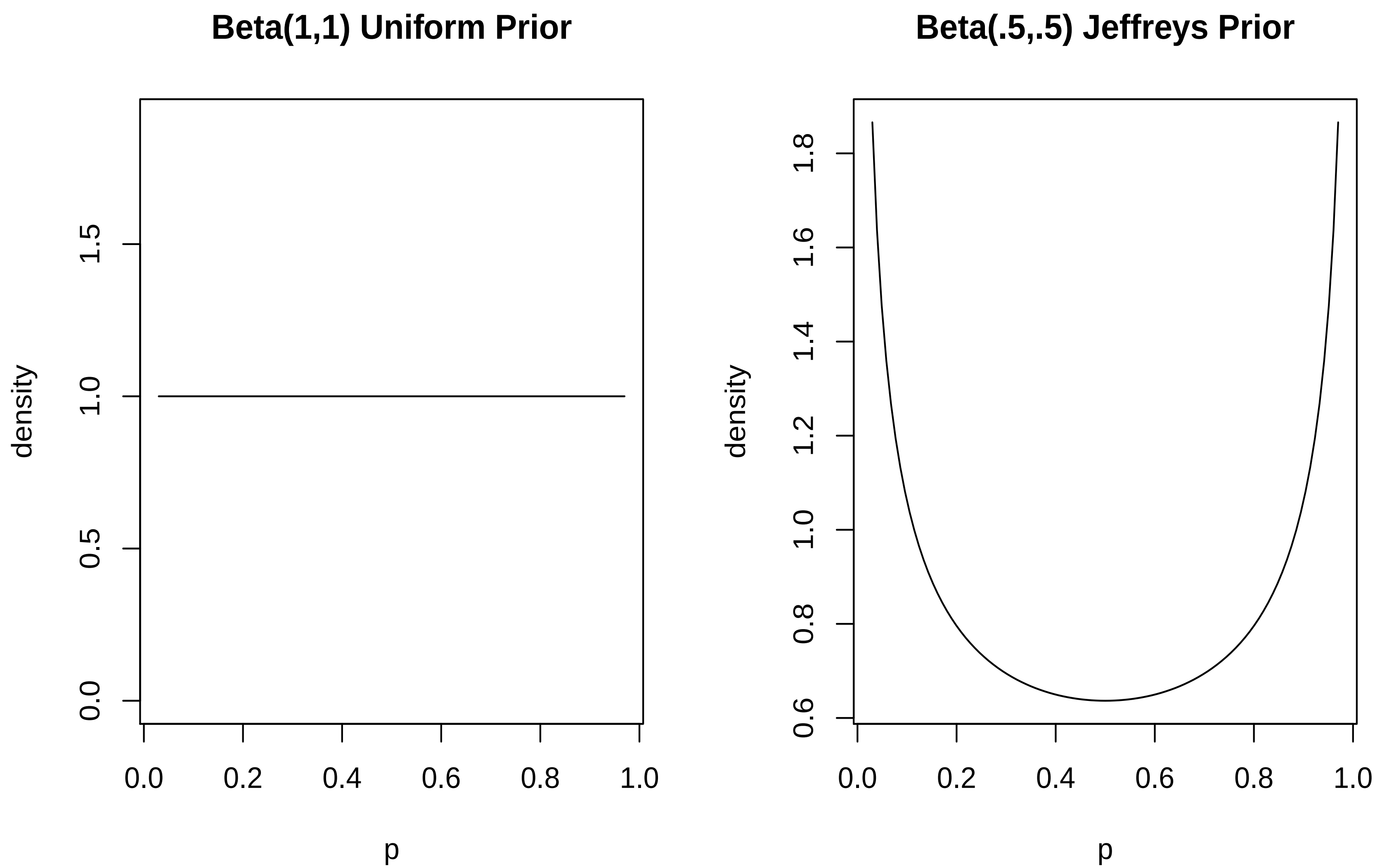
(1). Suppose $Y_a \sim$ Bernoulli(p), $a = 1, \ldots 50$, $N_1 = 13$. Consider priors: **Jeffreys** Beta$(1/2, 1/2)$, or **Uniform** Beta$(1, 1)$.

Estimates: Jeffreys $13.5/51 = 0.265$, Uniform $14/52 = 0.269$

Intervals: Jeffreys `qbeta(c(.025,.975),13.5,37.5)` $= (.154, .393)$

Uniform `qbeta(c(.025,.975),14,38)` $= (.159, .396)$

Priors pictured on next slide


Beta(1,1) Uniform Prior
Beta(.5,.5) Jeffreys Prior
density
p
0.0
0.5
1.0
1.5
0.2
0.4
0.6
0.8
1.2
1.4
1.6
1.8

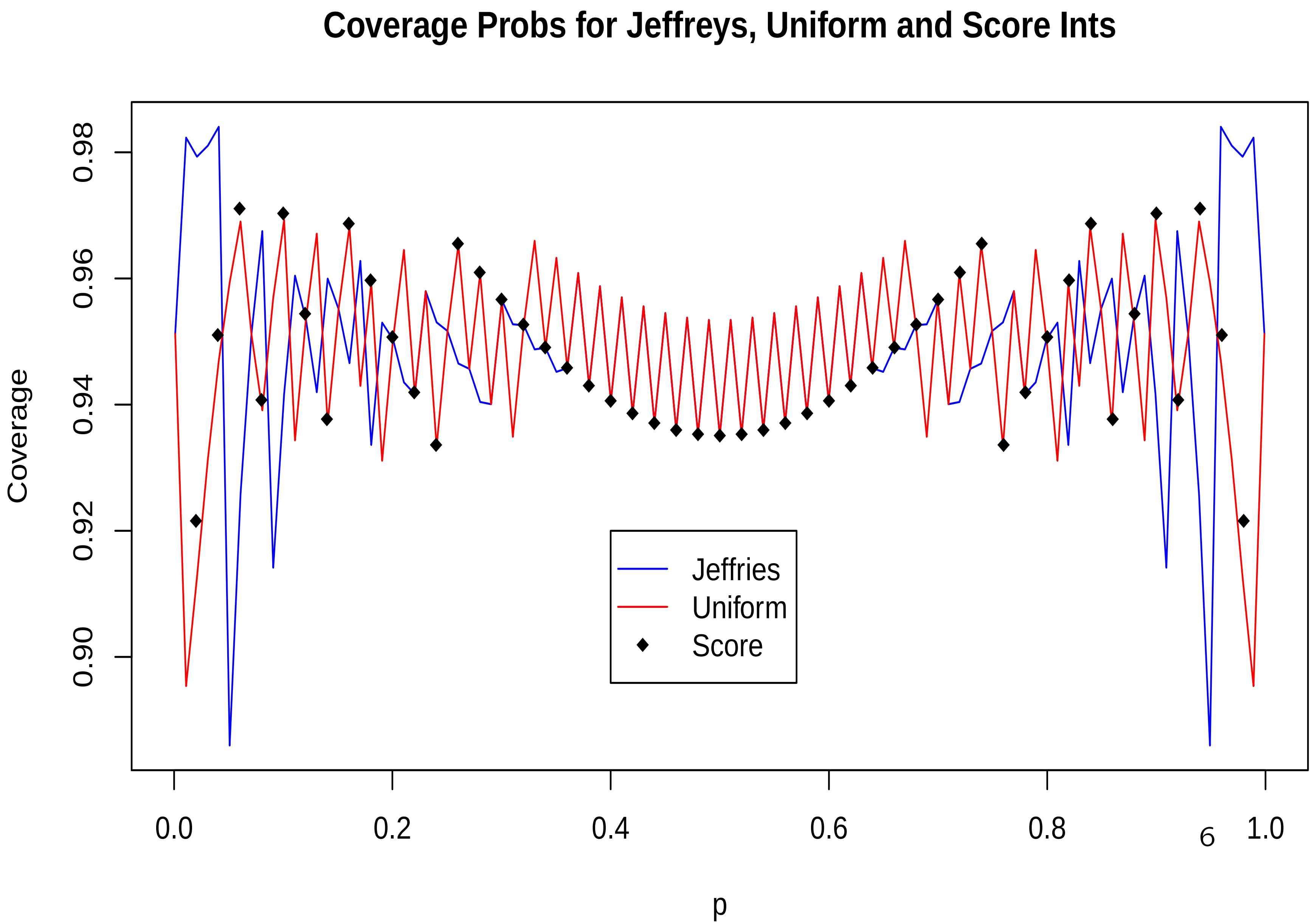
Bayes Credible Interval Coverage

(2). Coverage: as function of (n,p), for fixed $b = 1/2$ (Jeffreys) or $b = 1$ (Uniform)

$\sum_{k=0}^{n}$ `dbinom(k,n,p) I[qbeta(.025,k+b, n-k+b) <`

`p < qbeta(.975,k+b,n-k+b)]`

Plotted coverage together with that of Score Interval, for n=50, on next slide.


Coverage Probs for Jeffreys, Uniform and Score Ints
Coverage
0.90
0.92
0.94
0.96
0.98
p
0.0
0.2
0.4
0.6
0.8
1.0
Jeffries
Uniform
Score

Multinomial, Binomial, Poisson Likelihood (2×2)

Consider 2×2 table with counts N_{zx} with marginal totals, N_{z+}, etc. and underlying parameters π_{zx}

	X=0	1	Tot
Z=0	N_{00}	N_{01}	N_{0+}
1	N_{10}	N_{11}	N_{1+}
Tot	N_{+0}	N_{+1}	N_{++}

Three scenarios possible with likelihood $L(\underline{\pi}) \propto \prod_{z,x=0,1} \pi_{zx}^{N_{zx}}$

(1). n fixed, $(N_{zx},\ z,x = 0,1) \sim \text{Multinom}(n,\ \underline{\pi})$

(2). N_{z+} fixed, $N_{z1} \sim \text{Binom}(n,\ \pi_{z1})$, $\pi_{z+} = 1$

(3). all $N_{zx} \sim \text{Poisson}(n\pi_{zx})$ indep., $\pi_{++} = 1$

Scenario (3) is like the Poisson problem (B) in HW1, or eq. (1.5) on p.8.

Multinomial 2×2 LRT Example (Sec.2.1.6)

$Z_a \in \{0,1\}$ random (Seat-belt use), $X_a \in \{0,1\}$ (Fatal accident)

$$\beta = (\gamma, \lambda_1, \lambda_2) = \Big(p_{11}/(p_{+1}p_{1+}),\, p_{+1},\, p_{1+}\Big), \qquad K = 4$$

with $\gamma = p_{11}/(p_{+1}p_{1+}) = 1$ under row-column independence.

Mmodel is $\pi_{11}(\gamma, \lambda) = \gamma\lambda_1\lambda_2$, $\pi_{+1} = \lambda_1$, $\pi_{1+} = \lambda_2$, $\pi_{++} = 1$.

unrestricted MLE $\widehat{p}_{zc} = N_{zc}/n$, restricted MLE maximizes

$(\lambda_1\lambda_2)^{N_{11}}\,(\lambda_1 - \lambda_1\lambda_2)^{N_{01}}\,(\lambda_2 - \lambda_1\lambda_2)^{N_{10}}\,((1-\lambda_1)(1-\lambda_2))^{N_{00}}$

which occurs (**check it!**) at $(\widehat{\lambda}_1)_r = N_{+1}/n$, $(\widehat{\lambda}_2)_r = N_{1+}/n$

$$X^2 = \sum_{z,c} \frac{(N_{zc} - E_{zc})^2}{E_{zc}}, \; E_{11} = \frac{N_{+1}N_{1+}}{n}, \; E_{1+} = \frac{N_{1+}}{n}, \; E_{+1} = \frac{N_{+1}}{n}$$