STAT 770 Sep. 21 Lecture Part A
Testing & Estimation for 2-way Tables

Reading for this lecture:

Chapter 2 in Agresti, plus review of Delta Method in Sec. 16.1.

We collect LRTand MLE-based Hypothesis Testing examples for 2-way tables, under various conditioning and parameterization.

Along the way, we discuss the (Univariate) Delta Method as a general way to obtain limiting normal distribution for a transformed parameter MLE.

Recall: LRT in Contingency Table Setting

$Y_a = (Z_a, X_a)$ Multinomial with probabilities $p_{z,c}$

$\theta = \{p_{z,c} : \ (z,c) \in \mathcal{K}\}, \quad \beta = (\theta_1, \ldots, \theta_d), \ d = |\mathcal{K}| - 1$

$$L(\beta; \underline{\mathbf{Y}}) = (\text{multinom. coeff.}) \cdot \prod_{(z,c)\in\mathcal{K}} p_{z,c}^{N_{z,c}}$$

Lower dimensional model $p_{z,c} = \pi_{z,c}(\gamma_0, \lambda)$ is Null Hypothesis
(Many examples will follow !)

So LRT $\Lambda = G^2 = -2 \log \left[L(\{\pi_{z,c}(\gamma_0, \widehat{\lambda}_r\}) \Big/ L(\{\widehat{p}_{x,c}\}) \right]$

$$= 2 \sum_{(z,c)\in\mathcal{K}} N_{z,c} \log \left(\frac{N_{z,c}/n}{\pi_{z,c}(\gamma_0, \widehat{\lambda}_r)} \right)$$

Row-column independence in 2×2 Tables

Here $Z_a \in \{0,1\}$ are random, $\mathcal{K} = \{0,1\}^2$, $K = 4$ and

$$\beta = (\gamma, \lambda_1, \lambda_2) = \big(p_{11}/(p_{+1}p_{1+}),\, p_{+1},\, p_{1+}\big)$$

with $\gamma = p_{11}/(p_{+1}p_{1+}) = 1$ under row-column independence.

The model is $\pi_{11}(\gamma,\lambda) = \gamma\lambda_1\lambda_2$, $\pi_{+1} = \lambda_1$, $\pi_{1+} = \lambda_2$, $\pi_{++} = 1$.

The unrestricted MLE is $\widehat{p}_{zc} = N_{zc}/n$, $z, c = 0, 1$, while the restricted MLE maximizes the likelihood

$$(\lambda_1\lambda_2)^{N_{11}}\,(\lambda_1 - \lambda_1\lambda_2)^{N_{01}}\,(\lambda_2 - \lambda_1\lambda_2)^{N_{10}}\,((1-\lambda_1)(1-\lambda_2))^{N_{00}}$$

which occurs (**check it!**) at $(\widehat{\lambda}_1)_r = N_{+1}/n$, $(\widehat{\lambda}_2)_r = N_{1+}/n$

$X^2 \overset{\mathcal{D}}{\approx} \chi^2_1$ **from (II) above has the familiar form**

$\sum_{(z,c)} (O_{z,c} - E_{z,c})^2/E_{z,c}$, **with** $O_{z,c} = N_{z,c}$, $E_{z,c} = n\pi_{z,c}$

Estimation (or Wald Testing) in Multinomial 2×2 Tables

Parameter β on last slide is equivalent to $(\pi_{11}, \pi_{01}, \pi_{10})$, so MLE is the function of relative-frequency MLEs:

$$\hat{\lambda}_1 = \hat{\pi}_{+1} = N_{+1}/n\,, \quad \hat{\lambda}_2 = \hat{\pi}_{1+} = N_{1+}/n\,, \quad \hat{\gamma} = \frac{n\,N_{11}}{N_{+1}N_{1+}}$$

How to write down joint dist.'n of $\hat{\beta} - \beta$, or marginal of $\hat{\gamma}$?

This is a case of Jacobian change of variable:

$$\mathbf{b} \equiv (\pi_{11}, \pi_{01}, \pi_{01}) \mapsto \beta \equiv (\gamma, \lambda_1, \lambda_2)$$

$$(b_1, b_2, b_3) \ \mapsto \ \left(\frac{b_1}{(b_1+b_2)(b_1+b_3)},\, b_1 + b_2,\, b_1 + b_3\right)$$

(Univariate) Delta Method

Suppose θ scalar and $\sqrt{n}\,(\tilde{\theta} - \theta) \sim \mathcal{N}(0,\, \sigma^2)$

σ^2 is called **asymptotic variance** or **a.var** of $\tilde{\theta}$

If $\psi = g(\theta)$, $\tilde{\psi} = g(\tilde{\theta})$, with g known and continuously differentiable, then

$$\sqrt{n}\,(g(\tilde{\theta}) - g(\theta) \overset{P}{\approx} \sqrt{n}\, g'(\theta)\,(\tilde{\theta} - \theta) \overset{\mathcal{D}}{\approx} \mathcal{N}(0, (g'(\theta))^2\sigma^2)$$

Can make confidence intervals $\quad \psi \in \hat{\psi} \pm \frac{\Phi^{-1}(1-\alpha/2)}{\sqrt{n}}\,|g'(\tilde{\theta})|\,\tilde{\sigma}$

Multivariate Delta Method

asymptotic distribution: $\sqrt{n}\,(\hat{\mathbf{b}} - \mathbf{b}) \xrightarrow{\mathcal{D}} \mathcal{N}\Big(\mathbf{0},\, \mathtt{diag}(\mathbf{b}) \,-\, \mathbf{b}\mathbf{b}^{tr}\Big)$

smooth transform: $(b_1, b_2, b_3) \leftrightarrow \Big(\frac{b_1}{(b_1+b_2)(b_1+b_3)},\, b_1 + b_2,\, b_1 + b_3\Big)$

Jacobian: $$J^{tr} = \nabla_{\mathbf{b}}\, \beta^{tr} = \begin{pmatrix} \gamma(\frac{1}{\pi_{11}} - \frac{1}{\pi_{+1}} - \frac{1}{\pi_{1+}}) & 1 & 1 \\ -\gamma/\pi_{+1} & 1 & 0 \\ -\gamma/\pi_{1+} & 0 & 1 \end{pmatrix}$$

Taylor approx. $$\sqrt{n}\,(\hat{\beta} - \beta) \overset{P}{\approx} J\,\sqrt{n}\,(\hat{\mathbf{b}} - \mathbf{b})$$

$$\overset{\mathcal{D}}{\approx} \mathcal{N}\Big(\mathbf{0},\, J\Big[\mathtt{diag}(\mathbf{b}) - \mathbf{b}\mathbf{b}^{tr}\Big]J^{tr}\Big)$$

From upper-left of variance matrix, can read off

$$\text{a.var}(\hat{\gamma}) \,=\, \nabla'_{\mathbf{b}}\gamma\,\Big[\mathtt{diag}(\mathbf{b}) - \mathbf{b}\mathbf{b}^{tr}\Big]\,\nabla_{\mathbf{b}}\gamma$$

Testing Equality of Row Proportions in 2×2 Table

In this setting, Z_a values are fixed by design, so the row-totals $N_{z+} = n_z$ are nonrandom and known, and $N_{z1} \sim \text{Binom}(n_z, \pi_z)$, with $\pi_z = p_{z1}/p_{z+}$.

Here we can take $\beta = (\gamma, \lambda)$ in different ways,

with $H_0 : \gamma = 1$ and $\lambda = \pi_0$ under H_0.

Example 1. Relative Risk, RR: $\beta = (\pi_1/\pi_0, \, \pi_0)$

Example 2. Odds Ratio, OR: $\beta = \Big([\pi_1/(1-\pi_1)]/[\pi_0/(1-\pi_0)], \, \pi_0\Big)$

In RR, the restricted MLE (under $\gamma = 1$) maximizes

$$\Big[\prod_{z=0}^{1} \binom{n_z}{N_{z1}}\Big] \pi_0^{N_{11}+N_{01}} (1 - \pi_0)^{N_{10}+N_{00}} = c \cdot \pi_0^{N_{+1}} (1 - \pi_0)^{N_{+0}}$$

In both RR and OR, $\hat{\lambda} = N_{+1}/n$ **and** $E_{z,c} = n_z \pi_0^c (1 - \pi_0)^{1-c}$

Delta Methods for RR and OR Estimates

RR case: $b = \binom{\pi_1}{\pi_0} \mapsto \beta = \binom{\pi_1/\pi_0}{\pi_0}, \quad \nabla_{\mathbf{b}}\beta_1 = \binom{1/\pi_0}{-\pi_1/\pi_0^2}$

$\widehat{\pi}_z = N_{z1}/n_z \sim \mathcal{N}\Big(\pi_z,\, n_z^{-1}\pi_z(1-\pi_z)\Big)$ **independent**

$$\text{a.var}(\widehat{\beta}_1) = \left(\tfrac{1}{\pi_0}\right)^2 (1, -\beta_1)\begin{bmatrix} \pi_1(1-\pi_1)/n_1 & 0 \\ 0 & \pi_0(1-\pi_0)/n_0 \end{bmatrix}\binom{1}{-\beta_1}$$

So can read off a.var and use approximate normal distribution to construct (Wald-type) CIs.

Log OR in place of OR, two sets of Binomial Trials

logOR: $\mathbf{b} = \binom{\pi_1}{\pi_0} \mapsto \beta = \Big(\log\Big\{\frac{\pi_1(1-\pi_0)}{(1-\pi_1)\pi_0}\Big\},\ \pi_0 \Big)\,, \qquad \frac{\partial \beta_1}{\partial b_z} = \frac{1}{\pi_z(1-\pi_z)}$

$\widehat{\pi}_z \;=\; N_{z1}/n_z \;\sim\; \mathcal{N}\Big(\pi_z,\, n_z^{-1}\pi_z(1-\pi_z)\Big)$ **independent**

$$\text{a.var}(\widehat{\beta}_1) \;=\; \sum_{z=0}^{1} \frac{1}{(\pi_z(1-\pi_z))^2}\,\pi_z(1-\pi_z)\,\frac{n}{n_z} \;=\; \sum_{z=0}^{1} \frac{n}{n_z\pi_z(1-\pi_z)}$$

Again read off a.var and use approx. $\mathcal{N}$ to construct (Wald-type) CIs.

Note. a.var is the variance for the limiting dist'n for $\sqrt{n}(\widehat{\beta}-\beta)$