STAT 770 Feb. 21 Lecture
Misc. Topics on Testing and CI's in 2-way Tables

Reading and Topics for this lecture:

(1) Fisher exact test, Section 3.5.1–3.5.3 (Problem D on HW2)

(2) Nested hypotheses and Partitioning of LRT statistic & X^2, Sec. 3.3.3

(3) Conditional versus marginal independence (Sec. 2.3.4) and use of OR's in 2×2 subtables (Sec. 2.4.1)

(4) Pearson and Standardized Residuals, Sec. 3.3.1-3.3.2

Fisher Exact Test

In 2×2 tables with small n, RA Fisher suggested to condition on all marginals in testing for row-column independence. We saw in HW1(C)(a) that then $N_{11} \sim \text{Hypergeom}(n, N_{1+}, N_{+1})$

$$P(N_{11} = k) \;=\; \binom{N_{1+}}{k}\binom{N_{2+}}{N_{+1}-k}\Big/\binom{n}{N_{+1}} \;=\; \texttt{dhyper}(k, N_{1+}, N_{2+}, N_{+1})$$

Example 1. $N_{1+} = \#$ Defective in n manufactured items, $N_{+1} = m$ sampled, N_{11} defective in sample. **Quality inspection**

Example 2. Tea-tasting or HW 2 (D): do experiment drawing from fixed numbers $D, n - D$ where total number to be drawn is fixed.

In Examples, conditioning on marginals is unnecessary: the experiment fixes them !

Exact Test or CI, Illustration

If a set of 36 people contains 20 who favor Trump and 16 for Biden, but a random sample of 10 has 8 for Biden, the p-value for a 2-sided test that the candidates are tied in this group is

`2*(1-phyper(7,18,18,10)) = 0.060 = 2 * 7613892/choose(36,10)`

and the power of the test that rejects on $[3,7]^c$ is

`phyper(2,16,20,10)+1-phyper(7,16,20,10) = 0.081`

while `phyper(2,30,6,10)+1-phyper(7,30,6,10) = 0.801`

Random sampling enforces row-column independence.

The 1-sided CI for k based on $(H_0 :\geq k$ for Trump) is

$$\{k : 1 - \text{phyper}(7, 36 - \text{k}, \text{k}, 10) > 0.05\} = [0, 16]$$

Idea of Nested Hypothesis Tests

Example: H_A : Ind/Repub row-col independent
H_B : Dem $\times$ Not-Dem row-col independent

```
  PartyTabl
       Dem Indep Repub
Female 762   327   468
Male   484   239   477
```

Nested hypotheses: $H_0 = H_A \cap H_B$ full row-column indep.

$H_1 = H_A$, $H_2 =$ unrestricted multinomial, $H_0 \subset H_1 \subset H_2$

Remark: testing H_B vs H_B^c is just like testing H_0 vs H_1

Model dimensions: $\dim(H_0) = 3$, $\dim(H_1) = 4$, $\dim(H_2) = 5$

Behavior of LRTs in nested setting

Denote restricted MLEs by $\widehat{\theta}(H_k)$, likelihoods by $L(\theta)$

LRT for H_0 versus $H_1 \backslash H_0$: $\quad -2\ \log \left[L(\widehat{\theta}(H_0))/L(\widehat{\theta}(H_1)) \right]$

LRT for H_1 versus $H_2 \backslash H_1$: $\quad -2\ \log \left[L(\widehat{\theta}(H_1))/L(\widehat{\theta}(H_2)) \right]$

LRT for H_0 versus $H_2 \backslash H_0$: $\quad -2\ \log \left[L(\widehat{\theta}(H_0))/L(\widehat{\theta}(H_2)) \right]$

Note that 3rd LRT is the sum of the first two LRT's, and degrees of freedom add ! See `RscriptNested.RLog`

Book relates exact additive partitioning to 'orthogonal Odds Ratios': large-sample independence of partitioned LRTs.

Marginal versus Conditional Indep.

Consider categorical factors X, Y, Z: X, Y conditionally indep given Z if for each i, j, k, $P(X = i \,|\, Y = j,\, Z = k)$ is free of j, or

$$P(X = i,\, Y = j \,|\, Z = k) = P(X = i \,|\, Z = k) \cdot P(Y = j \,|\, Z = k)$$

But generally X, Y are not independent when this is true !

$$P(X = i,\, Y = j) = \sum_k P(Z = k)\, P(X = i|Z = k)\, P(Y = j|Z = k)$$

An important reason why aggregating an inhomogeneous population over an important stratifying variable may mis-lead us about dependence between categorical factors !