STAT 770 Sep. 30 Lectures
HW Topics and Chapter 3 Material

Reading and Topics for this lecture:

(1) HW 2.**(B)** on Delta Method: reading from Secs. 3.1 & 16.1. Material on 2.**(F)** LRT for fit in grouped-data 1-way layout.

(2) LRT, X^2 in 2-way tables: degrees of freedom, Sec. 3.2.1

(3) Wald, Score and LRT for $\widehat{\pi}_1 - \widehat{\pi}_0$, Secs. 3.2.4-3.2.6.

(4) Mention 'ordered classifications', trend tests, Sec. 3.4

(5) Bayes for 2x2 Table proportions, Sec. 3.6

Delta Method Recap

$\beta_1 = g(\pi_1, \pi_0)$ in indep. $N_{z1} \sim \text{Binom}(n_z, \pi_z)$, $\widehat{\pi}_z = N_{z1}/n_z$

$$\nabla\beta_1 = \begin{pmatrix} \partial\beta_1/\partial\pi_1 \\ \partial\beta_1/\partial\pi_0 \end{pmatrix}, \quad \begin{pmatrix} \widehat{\pi}_1 - \pi_1 \\ \widehat{\pi}_0 - \pi_0 \end{pmatrix} \sim \mathcal{N}\Big(\mathbf{0}, \begin{bmatrix} \frac{\pi_1}{n_1}(1-\pi_1) & 0 \\ 0 & \frac{\pi_0}{n_0}(1-\pi_0) \end{bmatrix}\Big)$$

$$\widehat{\beta}_1 - \beta_1 \overset{\mathcal{D}}{\approx} \mathcal{N}\Big(0, \, (\nabla\beta_1)^{tr} \begin{bmatrix} \frac{\pi_1}{n_1}(1-\pi_1) & 0 \\ 0 & \frac{\pi_0}{n_0}(1-\pi_0) \end{bmatrix} \nabla\beta_1\Big)$$

Examples: $g(\pi_1, \pi_0) = \quad \frac{\pi_1}{\pi_0}$ or $\frac{\pi_1(1-\pi_0)}{\pi_0(1-\pi_1)}$ or $\pi_1 - \pi_0$

Extension: vectors of more cell probabililty arguments, with $\text{Cov}(\widehat{\pi}_{zx}, \widehat{\pi}_{z'x'}) = -\frac{1}{n}\pi_{zx} \cdot \pi_{z'x'}$ for $(z, x) \neq (z', x')$

Grouped Data LRT's with Estimated Parameters

Goodness of fit of $f(x,\theta)$ with counts $N_j = \sum_{i=1}^{n} I_{\{X_i \in (a_{j-1},a_j]\}}$

Cells $\left\{(a_{j-1},a_j]\right\}_{j=1}^{K}$, Prob.'s $\pi_j = \pi_j(\theta) = \int_{a_{j-1}}^{a_j} f(x,\theta)\,dx$

$H_0 : f(x,\theta)$, some θ. Restricted MLE $=$ argmax$_\theta \prod_{j=1}^{K} p_j(\theta)^{N_j}$

$$LRT = 2 \sum_{j=1}^{K} N_j \log \left[N_j \Big/ (n\,\pi_j(\widehat{\theta}_r))\right] \approx \chi^2_{K-1-\dim(\theta)}$$

Row-Column Independence in $R \times C$ Tables

$$(N_{zx},\ z = 1,\ldots,R,\ x = 1,\ldots,C) \sim \text{Multinom}(n,\ \{\pi_{zx}\})$$

$H_0 : \pi_{zx} = \pi_{z+}\,\pi_{+x},\quad (\widehat{\pi}_{zx})_r \;=\; N_{z+}\,N_{+x}/n^2,\quad$ dim$= R + C - 2$

$L(\{\pi_{zx}\}) = \prod_{z,x} \pi_{zx}^{N_{zx}},\quad$ Unrestricted $\;\widehat{\pi}_{zx} = N_{zx}/n,\;$ dim $RC - 1$

So $\;G^2 = 2\sum_{z,x} N_{zx} \log\left(\frac{N_{zx}}{n(\widehat{\pi}_{zx})_r}\right) \approx X^2 = \sum_{z,x} \frac{(N_{zx} - n(\widehat{\pi}_{zx})_r)^2}{n\,(\widehat{\pi}_{zx})_r}$

referred to χ^2_{df} distribution, $df = RC - R - C + 1 = (R-1)(C-1)$

Various CI's for Differences of Proportions $\widehat{\pi}_z$

Start with Wald, using the Delta Method given on earlier slide:

$$\widehat{\pi}_1 - \widehat{\pi}_0 - (\pi_1 - \pi_0) \sim \mathcal{N}\Big(0, \frac{\pi_1(1-\pi_1)}{n_1} + \frac{\pi_0(1-\pi_0)}{n_0}\Big)$$

implies CI $\pi_1 - \pi_0 \in \widehat{\pi}_1 - \widehat{\pi}_0 \pm \Phi^{-1}(1-\alpha/2)\Big[\frac{\widehat{\pi}_1(1-\widehat{\pi}_1)}{n_1} + \frac{\widehat{\pi}_0(1-\widehat{\pi}_0)}{n_0}\Big]^{1/2}$

Next, Score & LRT: let $\Delta = \pi_1 - \pi_0$, then $\log L(\Delta, \pi_0) = c +$

$N_{11}\log(\pi_0+\Delta) + N_{10}\log(1-\pi_0-\Delta) + N_{01}\log(\pi_0) + N_{00}\log(1-\pi_0)$

$$\frac{\partial}{\partial\Delta}\log L(\Delta,\pi_0) = (N_{11} - (\pi_0+\Delta)n_1) \Big/ \Big\{(\pi_0+\Delta)(1-\pi_0-\Delta)\Big\}$$

$$\frac{\partial}{\partial\pi_0}\log L(\Delta,\pi_0) = \frac{N_{11} - (\pi_0+\Delta)n_1}{(\pi_0+\Delta)(1-\pi_0-\Delta)} + \frac{N_{01} - \pi_0 n_0}{\pi_0(1-\pi_0)}$$

CI's Inverting LRT for $\pi_1 - \pi_0$

Let $\hat{\pi}_{r,0} = (\hat{\pi}_0(\Delta_0, \pi_0))_r$ maximize $\log L(\Delta_0, \cdot)$

LRT of $H_0 : \Delta = \Delta_0$ accepts when

$$N_{11} \log\left(\frac{N_{11}}{n_1(\Delta_0 + \hat{\pi}_{r,0})}\right) + N_{10} \log\left(\frac{N_{10}}{n_1(1 - \hat{\pi}_{r,0} - \Delta_0)}\right)$$

$$+ N_{01} \log\left(\frac{N_{01}}{n\hat{\pi}_{r,0}}\right) + N_{00} \log\left(\frac{N_{00}}{n(1 - \hat{\pi}_{r,0})}\right) \leq \frac{1}{2} \chi^2_{1,\alpha}$$

Inverted LRT CI is the set of Δ_0 for which the Accept event occurs

CI's Inverting Score Test for $\pi_1 - \pi_0$

Note: "Score Test" (incorrect on Agresti p.79) more complicated than for binomial due to nuisance parameter $\pi_0 \mapsto \widehat{\pi}_{r,0}$.

Score Test for $\Delta = \Delta_0$ rejects when $\Big| N_{11}/n - (\widehat{\pi}_{r,0} + \Delta_0) \Big| \ge \widehat{C}_\alpha$

It is shown in detail [equations (7)-(8)] in the first Asymptotics Handout **(3)** on Stat 770 web-page that

$$\frac{1}{\sqrt{n}} \frac{\partial}{\partial \Delta} \log L(\Delta_0, \, \widehat{\pi}_{r,0}) \; = \; \frac{1}{\sqrt{n}} \Big(\frac{\partial}{\partial \Delta} \; - \; J_{12} J_{22}^{-1} \frac{\partial}{\partial \pi_0} \Big) \log L(\Delta_0, \pi_0)$$

$$\text{a.var}\Big(n^{-1/2} \frac{\partial}{\partial \Delta} \log L(\Delta_0, \widehat{\pi}_{r,0}) \Big) \; = \; J_{11} - J_{12} J_{22}^{-1} J_{21} \qquad \text{where}$$

$$J(\theta) \; = \; \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \; = \; \frac{-1}{n} E\Big[\nabla^{\otimes 2}_{(\Delta, \pi_0)} \log L(\Delta_0, \pi_0) \Big]$$

Calculating the Fisher Info for Score Test

We saw on slide 5 that $\log L(\Delta, \pi_0) =$

$N_{11}\log(\pi_0+\Delta)+N_{10}\log(1-\pi_0-\Delta)+N_{01}\log(\pi_0)+N_{00}\log(1-\pi_0)$

So Fisher Information matrix at (Δ_0, π_0)

using $EN_{z1} = n_z\pi_z, \quad EN_{z0} = n_z(1-\pi_z)$ [with $\pi_1 = \pi_0+\Delta_0$] is

$$J = \begin{pmatrix} n_1/(n\pi_1(1-\pi_1)) & n_1/(n\pi_1(1-\pi_1)) \\ n_1/(n\pi_1(1-\pi_1)) & n_1/(n\pi_1(1-\pi_1)) + n_0/(n\pi_0(1-\pi_0)) \end{pmatrix}$$

Score Test: $\dfrac{(N_{11}-n_1\widehat{\pi}_{r,1})/(\widehat{\pi}_{r,1}(1-\widehat{\pi}_{r,1}))^2}{n_1/(\widehat{\pi}_{r,1}(1-\widehat{\pi}_{r,1}))+n_0/(\widehat{\pi}_{r,0}(1-\widehat{\pi}_{r,0}))} \geq \chi^2_{1,\alpha}$

where $\widehat{\pi}_{r,1} = \Delta_0 + \widehat{\pi}_{r,0}$

Model Parameters and ‘Tests for Trend’

For $R \times C$ **ordinal** tables: think in terms of ‘models’ for π_{zx}

Row-column indep. says: $\pi_{zx} = a_z b_x$ $R + C - 2$ parameters

Alternatives have many more parameters: $(R-1)(C-1)$ more

Assume ordered categories z, x, with ‘scores’ u_z, v_x,

consider models $\pi_{zx} = \exp(\alpha_z + \beta_x + \gamma\, u_z\, v_x)$

To test presence of γ, can look for **correlation** between u_{Z_a}, v_{X_a}

Statistic $M = (n-1)r^2$, $r = \sum_{z,x} u_z v_x N_{zx}/n$
with u,v centered and scaled (mean 0 and s.d. 1)

Bayes and Joint Credible Regions for π_1, π_0

How to test $H_0 : \pi_1 = \pi_0$ using independent binomials ?

Method 1: start with indep. Beta$(\alpha_{z1}, \alpha_{z0})$ priors for π_z. Find quantiles for credible region for π_1/π_0, using indep. posterior. Find quantiles for π_1/π_0 from the indep. posterior densities (given data) $\pi_z \sim$ Beta$(\alpha_{z1} + N_{z1},\, \alpha_{z0} + N_{z0})$.

Example: $N_{11} = 13, N_{10} = 50, N_{01} = 20, N_{00} = 41$, all $\alpha_{zx} = 2$. Posteriors $\pi_1 \sim$ Beta$(15, 52)$, $\pi_0 \sim$ Beta$(22, 43)$. Quantiles can be obtained from integrals or

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> quantile( rbeta(1e6,15,52)/rbeta(1e6,22,43),
                   prob=c(.025,.05,.95,.975) )
  2.5%     5%     95%   97.5%
0.363   0.402   1.046   1.142          Post.Exp. 0.683, MLE 0.629
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Other Approach to Testing $\pi_1 = \pi_0$

Issue is that indep. priors assign prob. 0 to $\pi_1 = \pi_0$.

Prior must allow dependence of π_1, π_0 to handle this.

Example. Joint prior mixture of

indep. π_1 and π_0 (mass p) and $\pi_1 = \pi_0$ (mass $1 - p$).

Mixtures are particularly easy to hand in calculating posteriors: denominator is

$$p \int\int g_1(\pi_1, \pi_0)\, L(\pi_1, \pi_0, \underline{\mathbf{N}}) d\pi_1 d\pi_0 + (1-p) \int g_2(\pi)\, L(\pi, \pi, \underline{\mathbf{N}}) d\pi$$