Automorphic coefficient sums and the quantum ergodicity question

Scott A. Wolpert*

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1 Introduction

The arithmetic functions of elementary number theory have statistical distributions, [7]. In 1849 Dirichlet showed that the divisor function satisfies

$$\sum_{1\le n\le x} d(n) = x\log x + (2\gamma - 1)x + O(x^{1/2})$$

for γ Euler's constant; more generally for $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$ then for α positive, real

$$\sum_{1\le n\le x} \sigma_\alpha(n) = \zeta(\alpha+1)\frac{x^{\alpha+1}}{\alpha+1} + O(x^{\max\{1,\alpha\}}).$$

In 1915 Ramanujan [7] presented the formula

$$\sum_{1\le n\le x} d(n)^2 \sim \frac{x}{\pi^2}(\log x)^3$$

which when combined with the formula of Ingham [4, 12] provides that the coefficient sum

$$S(t,\hat{x}) = \sum_{1\le n\le t} d(n)e^{2\pi i n\hat{x}}$$

satisfies

$$|S(t,\hat{x})|^2 \sim \frac{t}{\pi^2}(\log t)^3$$

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as a positive measure in $\hat{x}$. The formula will serve as our paradigm for coefficient sums. The sums are associated with automorphic eigenfunctions. The multiplicative arithmetic function $\sigma_\alpha(|n|)$ occurs as the Fourier coefficients of the modular Eisenstein series

$$E(z;s) = \frac{1}{2} \sum_{(c,d)=1} \frac{y^s}{|cz+d|^{2s}} \tag{1}$$

for $Re\, s > 1$ and $z = x + iy,\ y > 0$, [2]. The Eisenstein series provides a basic example of an automorphic (non-square integrable) eigenfunction for the Laplace-Beltrami operator associated to the upper half plane $\mathbb{H}$.

We are interested in the statistical properties of automorphic eigenfunctions, particularly of ensembles of Fourier coefficients. The statistics of a large-eigenvalue limit of eigenfunctions presents a model for the transition between quantum and classical mechanics, [3, 6, 8, 10, 15, 17, 18, 23, 24, 25]. The geodesic flow represents time evolution for the classical mechanical system; the flow is ergodic for quotients of hyperbolic space. The *quantum ergodicity question* is to understand the transition between quantum and classical mechanics in the presence of a classical ergodic flow, [1, 3, 9, 16, 17, 18, 23, 24].

We are intrigued by the transition mechanism on the upper half plane. The mechanism involves automorphic eigenfunctions, coefficient sums and geodesic flow. The correspondence principle provides that high-energy eigenfunctions of the hyperbolic Laplacian concentrate along geodesics. Egorov's Theorem provides that a high-energy eigenfunction on a quotient $\Gamma\backslash\mathbb{H}$ gives rise to an almost measure (a distribution) on the unit (co)tangent bundle of the quotient, that is almost geodesic flow invariant, [6, 18]. For Γ a cofinite, non-cocompact, subgroup of $SL(2;\mathbb{R})$ a square integrable automorphic eigenfunction has a Fourier expansion

$$\phi(z) = \sum_{n\neq 0} a_n (y \sinh \pi r)^{1/2} K_{ir}(2\pi|n|y) e^{2\pi i n x} \tag{2}$$

for $z = x+iy \in \mathbb{H}$, eigenvalue $-\lambda = -(\frac{1}{4}+r^2) < -\frac{1}{4}$ for the hyperbolic Laplacian $y^2(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})$ and K_{ir} the Macdonald-Bessel function, [19, 21].

A theme of our investigations is that at high-energy the Egorov concentration measure on the space of geodesics is approximately $|S_\phi|^2$ for the coefficient sum

$$S_\phi(t,\hat{x}) = r^{-1/2} \sum_{1\leq|n|\leq rt(2\pi)^{-1}} a_n e^{2\pi i n \hat{x}} \tag{3}$$

for $(\hat{x}, t)$ certain elementary coordinates on the space of geodesics for $\mathbb{H}$, [22]. We describe applications of our results to coefficients sums.

2 The $SL(2;\mathbb{R})$ formalism

An element $B \in SL(2;\mathbb{R})$ has the unique Iwasawa decomposition

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

which provides for an equivalence of $SL(2;\mathbb{R}) = NAK$ with $S^*(\mathbb{H})^{1/2}$ the square root (double cover) of the unit cotangent bundle to the upper half plane by the rule

$$x + iy = y^{1/2}e^{i\theta}(ai + b), \quad y^{-1/2}e^{i\theta} = d - ic$$

for $z = x + iy \in \mathbb{H}$ and θ the argument for the root cotangent vector measured from the positive vertical, [14]. A symmetric k-tensor $f(z)dz^k$ on $\mathbb{H}$ is lifted to $SL(2;\mathbb{R})$ by first considering the *balanced* tensor $f(z)y^k dz^{k/2}d\bar{z}^{-k/2}$ (the hyperbolic metric is $ds = y^{-1}dz^{1/2}d\bar{z}^{1/2}$) and then associating the function $\tilde{f}(B) = \tilde{f}(z,\theta) = f(z)y^k e^{2ik\theta}$ on $SL(2;\mathbb{R})$. The complex exterior differential ∂ maps forms of type $d\bar{z}^k$ to forms of type $dzd\bar{z}^k$; the product $\partial_{hyp} = y^2\partial$ maps forms of type $d\bar{z}^k$ to forms of type $d\bar{z}^{k-1}$; ∂_{hyp} commutes with the action of $SL(2;\mathbb{R})$ translation. A generalization of the setup is as follows. Functions or symmetric tensors on $\mathbb{H}$ lift to functions on $SL(2;\mathbb{R})$; $SL(2;\mathbb{R})$ acts on functions on $SL(2;\mathbb{R})$ by left translation, the Lie algebra $sl(2;\mathbb{R})$ (containing generalizations of ∂_{hyp} and $\bar{\partial}_{hyp}$) acts by right translation. The action of $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $E^{\pm} = H \pm iV$ are basic to our considerations. The infinitesimal generator of geodesic flow is $H = \frac{1}{2}(E^+ + E^-)$; W is the infinitesimal generator of K, the fiber rotations of $S^*(\mathbb{H})^{1/2}$. In terms of the coordinates (x,y,θ) for $SL(2;\mathbb{R})$ the operator E^+ is simply $E^+ = 4iye^{2i\theta}\frac{\partial}{\partial z} - ie^{2i\theta}\frac{\partial}{\partial\theta}$ (E^+ is the *raising* operator and is closely related to the derivative ∂_{hyp}). We will also consider the Casimir operator $\mathfrak{C} = E^-E^+ - W^2 - 2iW$ and the $SL(2;\mathbb{R})$-invariant volume form $d\mathfrak{V} = y^{-2}dxdyd\theta$ (Haar measure).

3 Helgason's Fourier representation and Zelditch's equation

Helgason's representation theorem for eigenfunctions of the hyperbolic Laplacian is readily presented in terms of the Klein disc model $\mathbb{D}$ [11, 24]. For z in the unit disc $\mathbb{D}$ and b on the boundary $\mathbb{B}$ let $< z, b >$ denote the signed distance from the origin to the horocycle joining z to b. The functions $e^{(2ir+1)<z,b>}$ give a complete set of generalized eigenfunctions for $L^2(\mathbb{D})$

as (r, b) ranges over $\mathbb{R}^+ \times \mathbb{B}$. A smooth eigenfunction u of the hyperbolic Laplacian with eigenvalue $-(\frac{1}{4} + r^2)$ is represented as

$$u(z) = \int_{\mathbb{B}} e^{(2ir+1)<z,b>} dT(b)$$

for a distribution $T \in \mathcal{D}(\mathbb{B})$. Zelditch observed [24] that the integrand can be factored $e^{(2ir+1)<z,b>} dT = u^\infty(z, b) e^{2<z,b>} db$ with the distribution u^∞ having special properties:

1. u^∞ is A-invariant if and only if u is A-invariant for $A \in SL(2; \mathbb{R})$;
2. $Hu^\infty = (2ir - 1)u^\infty$;
3. $Xu^\infty = 0$;
4. u^∞ has the K-expansion $u^\infty = \sum_m u_{2m}$ where $Wu_{2m} = 2imu_{2m}$, $u_0 = u$ (the original eigenfunction) and $E^\pm u_{2m} = (2ir \pm 2m + 1)u_{2m\pm 2}$.

The distribution u^∞ encodes the oscillation of u. Modulo powers of the hyperbolic metric for m positive the term u_{2m} is simply the derivative $\partial^m_{hyp} u$ with the conjugate derivative for m negative. Motivated by considerations of the calculus of pseudo-differential operators Zelditch introduced the following, [24].

Definition 1 *For u, v eigenfunctions of the hyperbolic Laplacian on $\mathbb{H}$ with eigenvalue $-(\frac{1}{4} + r^2)$ set $Q(u, v) = u\overline{v^\infty}$, the microlocal lift of the pair (u, v).*

Zelditch discovered that Q satisfies a second-order differential equation $(H^2 + 4X^2 + 4irH)Q(u, v) = 0$, [24]. A proof based on the above properties is given in [22] (the argument does not involve growth conditions on u or v).

Lemma 2 *$Q(u, v)$ is a distribution on $SL(2; \mathbb{R})$. The geodesic flow derivative $HQ(u, v)$ has magnitude $O(r^{-1})$.*

In particular for $\chi \in C_c(SL(2; \mathbb{R}))$ from Zelditch's equation and integration by parts

$$\int_{SL(2;\mathbb{R})} HQ(u, v)\chi d\mathfrak{V} = \frac{i}{4r} \int_{SL(2;\mathbb{R})} Q(u, v)(H^2 + 4X^2)\chi d\mathfrak{V}.$$

For the right hand integrand $(H^2 + 4X^2)\chi$ is smooth and the contribution of $Q(u, v)$ is bounded by $\|u\|_2 \|v\|_2$ for L^2-norms over a neighborhood of $supp(\chi)$; the right hand side is $O(r^{-1})$.

4 The Macdonald-Bessel functions and the geodesic-indicator measure

We introduce the microlocal lift of the normalized Macdonald-Bessel functions.

Definition 3 *For $t = 2\pi n r^{-1}$, $n \in \mathbb{Z}$, set*

$$\mathcal{K}(z,t) = (y\, r\, sinh\, \pi r)^{1/2} K_{ir}(2\pi|n|y)e^{2\pi i n x}$$

and

$$\mathcal{K}^{\infty,even} = \sum_{m\ even} \mathcal{K}(z,t)_{2m}$$

and for $\Delta t = 2\pi r^{-1}$, set

$$Q(t) = \sum_{k\in\mathbb{Z}} \mathcal{K}(z, t + k\Delta t)\overline{\mathcal{K}^{\infty,even}(z,t)}.$$

$Q(t)$ is the microlocal lift of the Macdonald-Bessel function; Q encodes the oscillation of $\mathcal{K}$ by the sequence of all derivatives; Q satisfies the Zelditch differential equation.

Lemma 4 *$Q(t)$ is an order-four tempered distribution on $SL(2;\mathbb{R})$. Given $0 < t_0 < t_1$, $Q(t)$ is uniformly bounded for $t_0 \leq |t| \leq t_1$ and r large.*

Pre compactness plays a structural role in our considerations. The way is prepared to consider limits.

We first consider geodesic flow invariant measures. To each point of a complete geodesic on $\mathbb{H}$ are associated the forward and backward directed (co)tangents. Each cotangent vector in turn has two square roots in $S^*(\mathbb{H})^{1/2} \approx SL(2;\mathbb{R})$. The association of the four root cotangent vectors to a point of a geodesic provides a lift of the geodesic to $SL(2;\mathbb{R})$ (the lift consists of four complete right action orbits of the subgroup A of the NAK-decomposition).

Definition 5 *For $\widehat{\alpha\beta}$ a geodesic on $\mathbb{H}$ let $\Delta_{\widehat{\alpha\beta}}$ be the Dirac delta measure of flow-time (lifted arc-length) integration over the four root cotangent fields of $\widehat{\alpha\beta}$.*

A non-vertical geodesic on the upper half plane is a Euclidean circle. For $y = (t^{-2} - (x - \hat{x})^2)^{1/2}$ with center $(\hat{x}, 0)$ and radius t^{-1}, then $(\hat{x}, t)$ provides a coordinate for $\mathbb{G}$ the space of non-vertical geodesics. The $SL(2;\mathbb{R})$ invariant area element on $\hat{\mathbb{G}}$, the full space of geodesics on $\mathbb{H}$, is simply $d\hat{x}dt$. We combine a sequence of integral identities and an induction argument on the K-weight using Zelditch's equation to obtain the main result, [22]. Denote the group of integer translations by $\Gamma_{\mathbb{Z}} = \{\left(\begin{smallmatrix} 1 & n \\ 0 & 1\end{smallmatrix}\right) | n \in \mathbb{Z}\}$.

Theorem 6 *Given $0 < t_0 < t_1$ for the geodesic $\widehat{\alpha\beta}$: $y = (t^{-2} - x^2)^{1/2}$ on $\mathbb{H}$*

$$\lim_{r\to\infty} Q(t)\, d\mathfrak{V} = \frac{\pi^2}{8} \sum_{\gamma\in\Gamma_{\mathbb{Z}}} \Delta_{\gamma(\widehat{\alpha\beta})}$$

in the sense of tempered distributions on $SL(2;\mathbb{R})$. The convergence is uniform for $t_0 \leq |t| \leq t_1$ as r becomes large.

The Macdonald-Bessel functions with the proper scaling of parameters concentrate along a single geodesic. The result provides an instance of the correspondence principle independent of the calculus of pseudo-differential operators. The uniform convergence will be used to consider sums of Macdonald-Bessel functions.

5 Applications

We consider four applications of the results to automorphic eigenfunctions. Let Γ be a cofinite non-cocompact subgroup of $SL(2;\mathbb{R})$ containing $\Gamma_{\mathbb{Z}}$ as the stabilizer of infinity.

5.1 General equivalences

For automorphic eigenfunctions ϕ the coefficient sums (3) play a basic role. We begin with a coefficient summation scheme for studying quantities quadratic in the eigenfunction. The scheme provides a positive measure. For $(\hat{x}, t)$ in $\mathbb{R} \times \mathbb{R}^+$ introduce the measure (a tempered distribution)

$$\Omega_{\phi,N} = d_t \mathcal{F}_N * |S_\phi(t, \hat{x})|^2$$

for d_t the Lebesgue-Stieljes derivative in t and for convolution in $\hat{x}$ with the Fejér kernel $\mathcal{F}_N$. The tempered distribution $\Omega_{\phi,N}$ is bounded by $\|\phi\|_2^2$. In fact for $\{\phi_j\}$ a sequence of unit-norm automorphic eigenfunctions the sequences $\{Q(\phi_j, \phi_j)\}$ and $\{\Omega_{\phi_j,N}\}$ are precompact. We can consider a convergent sequence $\{\phi_j\}$ and write $Q_{limit} = \lim_j Q(\phi_j, \phi_j) d\mathfrak{V}$ and $\mu_{limit} = \lim_N \lim_j \Omega_{\phi_j,N}$. For $(\hat{x}, t)$ the above described coordinates the distributions $\Omega_{\phi,N}$ and μ_{limit} are given on $\mathbb{G}$ the space of non-vertical geodesics. An application of our formula is the following relationship

$$Q_{limit} = \frac{\pi^2}{8} \int_{\mathbb{G}} \Delta_{\widehat{\alpha\beta}}\, \mu_{limit}$$

in the sense of tempered distributions on $\Gamma\backslash SL(2;\mathbb{R})$, [22]. A consequence of the construction of $\Omega_{\phi,N}$ is that μ_{limit} is positive; $\Delta_{\widehat{\alpha\beta}}$ is positive and it follows that Q_{limit} is a positive measure. Q_{limit} is Γ-invariant; it follows that the extension (by vertical geodesics forming a null set) of

μ_{limit} to $\hat{\mathbb{G}}$ is Γ-invariant. The action of Γ on $\hat{\mathbb{G}}$ is ergodic relative to the $SL(2;\mathbb{R})$-invariant measure; the Γ-invariance of μ_{limit} is a strong condition on the limits of coefficient sums. The equality $Q_{limit} = d\mathfrak{V}$ is equivalent to the weak* convergence of $|S_{\phi_j}(t,\hat{x})|^2$ to $4\pi^{-2}t$. The *quantum unique ergodicity* conjecture poses that every Q_{limit} is indeed a constant multiple of $d\mathfrak{V}$ [17]; in particular that high-energy microlocal approximate the uniform density.

Good has a related result on coefficient sums for a *fixed* automorphic form, [5]. For Ramanujan's function $\tau(n)$ defined by the weight 12 modular form

$$\Delta = (2\pi)^{12} q \prod_{n=1}^{\infty}(1-q^n)^{24} = (2\pi)^{12}\sum_{n=1}^{\infty}\tau(n)q^n, \ q = e^{2\pi iz}$$

Good considered the coefficient sums

$$S(t,\hat{x}) = \sum_{1\le n\le t}\tau(n)e^{2\pi in\hat{x}}$$

and found an explicit form of weak* convergence involving the Petersson inner product

$$\int_{\hat{x}_1}^{\hat{x}_2}|S(t,\hat{x})|^2 d\hat{x} = \frac{3}{12!\pi^{13}} < \Delta,\Delta >_P (\hat{x}_2-\hat{x}_1)t^{12} + O(t^{12-1/3+\epsilon}).$$

The form is fixed and so the limit in the sum length t is the analog of the high-energy limit. The limit of $|S|^2$ is the uniform density in $\hat{x}$.

5.2 The modular Eisenstein series

Luo and Sarnak followed by Jakobson considered the high-energy limit of the analytic continuation of the modular Eisenstein series on the spectral line, [13, 15]. Even though the Eisenstein series is not square integrable the analysis can be effected. From the Maass-Selberg relation the microlocal lift $Q_E(r) = E(z;\frac{1}{2}+ir)\overline{E(z;\frac{1}{2}+ir)}^{\infty,even}$ has magnitude $\log|r|$. Luo-Sarnak and Jakobson used the hard-analysis estimates available for L-functions and for Kloosterman (exponential) sums to obtain their results. The authors found that $(\log|r|)^{-1}Q_E(r)$ weak* converges to $48\pi^{-1}d\mathfrak{V}$. For limits $Q_{E,limit} = \lim_j(\log|r_j|)^{-1}Q_E(r_j)\,d\mathfrak{V}$ and $\mu_{E,limit} = \lim_N\lim_j(\log|r_j|)^{-1}d_t\mathfrak{F}_N * |S_E(t,\hat{x})|^2$ we find the relationship

$$Q^{symm}_{E,limit} = \frac{\pi^2}{8}\int_{\mathbb{G}}\Delta_{\widehat{\alpha\beta}}\,\mu_{E,limit}.$$

We then find that $(\log|r|)^{-1}Q_E(r)$ converging to $48\pi^{-1}d\mathfrak{V}$ is equivalent to

$$(|\zeta(1+2ir)|^2|r|\,\log|r|)^{-1}|\sum_{1\le n\le rt}\sigma_{2ir}(n)n^{-ir}e^{2\pi in\hat{x}}|^2 \textit{ converging to } 48\pi^{-2}t$$

weak* in $\hat{x}$ for each positive t. The normalization of the sum by the Riemann zeta function is significant since $|\zeta(1+2ir)|$ is known to at least vary between $(\log\log|r|)^{-1}$ and $\log\log|r|$, [20]. The formula can be compared to the Ramanujan and Ingham formulas. The convergence is also suggestive of Good's formula and of the residue formula at $s = 1$ for the Ramanujan identity [20]

$$\sum_{n=1}^{\infty}\frac{|\sigma_{2ir}(n)|^2}{n^s}=\frac{\zeta^2(s)\zeta(s+2ir)\zeta(s-2ir)}{\zeta(2s)},$$

5.3 The spectral average of modular eigenfunctions

Zelditch considers for Γ a cofinite, non-cocompact subgroup of $SL(2;\mathbb{R})$ with orthonormal basis for $L^2(\Gamma\backslash\mathbb{H})$-eigenfunctions $\{(\phi_j,\lambda_j)\}$ and a basis of Eisenstein series $\{E_k\}$ the joint spectral average

$$\sigma_T=\sum_{0\le r_j\le T}Q(\phi_j,\phi_j)+\frac{1}{4\pi}\sum_k\int_{-T}^{T}Q_{E_k}(r)dr\ [25].$$

From the Selberg-Weyl law the spectral contribution in the interval $[-T,T]$ is given as $(4\pi)^{-1}Area(\Gamma\backslash\mathbb{H})\,T^2$. Zelditch shows that

$$\sigma_T\sim T^2$$

in the sense of distributions [25, Theorem 5.1]. For the case of congruence subgroups the spectral contribution of the Eisenstein series has a smaller order of magnitude and thus effectively σ_T is given by the sum $\sum_{r_j\le T}Q(\phi_j,\phi_j)$. It follows from Zelditch's result and our considerations that for congruence subgroups the spectral average of the coefficient sums

$$T^{-2}\sum_{0\le r_j\le T}|S_{\phi_j}(t,\hat{x})|^2\ \textit{converges to}\ \frac{4t}{\pi^2}$$

weak* in $\hat{x}$ for each positive t. The spectral average of the coefficient sums is the uniform density in $\hat{x}$.

5.4 Renormalization of semi-classical limits

It is an open question if high-energy limits are necessarily non trivial; in the absence of unique quantum ergodicity all mass could in the limit escape into the cusps resulting in Q_{limit} and μ_{limit} being trivial. To investigate this possibility we renormalize the eigenfunctions to unit L^2-mass on a compact set and consider the limit of corresponding microlocal lifts [22, Chapter 5]. The first matter is to compare normalizations: we show that the resulting L^2-norms are bounded by $\log \lambda$. The bound is used to establish pre compactness for a sequence of renormalized microlocal lifts and that a high-energy limit has the expected basic properties. The limit is necessarily non trivial. The limit of square coefficient sums is found to be the zeroth Fourier-Stieljes coefficient of a Γ-invariant measure on the space of geodesics. The action of Γ on the space of geodesics is noted to have a compact fundamental set. A lower bound for square coefficient sums results. Given Γ there exist positive constants $t_0 < t_1$ such that for large eigenvalues the mapping $\phi \to (S_\phi(t_1, \theta) - S_\phi(t_0, \theta))$ from eigenfunctions to linear coefficient sums twisted by an additive character is a uniform quasi-isometry relative to the L^2-norms for a suitable compact set and the unit circle. In particular the mapping from eigenfunctions to twisted linear coefficient sums $S_\phi(t_1, \theta)$ is an injection.

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