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Hyperbolic 3-Manifolds With Nonintersecting Closed Geodesics

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Abstract. A hyperbolic 3-manifold is said to have the spd-property if all its closed geodesics are simple and pairwise disjoint. For a 3-manifold which supports a geometrically finite hyperbolic structure we show the following dichotomy: either the generic hyperbolic structure has the spd-property or no hyperbolic structure has the spd-property. Both cases are shown to occur. In particular, we prove that the generic hyperbolic structure on the interior of a handlebody (or a surface cross an interval) of negative Euler characteristic has the spd-property. Simplicity and disjointness are consequences of a variational result for hyperbolic surfaces. Namely, the intersection angle between closed geodesics varies nontrivially under deformation of a hyperbolic surface.

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1. Introduction and Basics

All Riemannian manifolds are assumed to be complete and orientable. A Riemannian manifold M is said to have the *spd-property* if all the closed geodesics in M are simple and pairwise disjoint. More generally, we say that a subgroup Γ of isometries of hyperbolic (Euclidean) space has the *spd-property* if the axes are disjoint for each pair of loxodromics (translations) α and β in Γ not lying in a common cyclic subgroup. In the case that Γ is discrete and torsion free this is equivalent to the quotient hyperbolic (Euclidean) manifold having the spd-property.

A complete Riemannian manifold of constant positive curvature cannot have the spd-property since $\mathbb{S}^3$ has great circles which intersect. Also, as a consequence of the Bieberbach theorems, a flat Riemannian manifold with noncyclic infinite fundamental group cannot have the spd-property (cf. [Wf]). Trivially, any hyperbolic 3-manifold with Abelian fundamental group has the spd-property. Kerry Jones and Alan Reid ([Jn-R]) have shown that all closed arithmetic hyperbolic 3-manifolds have intersecting closed geodesics; in particular, they do not have the spd-property (see also [C-R]). In this paper, we produce the first nontrivial examples of hyperbolic

3-manifolds which possess the spd-property. This is done by showing that the generic hyperbolic structure on the interior of a handlebody (or a surface cross an interval) of negative Euler characteristic has the spd-property. (See Theorem 3.2 for a precise statement.) For a general 3-manifold which admits a geometrically finite hyperbolic structure, we show that either the generic hyperbolic structure has the spd-property or no hyperbolic structure has the spd-property (for a precise formulation see Theorem 2.2).

For the basics on hyperbolic geometry, hyperbolic manifolds, and Kleinian groups the reader is referred to the following books ([Be, Mk], and [R]).

A *Kleinian group* is a discrete subgroup Γ of orientation preserving isometries of hyperbolic 3-space, $\mathbb{H}^3$. The action of Γ on the boundary of $\mathbb{H}^3$ breaks up into two sets: the *limit set*, $\Lambda(\Gamma)$, and the *set of discontinuity*, $\Omega(\Gamma)$. The Kleinian group is said to be of the *second kind* if the set of discontinuity is not empty, otherwise it is said to be of the *first kind*. A *Fuchsian group* is a Kleinian group which keeps a round disc invariant.

Let M be the Riemannian manifold of constant negative curvature, $\mathbb{H}^3/\Gamma$. The *convex core* of M is $\mathrm{CH}(\Lambda(\Gamma))/\Gamma$, where $\mathrm{CH}(\Lambda(\Gamma))$ is the convex hull of $\Lambda(\Gamma)$. A finitely generated Γ is said to be *geometrically finite* if the convex core of M has finite volume.

2. The Representation Space and the spd-Property

Let Γ be a finitely generated group. Let $\mathrm{Hom}(\Gamma)$ be the space of representations of Γ into $\mathrm{PSL}(2, \mathbb{C})$ and $D(\Gamma)$ the (possibly empty) subspace of discrete faithful representations. Let $\mathrm{Hom}(\Gamma)$ and $\mathcal{D}(\Gamma)$, respectively, be the quotients of $\mathrm{Hom}(\Gamma)$ and $D(\Gamma)$ by the action of $\mathrm{PSL}(2, \mathbb{C})$ conjugation of representations.

We begin with observations on the cross ratio. For a quadruple of distinct points on $\mathbb{P}^1$

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)},$$

we find that

$$[z_1, z_2, z_3, z_4] + \frac{1}{[z_1, z_2, z_3, z_4]} = p/q,$$

for

$$p = (z_1 - z_3)^2(z_2 - z_4)^2 + (z_1 - z_4)^2(z_2 - z_3)^2$$

and

$$q = (z_1 - z_4)(z_2 - z_3)(z_1 - z_3)(z_2 - z_4).$$

The polynomials p and q are fixed on interchanging z_1 with z_2 and z_3 with z_4. The quantities p and q are then polynomials in the elementary symmetric functions in $\{z_1, z_2\}$ and $\{z_3, z_4\}$.

Now for $\gamma \in \Gamma$ the elementary symmetric functions in the fixed points of $\rho(\gamma)$, $\rho \in \mathrm{Hom}(\Gamma)$, are actually holomorphic maps from $\mathrm{Hom}(\Gamma)$ to $\mathbb{P}^1$. To see this observe that

$$\rho(\gamma) = \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\} \in \mathrm{PSL}(2, \mathbb{C})$$

varies holomorphically and that the fixed points of $\rho(\gamma)$ are the roots of

$$cz^2 + (d - a)z - b = 0.$$

The quotients $(a - d)/c$ and b/c are holomorphic maps from $\mathrm{Hom}(\Gamma)$ to $\mathbb{P}^1$ and are generators for the elementary symmetric functions in the roots.

We have the following dichotomy:

LEMMA 2.1. *Let Γ be a finitely generated group with a subset $\Gamma_* \subset \Gamma$ and a representation ϕ, such that the transformations $\phi(\alpha)$, $\alpha \in \Gamma_*$, are all loxodromic with disjoint fixed points. For $\mathcal{C}$ the component of* $\mathrm{Hom}(\Gamma)$ *containing ϕ either:*

(1) *there exists a generic subset $V \subset \mathcal{C}$ (a countable intersection of open dense subsets) such that for each $\rho \in V$ and each pair $\alpha, \beta \in \Gamma_*$, $\rho(\alpha)$ and $\rho(\beta)$ have disjoint axes if they are loxodromic, or*

(2) *there exists a pair of elements α and β in Γ_*, such that for each $\rho \in \mathcal{C}$, whenever $\rho(\alpha)$ and $\rho(\beta)$ are loxodromic with disjoint fixed points, then their axes have an intersection with an angle independent of the particular representation.*

Proof. We start by defining a rational function $\mathcal{C}_{\alpha\beta}$, the complex-distance between representing axes. From the above considerations the rule

$$\mathcal{C}_{\alpha\beta} : \rho \mapsto t_{\alpha\beta} + \frac{1}{t_{\alpha\beta}} \quad \text{for } t_{\alpha\beta} = \left[a_{\rho(\alpha)}, r_{\rho(\alpha)}, a_{\rho(\beta)}, r_{\rho(\beta)}\right]$$

can also be expressed as a quotient $p_{\alpha\beta}/q_{\alpha\beta}$ of two polynomials in the elementary symmetric functions in the fixed points of $\rho(\alpha)$ and $\rho(\beta)$. Here, $a_{\rho(\alpha)}$ and $r_{\rho(\alpha)}$ represent the attracting and repelling fixed points, respectively, of $\rho(\alpha)$. Let $Z_{\alpha\beta} \subset \mathcal{C}$ be the union of the zero sets and polar sets of $p_{\alpha\beta}$ and $q_{\alpha\beta}$. $Z_{\alpha\beta}$ is a subvariety (possibly $\mathcal{C}$ itself) that we wish to be *nongeneric*. To this end consider that $q_{\alpha\beta}$ might be the constant 0; this is precluded for α, $\beta \in \Gamma_*$ and ϕ (by disjointness of the fixed points). For $p_{\alpha\beta}$ or $q_{\alpha\beta}$ the constant ∞, then for all $\rho \in \mathcal{C}$ either $\rho(\alpha)$ or $\rho(\beta)$ would have ∞ as a fixed point; this is not possible since $\mathcal{C}$ includes all $\mathrm{PSL}(2, \mathbb{C})$ conjugates of a representation. In brief the zero set of $q_{\alpha\beta}$ and the polar sets of $p_{\alpha\beta}$, $q_{\alpha\beta}$ are all proper subvarieties. Consider next that $p_{\alpha\beta}$ is identically zero, then $t_{\alpha\beta} = \pm\, i$ and the axes of the representing transformations do not lie on a common geodesic plane, hence are necessarily disjoint. In summary for α, β in Γ_* then either the representing transformations always have disjoint axes, or $Z_{\alpha\beta}$ is a proper subvariety.

For the second situation consider the variety $\mathcal{C} - Z_{\alpha\beta}$ with the restriction of $\mathcal{C}_{\alpha\beta}$ (the restriction is defined and holomorphic throughout $\mathcal{C} - Z_{\alpha\beta}$). $\mathcal{C} - Z_{\alpha\beta}$ is connected and thus either:

(1) $\mathcal{C}_{\alpha\beta}$ does not have a constant value in $(-\infty, -2]$, or
(2) $\mathcal{C}_{\alpha\beta}$ does have a constant value in $(-\infty, -2]$.

Observe that $\mathcal{C}_{\alpha\beta}$ has a value in $(-\infty, -2]$ if and only if the axes of $\rho(\alpha)$ and $\rho(\beta)$ intersect. For alternative (1) the preimage $\mathcal{C}_{\alpha\beta}^{-1}((-\infty, -2])$ is a real semi-algebraic set with real codimension at least one. It follows that $G_{\alpha\beta} = \mathcal{C} - Z_{\alpha\beta} - \mathcal{C}_{\alpha\beta}^{-1}((-\infty, -2])$ is an open dense subset of $\mathcal{C}$. The generic set is simply the intersection of the $G_{\alpha\beta}$, for all distinct pairs α, $\beta \in \Gamma_*$ having a representation in $\mathcal{C}$ with intersecting axes. The alternative is given by (2). □

THEOREM 2.2. *Let Γ be a finitely generated torsion-free group with a discrete faithful geometrically finite representation ϕ. Let $\mathcal{C}(\Gamma)$ be the interior of the component (in* Hom(Γ)*) containing ϕ of the discrete faithful geometrically finite representations of Γ where for $\rho \in \mathcal{C}(\Gamma)$, $\rho(\gamma)$ is parabolic if and only if $\phi(\gamma)$ is parabolic. Then either:*

(1) *there exists a subset $V \subset \mathcal{C}(\Gamma)$, a countable intersection of open dense sets, such that for $\rho \in V$, $\mathbb{H}^3/\rho(\Gamma)$ has the spd-property, or*
(2) *there exist a pair of elements α and β in Γ, so that $\rho(\alpha)$ and $\rho(\beta)$ are loxodromic elements whose corresponding closed geodesics have an intersection in $\mathbb{H}^3/\rho(\Gamma)$ with angle constant for all $\rho \in \mathcal{C}(\Gamma)$. In particular, for no $\rho \in \mathcal{C}(\Gamma)$, does $\mathbb{H}^3/\rho(\Gamma)$ have the spd-property.*

Proof. Set $M = \mathbb{H}^3/\phi(\Gamma)$ and note that all representations in $\mathcal{C}(\Gamma)$ are geometrically finite, since this is a quasiconformal deformation space ([Md]). If M has finite volume, Mostow–Prasad rigidity provides that the hyperbolic structure on M is unique and the conclusion trivially follows.

Consider the situation for M having infinite volume. Since the representation ϕ is discrete faithful geometrically finite, the subset $\Gamma_* \subset \Gamma$ of primitive nonparabolic elements is represented by loxodromic transformations with disjoint fixed points. Lemma 2.1 provides for a generic subset V of the ϕ-component of Hom(Γ) or the existence of a pair of elements with axes having fixed intersection angle. In the first case the subset $V \cap C(\Gamma)$ is generic in $C(\Gamma)$ (a countable intersection of open dense sets) since $C(\Gamma)$ is open in Hom(Γ) and it thus follows that the projection of $V \cap C(\Gamma)$ to $\mathcal{C}(\Gamma)$ is likewise generic. Thus one of the two alternatives must occur. □

3. Manifolds with the spd-Property

Clearly the spd-property is inherited by covers. For the converse we have,

PROPOSITION 3.1 *A geometrically finite hyperbolic* 3*-manifold finitely covered by a manifold with the spd-property has the spd-property. In particular, having the spd-property is a commensurability invariant.*

Proof. Let G be a finite index subgroup having the spd-property in the Kleinian group Γ, and let γ and β be loxodromic elements in Γ. Since loxodromic elements have infinite order, it must be that powers of γ and β lie in G. Since a power of a loxodromic element has the same axis as the loxodromic element, the hypothesis provides that γ and β have disjoint axes. □

THEOREM 3.2. *Let $\phi : \Gamma \to \mathrm{PSL}(2, \mathbb{R})$ be a finitely generated torsion free faithful Fuchsian representation with no parabolic elements. Let $\mathcal{C}(\Gamma)$ be the interior of the component of discrete faithful representations containing ϕ. There exists a subset $V \subset \mathcal{C}(\Gamma)$, a countable intersection of open dense sets, so that for any $\rho \in V$, $\mathbb{H}^3/\rho(\Gamma)$ has the spd-property.*

In the case of Γ a cocompact surface group, $\mathcal{C}(\Gamma)$ is quasi-Fuchsian space and in the case of Γ a free group, then $\mathcal{C}(\Gamma)$ is Schottky space. Theorem 3.2 provides that a generic representation in quasi-Fuchsian or Schottky space has the spd-property.

In order to prove the theorem, we consider plane hyperbolic geometry. In the following, by a triangle in $\mathbb{H}^2$, we will mean a three sided polygon with geodesic sides and possibly ideal vertices.

LEMMA 3.3. *Let α and β be (possibly the same) closed geodesics on a compact hyperbolic surface S which transversally intersect at the point p. There exists a simple closed geodesic γ which intersects α and β such that either γ passes through p or there exist geodesic subarcs of α, β, and γ which bound a (possibly non embedded) triangle on S.*

We remark that the lemma would be immediate if the tangent directions to the set of simple closed geodesics were dense in the unit tangent bundle of S. However, for closed hyperbolic surfaces, the totality of simple closed geodesics are nowhere dense even on the surface S, [Jr].

Proof of Lemma. Let μ be a measured geodesic lamination which is complete, that is, the complementary regions are ideal triangles, [B-C]. A point of S lies either interior to a complementary triangle or on a leaf of the lamination. We consider the two possibilities for the intersection point p. Consider first that p is interior to a triangle Δ. Arcs of α and β enter and leave Δ by crossing distinct boundary edges of the triangle. It follows that there are subarcs of α and β (with vertex angle in $(0, \pi)$) which begin at p and cross a common boundary edge of Δ. Recall that each leaf of μ can be geometrically approximated by simple closed geodesics (no leaf is a closed curve; each leaf ℓ recurs infinitely often on some transverse arc; simple closed curves are formed by taking large subarcs of ℓ and suitable connectors along the transverse arc; by the *closing lemma* if the connectors are small the constructed curves are close to closed geodesics). A triangle is formed by the arcs along α, β and an arc along an approximating simple closed geodesic. The remaining possibility is that p lies on a leaf ℓ of the lamination. A totally degenerate triangle is given by the sequence of trivial arcs along α, β then ℓ.

Approximate ℓ in a neighborhood of p by a simple closed geodesic γ. A triangle is given by arcs along α, β and γ. $\square$

PROPOSITION 3.4. *Let G be a finitely generated torsion free Fuchsian group with no parabolic elements, and A, B hyperbolic elements in G with intersecting axes. There exist Fuchsian deformations of G for which the corresponding angle of intersection is arbitrarily close to zero.*

Proof. If G is not cocompact double the Nielsen convex core to obtain $\hat{G}$, cocompact, with $G \subset \hat{G}$. From Lemma 3.3 we can find a simple closed geodesic on $\mathbb{H}^2/\hat{G}$ which transversally intersects the geodesics corresponding to A and B. Lifting the configuration to the hyperbolic plane we find a simple hyperbolic element C in $\hat{G}$ whose fixed points separate the fixed points of A and B. We will deform $\hat{G}$ by Fenchel–Nielsen twisting (earthquaking) along C. Write D_t for the deformation of an element D in $\hat{G}$. The fixed points of A_t and B_t necessarily move relative to the fixed points of C_t, [K, Wp]. Choose the sense of the deformation so that the angle (in the triangle) between the axes of A_t and C_t increases. As the deformation parameter tends to infinity the configuration of fixed points for A_t and C_t degenerates; the intersection angle approaches π. Since the sum of angles of a triangle is bounded by π, the intersection angle between the axes of A_t and B_t must tend to zero. The family $\{D_t \mid D \in G\}$, a deformation of the subgroup G, provides the desired deformation. $\square$

The torsion-free assumption in the proposition is required; in the paper [Mt], Martin shows that there exist quite general Fuchsian groups with even-order torsion having hyperbolic elements whose axes intersect at an angle which remains constant in the PSL(2, $\mathbb{R}$) deformation space.

Proof of Theorem 3.2. Fix α and β in Γ. If the axes of $\phi(\alpha)$ and $\phi(\beta)$ do not intersect then we are done. Suppose that $\phi(\alpha)$ and $\phi(\beta)$ have intersecting axes. Then the holomorphic function $\mathcal{C}_{\alpha\beta}$ restricted to the locus of PSL(2, $\mathbb{R}$) representations is given by the angle of intersection between the axes. (The PSL(2, $\mathbb{R}$) representations are not contained in the polar set of $\mathcal{C}_{\alpha\beta}$.) Proposition 3.4 provides that $\mathcal{C}_{\alpha\beta}$ is not the constant function. Alternative (1) occurs in Theorem 2.2 and the conclusion follows. $\square$

4. Manifolds Without the spd-Property and an Application

We remark that there exist geometrically finite Kleinian groups of the second kind having the property that no discrete faithful deformation in the component containing the identity representation is spd. The construction of such a group involves starting with an infinite volume geometrically finite hyperbolic 3-manifold containing a totally geodesic thrice punctured sphere whose ends are contained in $Z + Z$ cusps of the ambient manifold. Since thrice punctured spheres have no moduli, any quasiconformal deformation of our manifold would not alter the totally geodesic character of the thrice punctured sphere. Hence, the deformed structure would continue to be non-spd.

The current considerations have consequences for Poincaré series. In general, the variational differential of the complex-distance between axes is represented by the Poincaré series of the rational function with poles at the axes ends. The nontrivial variation of the complex-distance provided in Proposition (3.4) provides for the generic nontriviality of the associated corresponding Poincaré series, [Kr,Wp].

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