Semi-classical limits for the hyperbolic plane

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1 Introduction

The correspondence principle of quantum mechanics provides that high-energy eigenfunctions of the Laplace-Beltrami operator concentrate along geodesics. The nature of the concentration for automorphic eigenfunctions and ergodic geodesic flow is of particular interest [2, 3, 4, 5, 6, 9, 17, 18, 20, 27, 32, 37, 38, 39, 40, 49, 50, 51, 52, 53]. For Γ a cofinite, non-cocompact, subgroup of $SL(2;\mathbb{R})$ a cuspidal automorphic eigenfunction for the hyperbolic Laplacian D has a Fourier series expansion

$$\varphi(z) = \sum_{n\neq 0} a_n (y \sinh \pi r)^{1/2} K_{ir}(2\pi|n|y) e^{2\pi inx} \tag{1.1}$$

for $z = x+iy,\ y > 0$, the variable for the upper half plane $\mathbb{H}$, eigenvalue $-\lambda = -(\frac{1}{4}+r^2) < -\frac{1}{4}$ and K_{ir} the Macdonald-Bessel function [42, 45]. Also associated to an eigenfunction φ are the coefficient sums

$$S_\varphi(t,\hat{x}) = (\pi r^{-1})^{1/2} \sum_{1\leq|n|\leq rt(2\pi)^{-1}} a_n e^{2\pi in\hat{x}};$$

the sums are of independent interest [8, 10, 14, 29, 41]. A theme of our investigations is that at high-energy the concentration measure on the space of geodesics is approximately given in terms of $|S_\varphi|^2$.

Our results have applications for general cofinite, non-cocompact subgroups, as well as, for congruence subgroups of $SL(2;\mathbb{Z})$. The first application concerns a congruence subgroup Γ and unit-norm eigenfunctions; the spectral average of the sum squares

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$|S_\varphi(t,\hat{x})|^2$ weak* converges in $\hat{x}$ to $8t(Area(\Gamma\backslash\mathbb{H}))^{-1}$. In a second application the results of W. Luo-P. Sarnak [32] and D. Jakobson [27] for the uniform distribution for the limit of the modular Eisenstein series are interpreted as a limit-sum formula for the elementary summatory function. In a third application we present a lower bound for square coefficient sums and note that a mapping from eigenfunctions to coefficient sums is a uniform quasi-isometry.

At the center of our investigation is the microlocal lift of an eigenfunction introduced by S. Zelditch [48, 49, 50, 51, 52]. Zelditch first observed that a ΨDO calculus can be based on Helgason's Fourier transform [48, Secs.1,2], [50, Secs.1,2]. The basic construction is the microlocal lift, a finite-order distribution that encodes the oscillation for a pair of eigenfunctions. For u, v functions on the upper half plane $\mathbb{H}$, each satisfying the differential equation $Df = -(\frac{1}{4}+r^2)f$, we write $\tilde{u}, \tilde{v}$ for the standard lifts to $SL(2;\mathbb{R})$ and consider the sequence

$$\begin{aligned} u_0 &= \tilde{u} \\ v_0 &= \tilde{v} \\ (2ir+2m+1)v_{2m+2} &= E^+ v_{2m} \\ (2ir-2m+1)v_{2m-2} &= E^- v_{2m} \end{aligned}$$

for E^+, respectively E^-, the $SL(2;\mathbb{R})$ raising, respectively lowering, operator; v_{2m} is in the weight $2m$ representation for the compact subgroup of $SL(2;\mathbb{R})$, [30]. The microlocal lift for the pair is defined by $Q(u,v) = \overline{u_0}\sum_m v_{2m}$. For $\sigma \in C^\infty(SL(2;\mathbb{R})\times\mathbb{R})$, a complete symbol for a ΨDO properly supported on $SL(2;\mathbb{R})$ ($\sigma(A,\tau)$ is asymptotically a sum of homogeneous terms in the frequency τ with bounded left invariant derivatives in A) the associated matrix element is

$$2\pi\langle Op(\sigma)v, u\rangle = \int_{SL(2;\mathbb{R})} \sigma_r Q(u,v)\, d\mathcal{V}$$

for σ_r the symbol evaluated at $\tau = r$ and $d\mathcal{V}$ Haar measure [48, 50]. We generalize a result of S. Zelditch [50, Prop.2.1] and show that the microlocal lift satisfies a partial differential equation; the equation is basic to our approach. A second microlocal lift is also considered. The Fejér sum $Q_M(u) = (2M+1)^{-1}|\sum_{m=-M}^{M} u_{2m}|^2$ provides a positive measure on $SL(2;\mathbb{R})$; the assignment u to $Q_M(u)$ is equivariant for the left action on $SL(2;\mathbb{R})$. An integration by parts argument provides a bound for the difference of $Q(u,v)$ and $Q_M(u)$ for M large of order $O(r^{-1})$ given uniform bounds for $\|u_{2m}\|$. The Fejér sum construction provides an alternative to introducing Friedrichs symmetrization, [9, 48]. Our initial interest is the microlocal lift of the Macdonald-Bessel functions.

The Macdonald-Bessel functions were introduced one-hundred years ago in a paper presented to the London Mathematical Society by H. M. Macdonald [33]. The paper includes a formula for the product of two Macdonald-Bessel functions in terms of the integral of a third Bessel function. We will develop formulas for the microlocal lift from such an identity

$$2K_{ir}(\beta y)K_{ir}(y) = \pi \operatorname{csch} \pi r \int_{\log \beta}^{\infty} J_0(y(2\beta \cosh \tau - 1 - \beta^2)^{1/2}) \sin r\tau d\tau, \ [31, \ 33].$$

To present our main result we first describe a family of measures on $SL(2;\mathbb{R})$. The square root (the double-cover) of the unit cotangent bundle of $\mathbb{H}$ is equivalent to $SL(2;\mathbb{R})$. A geodesic has two unit cotangent fields and four square root unit cotangent fields. The geodesic-indicator measure $\Delta_{\widehat{\alpha\beta}}$ on $SL(2;\mathbb{R})$ is the sum over the four lifts of the geodesic $\widehat{\alpha\beta}$ of the lifted infinitesimal arc-length element. We further write $\Delta_{\Gamma_\infty(\widehat{\alpha\beta})} = \sum_{\gamma\in\Gamma_\infty} \Delta_{\gamma(\widehat{\alpha\beta})}$ for the sum over Γ_∞, the discrete group of integer translations. We consider for $t = 2\pi n r^{-1}$ the microlocal lift of

$$\mathcal{K}(z,t) = (ry \sinh \pi r)^{1/2} K_{ir}(2\pi|n|y)e^{2\pi i n x},$$

and define the distribution

$$Q^{\text{symm}}(t) = \sum_{k,m\in\mathbb{Z}} \mathcal{K}(z, t + k\Delta t)\overline{\mathcal{K}(z,t)_{4m}}$$

for $\Delta t = 2\pi r^{-1}$. We show in Theorem 4.9 that for $\widehat{\alpha\beta}$ the geodesic on $\mathbb{H}$ with Euclidean center the origin and radius $|t|^{-1}$ and $d\mathcal{V}$ Haar measure then in the sense of tempered distributions

$$Q^{\text{symm}}(t)d\mathcal{V} \ \text{ is close to } \ \frac{\pi^2}{8}\Delta_{\Gamma_\infty(\widehat{\alpha\beta})}$$

uniformly for r large and $|t|$ restricted to a compact subset of $\mathbb{R}^+$. *At high-energy the microlocal lift of a Macdonald-Bessel function is concentrated along a single geodesic.* Accordingly, at high-energy the behavior of the microlocal lift of a sum of Macdonald-Bessel functions is explicitly a matter of the space of geodesics and sums of coefficients.

We introduce in Chapter 3 a coefficient summation scheme for studying quantities quadratic in the eigenfunction. The scheme provides a positive measure. In particular for $(\hat{x}, t) \in \mathbb{R} \times \mathbb{R}^+$ define the distribution

$$\Omega_{\varphi,N} = d_t\mathcal{F}_N * |S_\varphi(t,\hat{x})|^2$$

for d_t denoting the Lebesgue-Stieljes derivative in t and for convolution in $\hat{x}$ with the Fejér kernel $\mathcal{F}_N$. We use a slight improvement of the J. M. Deshouillers-H. Iwaniec coefficient sum bound [11] and show for a unit-norm automorphic eigenfunction that $\Omega_{\varphi,N}$ is a uniformly bounded tempered distributions. In fact for $\{\varphi_j\}$ a sequence of unit-norm automorphic eigenfunctions the sequences $\{Q(\varphi_j,\varphi_j)\}$ and $\{\Omega_{\varphi_j,N}\}$ are relatively compact. We consider a weak* convergent sequence and write $Q_{\text{limit}} = \lim_j Q(\varphi_j,\varphi_j)d\mathcal{V}$ and $\mu_{\text{limit}} = \lim_N \lim_j \Omega_{\varphi_j,N}$. For$(\hat{x}, t)$ connoting the geodesic on $\mathbb{H}$ with Euclidean center $\hat{x}$

and radius t^{-1} the distributions $\Omega_{\varphi,N}$ and μ_{limit} are given on $\mathbb{G}$ the space of non vertical geodesics on $\mathbb{H}$. The first application of our overall considerations is presented in Theorem 4.11

$$Q_{\text{limit}} = \frac{\pi}{8}\int_{\mathbb{G}} \Delta_{\widehat{\alpha\beta}}\mu_{\text{limit}} \tag{1.2}$$

in the sense of tempered distributions on $\Gamma_\infty\backslash SL(2;\mathbb{R})$. The integral representation provides basic information. In particular that Q_{limit} is a positive geodesic flow-invariant measure and that μ_{limit} extends to a Γ-invariant measure on $\widehat{\mathbb{G}}$ the space of geodesics on $\mathbb{H}$. Vertical geodesics on $\mathbb{H}$ are found to be null for Q_{limit}. Recall that the action of Γ on $\widehat{\mathbb{G}}$ is ergodic relative to the $SL(2;\mathbb{R})$-invariant measure. The Γ-invariance of μ_{limit} is a significant hypothesis for the limits of the coefficient sums S_φ. The Γ-action has compact fundamental sets and thus an invariant measure is compactly determined. We further find that the equality $Q_{\text{limit}} = d\mathcal{V}$ is equivalent to the convergence of $|S_{\varphi_j}(t,\hat{x})|^2$ to $4\pi^{-1}t$. For the modular group the convergence of $Q(\varphi_j,\varphi_j)$ to unity is found to agree with the residue formula at $s=1$ of the Rankin-Selberg convolution L-function $L_\varphi(s) = \int_{SL(2;\mathbb{Z})\backslash\mathbb{H}} \varphi^2(t)\mathcal{E}(z;s)dA$; $\mathcal{E}(z;s)$ the modular Eisenstein series.

We show in Chapter 5 that the above considerations can be extended to include the modular Eisenstein series. In particular from the Maass-Selberg relation the microlocal lift $Q_{\mathcal{E}}(r) = \mathcal{E}(z;\frac{1}{2}+ir)\sum_m \overline{\mathcal{E}(z;\frac{1}{2}+ir)_{2m}}$ has magnitude comparable to $\log|r|$. For limits $Q_{\mathcal{E},\text{limit}} = \lim_j (\log|r_j|)^{-1}Q_{\mathcal{E}}(r_j)d\mathcal{V}$ and $\mu_{\mathcal{E},\text{limit}} = \lim_N \lim_j (\log|r_j|)^{-1}d_t\mathcal{F}_N * |S_{\mathcal{E}}(t,\hat{x})|^2$ we find

$$Q^{\text{symm}}_{\mathcal{E},\text{limit}} = \frac{\pi}{8}\int_{\mathbb{G}} \Delta_{\widehat{\alpha\beta}}\mu_{\mathcal{E},\text{limit}}$$

in the sense of tempered distributions on $\Gamma_\infty\backslash SL(2;\mathbb{R})$ for $Q^{\text{symm}}_{\mathcal{E},\text{limit}}$the restriction of the distribution to functions right-invariant by $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$. The limit of the microlocal lift $Q_{\mathcal{E}}(r)$ is analyzed in the joint work of W. Luo-P. Sarnak [32] and the work of D. Jakobson [27]. The authors find that $(\log|r|)^{-1}Q_{\mathcal{E}}(r)$ weak* converges to $48\pi^{-1}d\mathcal{V}$ relative to $C_c(SL(2;\mathbb{Z})\backslash SL(2;\mathbb{R}))$. Their approach uses bounds for the Riemann zeta function [43], T. Meurman's bounds for the L-function of a cusp form [34], the H. Petersson-N. V. Kuznetsov trace formula [29], as well as, the work of H. Iwaniec [25] and J. Hoffstein-P. Lockhart [23]. We find that $(\log|r|)^{-1}Q_{\mathcal{E}}(r)$ converging to $48\pi^{-1}d\mathcal{V}$ is equivalent to

$$(|\zeta(1+2ir)|^2|r|\log|r|)^{-1}\Big|\sum_{1\le n\le rt}\sigma_{2ir}(n)n^{-ir}e^{in\nu}\Big|^2 \ \text{ converging to }\ 48\pi^{-2}t$$

weak* in ν for each positive t for the summatory function $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$ and r tending to infinity. The convergence is suggestive of the Ramanujan formula [16]

$$\sum_{1\le n\le x} d(n)^2 \sim \frac{x}{\pi^2}(\log x)^3$$

and the Ingham formula [24] for the divisor function (the additive divisor problem); the divisor-sum formulas provide that $S(t,\hat{x}) = \sum_{1\le n\le t} d(n)e^{2\pi i n\hat{x}}$ satisfies

$$|S(t,\hat{x})|^2 \sim \frac{t}{\pi^2}(\log t)^3$$

as a positive measure in $\hat{x}$. The convergence is also suggestive of the residue formula at $s=1$ for the Ramanujan identity

$$\sum_{n=1}^{\infty}\frac{|\sigma_{2ir}(n)|^2}{n^s} = \frac{\zeta^2(s)\zeta(s+2ir)\zeta(s-2ir)}{\zeta(2s)}, \ [36].$$

The *quantum unique ergodicity* conjecture postulates that in general Q_{limit} is a constant multiple of $d\mathcal{V}$, [37, 38]; also see [5, Chap. VIII Sec. 3] and the discussion of the effect of the automorphic condition on the *behavior of expansion coefficients*. The work of W. Luo - P. Sarnak and D. Jakobson establishes the conjecture for the modular Eisenstein series. In a previous work [47] we used a stationary-phase analysis of an $SL(2;\mathbb{R})$-translate of $\mathcal{K}(z,t)$, $z\in\mathbb{H}$, to study the coefficient sums S_{φ} [47, Theorem 5.4]. In the work [1] A. Alvarez-Parrilla extended the stationary-phase analysis to include an $SL(2;\mathbb{R})$-translate of $(\operatorname{Im} z)^s$, $z\in\mathbb{H}$, and studied the Eisenstein coefficient sums $S_{\mathcal{E}}$. Z. Rudnick and P. Sarnak established the result that for an arithmetic surface a Hecke-basis semi-classical limit cannot have projection to $\mathbb{H}$ with nontrivial singular support contained in a finite union of closed geodesics, [37].

The *renormalization* for formula (1.2) is considered in Section 5.4. In the possible absence of quantum unique ergodicity all L^2-mass could escape into the cusps for a semi-classical limit; Q_{limit} and μ_{limit} could possibly be trivial. We normalize eigenfunctions to unit L^2-mass on a compact set and reconsider the semi-classical limit. The L^2-norms can now be tending to infinity and the above considerations are not sufficient. In particular it is an open question if the microlocal lifts form a uniformly bounded family of distributions. We find though in Proposition 5.11 that the L^2-norms of renormalized eigenfunctions are bounded by $\log\lambda$. In Proposition 5.12 we use Fejér summation to construct the microlocal lift and to establish the existence of a non trivial semi-classical limit with the expected properties. In Theorem 5.14 we show that the resulting limit of square coefficient sums is the index zero Fourier-Stieljes coefficient of a Γ-invariant measure on the space of geodesics. The Γ-invariance has consequences for coefficient sums. The group action on the space of geodesics has a compact fundamental set as noted in Proposition 5.9. A positive lower bound for square coefficient sums is a consequence. Furthermore for large eigenvalues the mapping from eigenfunctions to scaled index interval linear coefficient sums twisted by an additive character is a uniform quasi-isometry relative to the L^2-norms for a suitable compact set and the unit circle (parameterizing the character). In particular the mapping from eigenfunctions to twisted linear coefficient sums is an injection.

We begin our analysis in Chapter 2 by considering the geodesic-indicator measures on $\mathbb{H}$ and the Radon transform; the adjoint is the weight-zero component of the integral transform (1.2). From a sequence of integral identities we show in Theorem 2.4 that suitable products of Macdonald-Bessel functions converge with rate r^{-1} to the Fourier-Stieljes coefficients of the geodesic-indicator. The focus of Chapter 3 is the analysis of sums of Macdonald-Bessel functions and the interchange of summation and spectral limits. We begin by introducing measures on $\mathbb{G}$ constructed from the Fourier coefficients of an automorphic eigenfunction. In Theorem 3.5 we establish (1.2) in effect for translation invariant test functions on $\mathbb{H}$. In Theorem 3.6 we give a new bound for coefficient sums and establish (1.2) in effect for general test functions on $\mathbb{H}$. The sum bound provides for $\|\varphi\|_2 = 1$ that $r^{-1} \sum_{1\leq n\leq rt} |a_n|^2$ is $o(1)$ for t small, uniformly in r. The bound plays an essential role in our arguments and apparently is the only available short-range sum bound sharp in the r-aspect, [26]. In Section 3.4 we show that the adjoint Radon transform is injective for translation invariant measures (the Radon transform is not surjective; translation invariance provides a special situation). The analysis of Chapter 4 concerns the microlocal lift on $SL(2;\mathbb{R})$ and the necessary bounds for considering sums of microlocalized Macdonald-Bessel functions. We begin with bounds for the raisings and lowerings of the Macdonald-Bessel functions. Then we use Zelditch's equation and an induction scheme on the $SL(2;\mathbb{R})$-weight to analyze the microlocal lift. Formula (1.2) is established in complete generality in Section 4.4. Initial consequences including the connections to limits of coefficient sums S_φ are presented in Sections 4.4 and 4.5. Chapter 5 is devoted to further applications. The analysis is extended to include the modular Eisenstein series; the analog of the relation (1.2) is presented in Theorem 5.6. Renormalization of the microlocal lift is considered in the final section.

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2 Products of Macdonald-Bessel functions and the Radon transform

2.1. The hyperbolic Radon transform is defined in terms of integration over geodesics in the upper half plane. We begin in Section 2.2 by introducing the Radon transform and the geodesic-indicator measure. The Fourier-Stieljes coefficients of the measure are presented in Proposition 2.1. In Section 2.3 we prescribe a test function and consider a sequence of integral identities to relate the product of the Macdonald-Bessel functions to the Fourier-Stieljes coefficients of the geodesic-indicator. The relation complete with a remainder is given in Theorem 2.4. An alternate formula is presented in Section 2.4 for the square of the Macdonald-Bessel function. The bounds for the remainders will be necessary in the consideration of sums in the following chapters.

2.2. We consider the space $\widehat{\mathbb{G}}$ of complete geodesics on $\mathbb{H}$ the upper half plane. $\widehat{\mathbb{G}}$ is naturally parameterized by considering end points on $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$; $\widehat{\mathbb{G}} \simeq \{\{\alpha, \beta\} \mid \alpha, \beta \in \widehat{\mathbb{R}},\ \alpha \neq \beta\}$. The Radon transform $\mathcal{R} : C_c(\mathbb{H}) \to C_c(\widehat{\mathbb{G}})$ for compactly supported functions is defined by

$$\mathcal{R}(f) = \int_\gamma f\ ds \quad \text{for} \quad f \in C_c(\mathbb{H}),\ \gamma \in \widehat{\mathbb{G}}$$

for ds the hyperbolic arc-length element, [22]. The space of geodesics $\widehat{\mathbb{G}}$ has an $SL(2;\mathbb{R})$-invariant area element $\omega = (\alpha - \beta)^{-2}|d\alpha \wedge d\beta|$. We wish to study the adjoint of the Radon transform. From the Riesz representation theorem the spaces $\mathcal{M}(\mathbb{H})$ and $\mathcal{M}(\widehat{\mathbb{G}})$ of regular Borel measures are the corresponding duals of the spaces of compactly supported continuous functions $C_c(\mathbb{H})$ and $C_c(\widehat{\mathbb{G}})$. The pairing for $g \in C_c(\widehat{\mathbb{G}})$ and $\nu \in \mathcal{M}(\widehat{\mathbb{G}})$ is $(g, \nu) = \int_{\widehat{\mathbb{G}}} g\nu$. The adjoint of the Radon transform $\mathcal{A} : \mathcal{M}(\widehat{\mathbb{G}}) \to \mathcal{M}(\mathbb{H})$ is prescribed by

$$\int_{\mathbb{H}} f\mathcal{A}(\nu) = \int_{\widehat{\mathbb{G}}\times\mathbb{H}} f(z)\nu(\{\alpha,\beta\})\delta_{\widehat{\alpha\beta}}(z) \tag{2.1}$$

for $f \in C_c(\mathbb{H})$, $z \in \mathbb{H}$ and $\delta_{\widehat{\alpha\beta}}$ the measure for arc-length integration along the geodesic $\widehat{\alpha\beta}$. In particular the geodesic-indicator $\delta_{\widehat{\alpha\beta}}$, a section of the trivial bundle $\widehat{\mathbb{G}} \times \mathcal{M}(\mathbb{H}) \to \widehat{\mathbb{G}}$, is the kernel for the integral representation of the adjoint $\mathcal{A}$.

We will study the kernel $\delta_{\widehat{\alpha\beta}}$ on $\mathbb{G} \subset \widehat{\mathbb{G}}$ the subspace of non vertical geodesics; $\mathbb{G} \simeq \{\{\alpha, \beta\} \mid \alpha, \beta \in \mathbb{R}, \alpha \neq \beta\}$. First we introduce alternate coordinates for $\mathbb{G}$. A non vertical geodesic is a Euclidean circle orthogonal to $\mathbb{R}$. Set $t = 2|\beta - \alpha|^{-1}$, the reciprocal radius, and $\hat{x} = (\alpha + \beta)/2$, the abscissa of the center. We have in terms of the $(\hat{x}, t)$-coordinates that $2\omega = |dt \wedge d\hat{x}|$. Define a function on $\mathbb{R}^+ \times \mathbb{R}^+$ by

$$S(t,y) = \begin{cases} 0 & \text{for} \quad ty \geq 1 \\ (y(1 - t^2y^2)^{1/2})^{-1} & \text{for} \quad ty < 1. \end{cases} \tag{2.2}$$

We are ready to consider the Fourier-Stieljes expansion for the kernel $\delta_{\widehat{\alpha\beta}}$, a quantity on $\mathbb{G}$ valued in positive measures on $\mathbb{H}$. For Γ_∞ the group of integer-translations denote the sum $\sum_{\gamma\in\Gamma_\infty} \delta_{\gamma(\widehat{\alpha\beta})}$ by $\delta_{\Gamma_\infty(\widehat{\alpha\beta})}$.

Proposition 2.1 *Notation as above. For $z = x + iy \in \mathbb{H}$, $(\hat{x}, t)$ coordinates for $\mathbb{G}$ then* $\delta_{\Gamma_\infty(\widehat{\alpha\beta})} = 2\sum_k e^{2\pi ik(x-\hat{x})} S(t,y) \cos(2\pi k(t^{-2} - y^2)^{1/2}) dxdy.$

Proof. The matter is to describe the measure $\delta_{\widehat{\alpha\beta}}(z)$ in terms of the prescribed coordinates. For a point (x, y) on the geodesic $\widehat{\alpha\beta}$ with coordinate $(\hat{x}, t)$ then $(x - \hat{x})^2 + y^2 = t^{-2}$

and the radius from the center $(\hat{x},0)$ to the point (x,y) has angle θ_0 to the positive x-axis with $\cos\theta_0 = \operatorname{sgn}(x-\hat{x})(1-t^2y^2)^{1/2}$. Thus in terms of the parameter y on $\mathbb{H}$ the element of hyperbolic arc-length along $\widehat{\alpha\beta}$ is simply $ds = |\sec\theta_0|y^{-1}\,|dy| = S(t,y)|dy|$. Furthermore the indicator measure of the geodesic is simply $\delta(x-\hat{x}+(t^{-2}-y^2)^{1/2})+\delta(x-\hat{x}-(t^{-2}-y^2)^{1/2})$ in terms of the one-dimensional Dirac delta. Thus from the Fourier-Stieljes expansion $\delta(x-a) = \sum_k e^{2\pi ik(x-a)}dx$ we find that $\sum_{\gamma\in\Gamma_\infty} \delta_{\widehat{\alpha\beta}}(\gamma z) = \sum_k 2e^{2\pi ik(x-\hat{x})}S(t,y)\cos(2\pi k(t^{-2}-y^2)^{1/2})dxdy$. The proof is complete.

2.3. We now show that certain products of Macdonald-Bessel functions converge to the Fourier-Stieljes coefficients of the kernel $\delta_{\Gamma_\infty(\widehat{\alpha\beta})}$. Throughout our considerations we will use the test function $h(u) = 2au^2e^{-au^2}$ for $a>0$. We write $K_{ir}(y)$for the Macdonald-Bessel function [31, Sec. 5.7] and start with an integral identity.

Lemma 2.2 *Notation as above. For $a>0$, $r>0$, $\alpha>0$ and $\beta\ge 1$ then*

$$\int_0^\infty K_{ir}(\beta y)K_{ir}(y)h(\alpha y)y^{-1}dy$$

$$= \pi\operatorname{csch}\pi r\int_0^\infty e^{-Y^2/(4a)}\sin\left(r\operatorname{arccosh}\left(1+\frac{P^2}{2}\right)\right)(4+P^2)^{-1/2}\frac{dP}{dY}dY$$

where $P^2 = (\alpha^2Y^2+(\beta-1)^2)\beta^{-1}$.

Proof. We start with the standard formula [31, Chap.5 Prob. 7]

$$K_{ir}(\beta y)K_{ir}(y) = \frac{\pi}{2}\operatorname{csch}\pi r\int_{\log\beta}^\infty J_0(yQ^{1/2})\sin r\tau d\tau$$

for $Q = 2\beta\cosh\tau - 1 - \beta^2$ and J_0 the order zero Bessel function. Next multiply by $h(\alpha y)y^{-1}$ and integrate to obtain

$$\int_0^\infty K_{ir}(\beta y)K_{ir}(y)h(\alpha y)y^{-1}dy = \tag{2.3}$$
$$\frac{\pi}{2}\operatorname{csch}\pi r\int_{\log\beta}^\infty\int_0^\infty J_0(yQ^{1/2})h(\alpha y)y^{-1}dy\sin r\tau d\tau;$$

the integrals are absolutely convergent since $Q = \beta e^\tau + O(1)$ for τ large and $J_0(x)$ is $O(x^{-1/2})$ for x large positive. Now from the tables of Hankel transforms [12, pg.29(10)] we have that $\int_0^\infty J_0(yQ^{1/2})h(\alpha y)y^{-1}dy = e^{-Q/(4a\alpha^2)}$ and thus the integral on the right hand side of (2.3) is

$$\frac{\pi}{2}\operatorname{csch}\pi r\int_{\log\beta}^\infty e^{-Q/(4a\alpha^2)}\sin r\tau d\tau. \tag{2.4}$$

We next set $\alpha^2Y^2 = Q$ and $P^2 = (\alpha^2Y^2+(\beta-1)^2)\beta^{-1}$ and observe that $1+\frac{P^2}{2} = \frac{\alpha^2Y^2+\beta^2+1}{2\beta} = \frac{Q+\beta^2+1}{2\beta} = \cosh\tau$. We accordingly have that $\tau = \operatorname{arccosh}(1+\frac{P^2}{2})$ and that $d\operatorname{arccosh}(1+\frac{u^2}{2}) = (4+u^2)^{-1/2}\,2du$. The desired integral results from (2.4) after a change of variables, since for $\beta \geq 1$, $Q(\tau)$ is monotone on $[\log\beta,\infty)$. The proof is complete.

The next matter is a simple identity for trigonometric integrals.

Lemma 2.3 *For C positive then*

$$\int_0^1 \frac{\cos(B(1-Y^2)^{1/2})}{Y(1-Y^2)^{1/2}} 2CY^2e^{-CY^2}dY = \int_0^\infty e^{-X^2/(4C)}\frac{\sin(X^2+B^2)^{1/2}}{(X^2+B^2)^{1/2}}XdX.$$

Proof. We first consider the left hand integral and substitute the series representation for the cosine and exponential to find the expansion for the integral

$$\sum_{p,q=0}^{\infty} \frac{(-1)^{p+q}}{(2p)!q!}B^{2p}C^{q+1}2\int_0^1 Y^{2q+1}(1-Y^2)^{p-1/2}dY$$

where $2\displaystyle\int_0^1 Y^{2q+1}(1-Y^2)^{p-1/2}dY = B(q+1,p+\frac{1}{2})$ is the Euler beta function [15, Sec. 8.380]. Now for the right hand side we use the series representation of the sine function to find

$$\frac{\sin(X^2+B^2)^{1/2}}{(X^2+B^2)^{1/2}} = \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}(X^2+B^2)^m$$

$$= \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}\sum_{p=0}^{m}\binom{m}{p}B^{2p}X^{2m-2p} = \sum_{p=0}^{\infty}\sum_{m=p}^{\infty}\frac{(-1)^m}{(2m+1)!}\binom{m}{p}B^{2p}X^{2m-2p}$$

$$= \sum_{p,q=0}^{\infty}\frac{(-1)^{p+q}}{(2p+2q+1)!}\binom{p+q}{p}B^{2p}X^{2q}.$$

We evaluate the integral $\displaystyle\int_0^\infty e^{-X^2/(4C)}X^{2q+1}dX = \frac{1}{2}(4C)^{q+1}q!$ and on substituting the identity $(2p+2q+1)!p!B(q+1,p+\frac{1}{2}) = (2p)!q!2^{2q+1}(p+q)!$ we have the desired equality of the two expansions. The proof is complete.

We are now ready to present the integral formula relating products of Macdonald-Bessel functions to the kernel $\delta_{\widehat{\alpha\beta}}$.

Theorem 2.4 *Notation as above. For $h(y) = 2ay^2e^{-ay^2}$, given t_0, t_1, k_0 positive, for $n, r > 0$ with $0 < t_0 < t = 2\pi nr^{-1} < t_1$, $0 \leq k \leq k_0$, $a > 0$ and r large then*

$$r\sinh\pi r\int_0^\infty K_{ir}(2\pi(n+k)y)K_{ir}(2\pi ny)h(y)y^{-1}dy =$$

$$\frac{\pi}{2}\int_0^{t^{-1}}\cos(2\pi k(t^{-2}-y^2)^{1/2})S(t,y)h(y)dy + O(r^{-2}a^2+r^{-1}a^{1/2})$$

for a remainder constant depending on t_0, t_1 and k_0.

Proof. We start with the left hand side. The integral equals

$$r \sinh \pi r \int_0^\infty K_{ir}((n+k)n^{-1}y)K_{ir}(y)h(y(2\pi n)^{-1})y^{-1}dy$$

which by Lemma 2.2 is

$$r\pi \int_0^\infty e^{-Y^2/(4a)} \sin(r \operatorname{arccosh}(1+\frac{P^2}{2}))(4+P^2)^{-1/2}\frac{dP}{dY}dY$$

for $P^2 = ((\frac{Y}{2\pi n})^2 + \left(\frac{k}{n}\right)^2)\frac{n}{n+k}$. We consider the factors in the integrand. For $t_0 \le t = 2\pi n r^{-1} \le t_1$ then $r^2P^2 = t^{-2}(Y^2 + (2\pi k)^2)(1+O(r^{-1}))$ and similarly $r\frac{dP}{dY} = t^{-1}(Y^2 + (2\pi k)^2)^{-1/2}Y(1+O(r^{-1}))$. Furthermore $\operatorname{arccosh}(1+\frac{P^2}{2}) = P + O(P^3)$ for all positive P and so $\sin(r \operatorname{arccosh}(1+\frac{P^2}{2})) = \sin rP + O(rP^3)$ for all positive r and P. Finally we have that $(4+P^2)^{-1/2} = \frac{1}{2} + O(P^2)$ for all P. Gathering the expansions we have that the original integral of Macdonald-Bessel functions is

$$\frac{\pi}{2}\int_0^\infty e^{-Y^2/(4a)} \sin(t^{-1}(Y^2 + (2\pi k)^2)^{1/2})(t(Y^2+(2\pi k)^2)^{1/2})^{-1}Y\,dY$$

$$+O\Big(\int_0^\infty e^{-Y^2/(4a)}(r^2P^3 + rP^2 + 1)\frac{dP}{dY}dY\Big).$$

To analyze the remainder we use the coarse bounds that P is $O((Y+1)r^{-1})$ and $\frac{dP}{dY}$ is $O(r^{-1})$ valid for Y positive and $t_0 \le t \le t_1$. The remainder simplifies to the quantity $O(\int_0^\infty e^{-Y^2/(4a)}(r^{-1}Y^3+1)r^{-1}dY)$, which is $O(r^{-2}a^2 + r^{-1}a^{1/2})$ by scaling considerations. Now we invoke the identity from Lemma 2.3 with $X = Yt^{-1},\ B = 2\pi k t^{-1}$ and $C = at^{-2}$ to find for the principal term

$$\frac{\pi}{2}\int_0^1 \cos(2\pi k(t^{-2} - Y^2t^{-2})^{1/2})(1-Y^2)^{-1/2}e^{-a(Y/t)^2}2at^{-2}Y\,dY.$$

The desired final integral results on substituting $Y = yt$. The proof is complete.

2.4. We are interested in a more detailed analysis of the large r behavior of the special integral

$$\mathcal{I}((4a\alpha^2)^{-1}, r) = \pi^{-1} r \sinh \pi r \int_0^\infty K_{ir}(y)^2 h(\alpha y) y^{-1} dy.$$

In the next chapter the refined analysis will be applied to the consideration of short-range coefficient sums. From Lemma 2.2

$$\mathcal{I}(A,r) = r\int_0^\infty e^{-AP^2}\sin\Big(r\operatorname{arccosh}\Big(1+\frac{P^2}{2}\Big)\Big)(4+P^2)^{-1/2}dP.$$

Set

$$G(u) = uF(u) = u\, e^{-u^2} \int_0^u e^{v^2} dv$$

where $F(z) = e^{-z^2} \displaystyle\int_0^z e^{v^2} dv$ is the probability integral [31, Sec.2.3]. The function $G(u)$ is positive for u positive, $O(u^2)$ for u small, and satisfies $G(u) = \frac{1}{2} + O(u^{-1})$ for u large positive (the method of [31, pg.20] can be used to establish the last expansion).

Lemma 2.5 *Notation as above. Given $A_0 > 0$ for $A \geq A_0$ and $r \geq 1$ then for $A^{1/2} \leq r$, $\mathcal{I}(A, r) = G(r(2A^{1/2})^{-1}) + O_{A_0}(A^{1/2}r^{-1})$ and given $\epsilon_0 > 0$ furthermore for $A^{1/2} \geq \epsilon_0 r$ then $\mathcal{I}(A, r) = G(r(2A^{1/2})^{-1}) + O_{A_0,\epsilon_0}(A^{-1})$.*

Proof. For the first expansion we start with the integral and integrate by parts twice: first with $u = \cos(r \operatorname{arccosh}(1+\frac{P^2}{2}))$, second with $u = r^{-1}\sin(r \operatorname{arccosh}(1+\frac{P^2}{2}))$ and then change variables with $A^{1/2}P = Q$. The resulting formula is

$$\mathcal{I}(A, r) = \frac{1}{2} + \frac{A^{1/2}}{2r} \int_0^\infty \sin\left(r \ \operatorname{arccosh}\left(1 + \frac{Q^2}{2A}\right)\right) d(e^{-Q^2} Q(4 + Q^2A^{-1})^{1/2}).$$

We then note that for $A \geq A_0$, $Q \geq 0$ it follows that $2 \leq (4 + Q^2A^{-1})^{1/2} \leq 2 + QA_0^{-1/2}$ and that $|\frac{d}{dQ}(4 + Q^2A^{-1})^{1/2}| \leq QA_0^{-1}$. It follows that the above integral is dominated by $\displaystyle\int_0^\infty e^{-Q^2}(1 + Q^3)dQ$, which is finite. The expression for the integral $\mathcal{I}$ and the expansion for the function G now give $\mathcal{I}(A, r) = G(r(2A^{1/2})^{-1}) + O(A^{1/2}r^{-1})$ for the first specified range.

For the second expansion we start with a change of variables for the integral: for $Q^2 = AP^2$ then

$$\mathcal{I}(A, r) = \frac{r}{A^{1/2}} \int_0^\infty e^{-Q^2} \sin\left(r \ \operatorname{arccosh}\left(1 + \frac{Q^2}{2A}\right)\right)(4 + \frac{Q^2}{A})^{-1/2} dQ.$$

Now we have that $\operatorname{arccosh}(1 + \frac{P^2}{2}) = P + O(P^3)$ for all positive P and thus for $A^{1/2} \geq \epsilon_0 r \geq \epsilon_0 > 0$ that

$$r \operatorname{arccosh}(1 + \frac{Q^2}{2A}) = rA^{-1/2}Q + O(rQ^3A^{-3/2}) = rA^{-1/2}Q + O(A^{-1}Q^3).$$

We also have the coarse bound that

$$2(4 + Q^2A^{-1})^{-1/2} - 1 \text{ is } O(A^{-1}Q^2).$$

Combining expansions it follows that

$$\mathcal{I}(A, r) = r(2A^{1/2})^{-1} \int_0^\infty e^{-Q^2} \sin(rQA^{-1/2})dQ + O(A^{-1} \int_0^\infty e^{-Q2}(Q^2 + Q^3)dQ).$$

The explicit integral is tabulated as

$$r^2(4A)^{-1}\,{}_1F_1(1,\frac{3}{2};-r^2(4A)^{-1}) = r(2A^{1/2})^{-1}F(r(2A^{1/2})^{-1}).$$

[13, pg.73(18) and pg.373], [31, formulas (9.9.1) and (9.13.3)]. The remainder integral is finite and the proof is complete.

3 Measures from Fourier coefficients and sums of Macdonald-Bessel functions

3.1. Our plan is to express the square of an automorphic eigenfunction as an integral over the space of geodesics. We use Fejér summation to define a family of positive measures from the Fourier coefficients of an eigenfunction. Quadratic expressions in the eigenfunctions are then represented as integrals of the measures. In Section 3.2 we introduce the measures, establish their uniform boundedness and consider their basic properties. In the next section we consider sums of the relation given in Theorem 2.4, sums of products of Macdonald-Bessel functions. Our goal is to find the limits of sums of products. Only the basic sum-square bound is available for the Fourier coefficients; to be able to interchange the spectral and summation limit a detailed analysis of the contribution of the Macdonald-Bessel functions is required. In Theorems 3.5 and 3.6 a high-energy limit of eigenfunction squares is presented as the integral of the geodesic-indicator and a limit of the constructed measures. In Corollary 3.7 we find that the high-energy limit is the adjoint Radon transform of the limit of the constructed measures. In the final section we use our formulation to show that the adjoint Radon transform is invertible for translation invariant measures.

3.2. Our plan is to construct and analyze *measures* describing the concentration properties of automorphic eigenfunctions. Let $\Gamma \subset SL(2;\mathbb{R})$ be a cofinite group with a width-one cusp at infinity. We consider Γ-invariant eigenfunctions of the hyperbolic Laplace-Beltrami operator with finite $L^2(\Gamma\backslash\mathbb{H})$ norm [42, 45]. For $D\varphi+\lambda\varphi=0,\ \lambda=(\frac{1}{4}+r^2)>\frac{1}{4}$, φ has the Fourier expansion

$$\varphi(z) = \sum_n a_n(y\sinh\pi r)^{1/2}K_{ir}(2\pi|n|y)e^{2\pi inx} \tag{3.1}$$

for $z=x+iy, a_0=0$, and the Macdonald-Bessel function (note the normalization of the Fourier coefficients). We start with a bound for the Fourier coefficients

$$\sum_{n=1}^{M}|a_n|^2 \le C_\Gamma\|\varphi\|_2^2(M+r),\ [47] \tag{3.2}$$

(the bound is a slight improvement of the Deshouillers-Iwaniec bound [11]).

Our constructions will involve the Fejér kernel, [28]. Recall that

$$\sum_{k=-2N}^{2N}\Big(1-\frac{|k|}{2N+1}\Big)X^k=\frac{1}{2N+1}(X^{-N}+\cdots+X^N)^2 \tag{3.3}$$

and in particular the Fejér kernel is simply

$$\mathcal{F}_N=\sum_{k=-2N}^{2N}\Big(1-\frac{|k|}{2N+1}\Big)e^{2\pi ik\hat{x}}.$$

$\mathcal{F}_N$ is positive and defines an operator by convolution on function spaces associated to $\mathbb{R}$, [28]. Convolution with $\mathcal{F}_N$ converges to the identity as N tends to infinity for $C_{\text{per}}(\mathbb{R})$ and $L^1_{\text{per}}(\mathbb{R})$, the spaces of continuous and integrable one-periodic functions. Convolution with $\mathcal{F}_N$ is also a formally self-adjoint operator.

We now give function space constructions of quantities from the Fourier coefficient sequence $\{a_n\}_{n\in\mathbb{Z}}$ of an eigenfunction φ, normalized by $\|\varphi\|_2=1$.

Definition 3.1 *For an eigenfunction φ, with eigenvalue $\lambda=\frac{1}{4}+r^2$, r positive, $(\hat{x},t)$ coordinates on $\mathbb{R}\times\mathbb{R}$, $\Delta t=2\pi r^{-1}$, set*

$$\chi_\varphi(\hat{x},t)=\big\{a_ne^{2\pi in\hat{x}},\quad (n-1)\Delta t\le t<n\Delta t$$

furthermore for $(\hat{x},t)$ coordinates on $\mathbb{G}$ set

$$\sigma_{\varphi,N}=(2N+1)^{-1}\pi\sum_{j,k=-N}^{N}\sum_{\epsilon=\pm1}\chi_\varphi(\hat{x},\epsilon t+j\Delta t)\overline{\chi_\varphi(\hat{x},\epsilon t+k\Delta t)}$$

$$\Phi_{\varphi,k}(t)=\pi\sum_{|n|\le rt(2\pi)^{-1}}a_{n+k}\overline{a_n}r^{-1}$$

(note $a_0=0$) and for the Lebesgue-Stieljes derivative $d\Phi_{\varphi,k}$ set

$$\mu_\varphi=\sum_k d\Phi_k(t)e^{2\pi ik\hat{x}}.$$

Comments and observations are in order. The quantity χ_φ is akin to a *probability amplitude* associated to the state φ. For $g\in C_c(\mathbb{R}^+)$ the one-dimensional integral $\int_{\mathbb{R}^+}g\sigma_{\varphi,\text{N}}$ in t already has the form $\langle Op_g\varphi,\varphi\rangle$ for a self-adjoint operator, since $\sigma_{\varphi,\text{N}}$ is a sum of Hermitian squares of $(2N+1)^{-1/2}\pi^{1/2}\sum_j\chi_\varphi(\hat{x},t+j\Delta t)$. In Section 4.4 we will find that the matrix elements $\langle Op_g\varphi, \varphi\rangle$ give the probability for observing classical trajectories. The quantities χ_φ and μ_φ are $\hat{x} \to \hat{x}+1$ invariant; $\sigma_{\varphi,N}$ is a Fejér sum of χ_φ and satisfies $\sigma_{\varphi,N} \geq 0$. From (3.2) the L^2-norm of χ_φ on $\{0 \leq \hat{x} \leq 1,\ 0 \leq t \leq t_0\}$ is bounded by $C_\Gamma(t_0+1)^{1/2}$ and similarly $|\Phi_k(t)|$ is bounded by $C_\Gamma(t^2+t(|k|+2)+1)^{1/2}$. It follows that μ_φ is at least a tempered distribution for $C^2_{\mathrm{per},c}(\mathbb{G})$, the space of one-periodic, compactly supported, twice-differentiable functions on $\mathbb{G}$ ($f \in C^2_{\mathrm{per},c}(\mathbb{G})$ has a Fourier expansion $\sum_k f_k(t)e^{2\pi ik\hat{x}}$ with $|f_k(t)| \leq C\|f\|(|k|+1)^{-2}$). The distribution μ_φ represents an elementary type of *microlocalization* of the eigenfunction square; μ_φ encodes the concentration and oscillation properties of the eigenfunction. Quadratic expressions in φ are integrals of μ_φ.

Our analysis will require an appropriately convergent sequence of eigenfunctions.

Definition 3.2 *For dA the hyperbolic area element a normalized sequence of eigenfunctions $\{\varphi_j\}$ with eigenvalues tending to infinity is $*$-convergent provided $\varphi_j^2 dA$ converges weak* relative to $C_c(\Gamma\backslash\mathbb{H})$ and provided for each k the Lebesgue-Stieljes derivatives $d\Phi_{\varphi_{j,k}}$ converge weak* relative to the continuous functions for each closed subinterval of* $[0,\infty)$.

Note that by weak* compactness of the unit ball of measures and diagonalization, a normalized sequence of eigenfunctions has a $*$-convergent subsequence. Weak* convergence of measures on $\mathbb{G}$ will be considered relative to the system of spaces $C_{\mathrm{per},t_0}(\mathbb{G})$, all positive t_0, of continuous one-periodic functions with support contained in $\{(\hat{x},t) \mid 0 < t \leq t_0\}$.

Proposition 3.3 *Notation as above. For $\{\varphi_j\}$ a $*$-convergent sequence then $\lim_j \sigma_{\varphi_j,N} = \mathcal{F}_N * \lim_j \mu_{\varphi_j}$ relative to each $C_{\mathrm{per},t_0}(\mathbb{G})$ and in particular each $\lim_j \mu_{\varphi_j}$ is a positive measure on $\mathbb{G}$ with $\|\lim_j \mu_{\varphi_j}\|$ bounded by $C(t_0+1)$.*

Proof. The basis of the considerations is the Fourier series expansion of a function in $C_{\mathrm{per}}(\mathbb{G})$. Functions with finite Fourier expansions in $\hat{x}$ are dense in $C_{\mathrm{per}}(\mathbb{G})$. Accordingly measures of uniformly bounded mass converge weak* provided their sequences of Fourier-Stieljes coefficients weak* converge. Each measure $\sigma_{\varphi,N}$ is positive and thus has its mass given by its zeroth coefficient which has integral bounded by $\Phi_{\varphi,0}(t+N\Delta t)$, which by (3.2) is bounded by $C(t+1+N\Delta t)$. Since as j tends to infinity, r tends to infinity and Δt tends to zero, we have for $g \in C_c(\mathbb{R}^+)$ that

$$\begin{aligned}
&\lim_j \int \sum_{\epsilon=\pm1} \chi_{\varphi_j}(\hat{x}, \epsilon t + (k+m)\Delta t)\overline{\chi_{\varphi_j}(\hat{x}, \epsilon t + m\Delta t)}g(t)e^{-2\pi ik\hat{x}}d\hat{x}dt \\
&= \lim_j \int \sum_{\epsilon=\pm1} \chi_{\varphi_j}(\hat{x}, \epsilon t + k\Delta t)\overline{\chi_{\varphi_j}(\hat{x}, \epsilon t)}\tilde{g}(t - m\Delta t)e^{-2\pi ik\hat{x}}d\hat{x}dt \\
&= \int g(t)d\Phi_{\varphi_{j,k}}(t)
\end{aligned}$$

since $\tilde{g}(t-m\Delta t)$ tends to $g(t)$ in $C_c(\mathbb{R}^+)$, where $\tilde{g}(\tau)$ is defined as $g(0)$ for $\tau<0$ and $g(\tau)$ for $\tau\geq 0$. The first conclusion follows and furthermore that $\mathcal{F}_N * \lim_j \mu_{\varphi_j}$ is a positive measure with mass bounded in terms of $C(t_0+1)$. It also follows that $\lim_j \mu_{\varphi_j}$ is a positive measure with mass bounded in terms of $C(t_0+1)$. The proof is complete.

3.3. We continue our preparations, introduce *test functions* and consider automorphic integrals. Let $\Gamma_\infty\subset\Gamma$ be the stabilizer of the width-one cusp at infinity.

Definition 3.4 *For $z=x+iy\in\mathbb{H}$ and k an integer set*

$$h_k(z)=2ay^2e^{-ay^2-2\pi ikx} \text{ and } H_k(z)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}h_k(\gamma z).$$

We now review certain basic bounds for incomplete theta series. Recall first that the $\Gamma_\infty\backslash\Gamma$ translates of a point intersect the horoball $\mathcal{B}=\{z\mid \operatorname{Im} z\geq 1\}$ at most once. Recall also the truncation at height-one of the Eisenstein series $\mathcal{E}^1(z;2)$, [7]; the function $\mathcal{E}^1(z;2)$ vanishes at each cusp. From the observation that $h(y)\leq 2ay^2$ for $0<a\leq 1$ it follows that $H_0(z)\leq 2a\mathcal{E}^1(z;2)$, provided $(\Gamma_\infty\backslash\Gamma)z\cap\mathcal{B}=\emptyset$ and $H_0(z)\leq 2a\mathcal{E}^1(z;2)+h(\operatorname{Im} z_\infty)$, provided $(\Gamma_\infty\backslash\Gamma)z\cap\mathcal{B}=\{z_\infty\}$. Basic bounds for $|H_k(z)|\leq|H_0(z)|$ now follow: the restriction of $H_k(z)$ to a compact set is bounded by a multiple of $a, 0<a\leq 1$; for a fixed $H_k(z)$ tends to zero as z tends to a cusp; also $H_k(z)$ is uniformly bounded for $0<a\leq 1$.

We begin our consideration of the $*$-convergent limit by analyzing the zeroth coefficients. Let ν (a non negative tempered distribution on $(0,\infty)$) be the zeroth Fourier-Stieljes coefficient of the lift to $\mathbb{H}$ of the $C_c(\Gamma\backslash\mathbb{H})$ weak* limit $\lim_j\varphi_j^2dA$ and let σ (a non negative tempered distribution on $[0,\infty)$ with possibly $\sigma(\{0\})>0$) be the weak* limit $\lim_j d\Phi_{\varphi_{j,0}}$ on each closed subset of $[0,\infty)$.

Theorem 3.5 *With the above notation.*

$$\int_0^\infty h(y)\nu(y)=\int_0^\infty G(a^{1/2}t^{-1})\sigma(t)$$

Proof. We must show that sequences of integrands have uniform majorants and that integrals converge. We begin with

$$\mathcal{J}(\alpha)=\int_{\substack{0\leq x\leq 1\\ 0<y\leq\alpha}}h\varphi^2dA \quad z=x+iy,$$

which can be integrated by parts for $u=h$ and

$$v(y_0)=\int_{\substack{0\leq x\leq 1\\ y_0\leq y}}\varphi^2dA.$$

The integral v is bounded as $v(y_0) \leq \|\varphi\|_2^2 \mathcal{V}(y_0)$ for $\mathcal{V}$ the counting function for translates of a standard fundamental domain intersecting $\{z \mid 0 \leq x \leq 1,\ y_0 \leq y\}$. The counting function $\mathcal{V}(y_0)$ is bounded by a multiple of $(y_0^{-1} + 1)$, [47]. We accordingly have that $\mathcal{J}(\alpha) = hv \ |_\alpha^0 + \int_0^\alpha vh' dy_0$ is uniformly bounded by a multiple of α. Now from the uniform bound for $\mathcal{J}(\alpha)$ and the convergence of $\varphi_j^2 dA$ on compact sets it follows that $\int h\varphi_j^2 dA$ limits to $\int h\nu$.

Next consider $\int_{t_0}^\infty G(a^{1/2}t^{-1}) d\Phi_{\varphi,0}(t)$ for G as introduced in Section 2.4 and integrate by parts to find the resulting integral $\int_{t_0}^\infty (\Phi_{\varphi,0}(t) - \Phi_{\varphi,0}(t_0)) G'(a^{1/2}t^{-1}) a^{1/2}t^{-2} dt$. Since $0 \leq \Phi_{\varphi,0}(t) - \Phi_{\varphi,0}(t_0) \leq \Phi_{\varphi,0}(t) \leq C(t+1)$ for $t \geq t_0$ from (3.2) and $G'(u)$ is $O(u)$ for u small, it follows that the last integral is uniformly bounded by a multiple of t_0^{-1}. The convergence of $\int G d\Phi_{\varphi_j}$ now follows from the pointwise convergence of Φ_{φ_j}.

We are ready to consider the equality of the integrals. We start with $\int_{\Gamma_\infty \backslash \mathbb{H}} \varphi^2 h dA = \pi \sum_n |a_n|^2 r^{-1} \mathcal{I}((2\pi n)^2/4a, r)$; the equation follows from Lemma 2.2 and the definition of $\mathcal{I}$ given in Section 2.4. The large-r expansion of $\mathcal{I}$ is given in Lemma 2.5 for $t = 2\pi|n|r^{-1}$, $A = (2\pi n)^2/4a$ as $\mathcal{I}((2\pi n)^2/4a, r) = G(a^{1/2}t^{-1}) + \mathcal{R}(n, r, a)$ where the remainder is $O_a(t)$ for $t \leq a^{1/2}$ and given t_0 positive the remainder is $O_{a,t_0}(n^{-2})$ for $t \geq t_0$. We will now show that the remainder sum $\sum_n |a_n|^2 r^{-1} \mathcal{R}$ tends to zero as r tends to infinity with a fixed. First note from the basic bound (3.2) that $\sum_{1 \leq n \leq M} |a_n|^2 r^{-1}$ is bounded for $2\pi M r^{-1}$ bounded. Since the remainder $\mathcal{R}$ is $O_a(2\pi n r^{-1})$ for $2\pi n r^{-1} \leq a^{1/2}$ it follows that $\sum_{1 \leq n \leq M} |a_n|^2 r^{-1} |\mathcal{R}|$ is bounded by a multiple of t_0 provided that $2\pi M r^{-1} \leq t_0 \leq a^{1/2}$, a suitable bound for the initial sum. For the tail sum observe that $\mathcal{R}$ is $O_{a,t_0}(n^{-2})$ for $n \geq M_0 = rt_0(2\pi)^{-1}$ and thus $\sum_{M_0 < n} |a_n|^2 r^{-1} |\mathcal{R}|$ is majorized by $\sum_{M_0 < n} |a_n|^2 n^{-2} r^{-1}$. The last sum can be evaluated by parts with $U(m) = \sum_{M_0 < n \leq m} |a_n|^2 r^{-1},\ v(m) = m^{-2}$ and bounded using (3.2) to note that $|U(m)| \leq C(mr^{-1} + 1)$. The resulting bound is by a multiple of M_0^{-2} which in turn is bounded by a multiple of r^{-2}, a suitable bound for the tail sum. The remainder sum tends to zero as r tends to infinity. The desired equality of integrals is now a consequence of the $*$-convergence, the existence of uniform majorants and the expansion for $\int \varphi^2 h dA$. The proof is complete.

We are ready to consider the adjoint Radon transform relation between the two positive measures $\Omega = \lim_j \varphi_j^2 dA$ and $\mu_{\text{limit}} = \lim_j \mu_{\varphi_j}$. We use the characterization of the transform given by formula (2.1).

Theorem 3.6 *Notation as above. For σ the limit $\lim_j d\Phi_{j,0}$ on $[0,\infty)$ then $\sigma(\{0\})=0$. For $z=x+iy\in\mathbb{H}$, k an integer, a positive and $h_k(z)=2ay^2e^{-ay^2-2\pi ikx}$ then*

$$4\int_{\Gamma_\infty\backslash\mathbb{H}} h_k\Omega=\int_{\Gamma_\infty\backslash\mathbb{H}\times\Gamma_\infty\backslash\mathbb{G}} h_k\delta_{\Gamma_\infty(\widehat{\alpha\beta})}\mu_{\text{limit}}$$

Proof. The basic matter is to show for the test function h_k that

$$(3.4)\qquad 4\int_{\Gamma_\infty\backslash\mathbb{H}} h_k\lim_j\varphi_j^2dA=\int_{\Gamma_\infty\backslash\mathbb{H}\times\Gamma_\infty\backslash\mathbb{G}} h_k\delta_{\Gamma_\infty(\widehat{\alpha\beta})}\lim_j d\Phi_{\varphi_{j,k}}e^{2\pi ik\hat{x}}$$

Majorants for the sequences of integrands are obtained from the following simple observations: $2|a_na_m|\le|a_n|^2+|a_m|^2$, $2|K_{ir}(2\pi|n|y)K_{ir}(2\pi|m|y)|\le K_{ir}(2\pi|n|y)^2+K_{ir}(2\pi|m|y)^2$ and from Proposition 2.1 that the absolute-value of the k^{th} Fourier-Stieljes coefficient of the kernel $\delta_{\Gamma_\infty(\widehat{\alpha\beta})}$ is bounded by the 0^{th} Fourier-Stieljes coefficient. As the first step of the proof we wish to show that given $0<t_0<t_1$ the combined contribution to (3.4) from the Fourier coefficients $a_{n+k}a_{-n}$ with either $2\pi|n|r^{-1}\le t_0$ or $2\pi|n|r^{-1}\ge t_1$ is bounded by $o(1)$. We start with the left hand side. From the above observations the contribution is bounded by the contribution considered in Theorem 3.5 from the products $|a_m|^2$, $|m|\le rt_0(2\pi)^{-1}+|k|$ and $|m|\ge rt_1(2\pi)^{-1}-|k|$. Theorem 3.5 can be applied for the restricted range of m (the contribution for all m serves as a majorant). The resulting contribution is simple $\int_{[0,t_0]\cup[t_1,\infty)}G(a^{1/2}t^{-1})\sigma(t)$. Now for the right hand side the contribution is similarly bounded by the contribution of the restricted m range with the zeroth Fourier-Stieljes coefficient of $\delta_{\Gamma_\infty(\widehat{\alpha\beta})}$. From Proposition 2.1, Lemma 2.3 and the citation of explicit integrals in the proof of Lemma 2.5 the bounding integral is again $\int_{[0,t_0]\cup[t_1,\infty)}G(a^{1/2}t^{-1})\sigma(t)$. Now suitable bounds for G and σ were provided in Theorem 3.5. For t_1 large positive $\int_{t_1}^{\infty}G(a^{1/2}t^{-1})\sigma(t)$ is $O(t_1^{-1})$ and for t_0 small positive $2\int_0^{t_0}G(a^{1/2}t^{-1})\sigma(t)=\sigma(\{0\})+O(t_0)$ (see Section 2.4). Since G and σ are positive it follows from Theorem 3.5 that $\sigma(\{0\})\le 2\int_{\Gamma_\infty\backslash\mathbb{H}}H_0\Omega$ for all a; as noted for a tending to zero then H_0 is uniformly bounded and tends to zero on compact sets. The integral $\int_{\Gamma_\infty\backslash\mathbb{H}}\Omega$ is finite by Fatou's lemma and so it follows that $\sigma(\{0\})=0$ and consequently that the combined t-tail contribution to (3.4) is small. For t in the restricted interval (t_0,t_1) the resulting contributions to (3.4) coincide by Proposition 2.1 and Theorem 2.4. The relation (3.4) is established. From the definition of the measure μ_{limit} the right hand integrand can be replaced with $h_k\delta_{\widehat{\alpha\beta}}\mu_{\text{limit}}$. The proof is complete.

Let $C_{y^{1+\epsilon}e^{-\epsilon y}}(\Gamma_\infty\backslash\mathbb{H})$ for ϵ positive denote the space of continuous functions bounded by a multiple of $y^{1+\epsilon}e^{-\epsilon y}$. We are ready to present the formula for the high-energy limit of eigenfunction squares.

Corollary 3.7 *With the above notation*

$$4\Omega = \int_{\Gamma_\infty\backslash\mathbb{G}} \delta_{\Gamma_\infty(\widehat{\alpha\beta})}\mu_{\text{limit}} = \int_{\mathbb{G}} \delta_{\widehat{\alpha\beta}}\mu_{\text{limit}}$$

in the sense of $C_{y^{1+\epsilon}e^{-\epsilon y}}(\Gamma_\infty\backslash\mathbb{H})$ *tempered distributions.*

Proof. The matter is reduced to the equality of Fourier-Stieljes coefficients provided the first two quantities are tempered distributions. We first consider Ω. The incomplete theta series (the sum over $\Gamma_\infty\backslash\Gamma$) is an operator from $C_{y^{1+\epsilon}e^{-\epsilon y}}(\Gamma_\infty\backslash\mathbb{H})$ to $C_0(\Gamma\backslash\mathbb{H})$ and thus since the integrals $\|\varphi_j\|^2$ are normalized it follows that Ω is tempered. By Proposition 3.3 the second quantity is a positive distribution and so it suffices to consider the pairing of a test function and the zeroth coefficient $\int_0^\infty y^{1+\epsilon}e^{-\epsilon y}(\delta_{\Gamma_\infty(\widehat{\alpha\beta})})_0$. The pairing (see Proposition 2.1) is bounded for t small and is $O(t^{-1-\epsilon})$ for t large, t the parameter for $\widehat{\alpha\beta}$. It now follows from (3.2) that $\int_{\Gamma_\infty\backslash\mathbb{G}} \delta_{\Gamma_\infty(\widehat{\alpha\beta})}\mu_{\text{limit}}$ is tempered.

We will use the approximation that linear combinations of $y^{1-\epsilon}e^{\epsilon y-ay^2}$ for positive values of a are dense in $C_0(\mathbb{R}^+) \subset C([0,\infty])$. For this assertion consider a Borel measure ν on $[0,\infty]$ with finite total mass that is orthogonal to each $y^{1-\epsilon}e^{\epsilon y-ay^2}$; since a test function is positive it follows that $\int y^{1-\epsilon}e^{\epsilon y-ay^2}\nu_+ = \int y^{1-\epsilon}e^{\epsilon y-ay^2}\nu_-$. The integrals are the Laplace transforms of the positive measures $y^{1-\epsilon}e^{\epsilon y}\nu_\pm$ in the variable y^2. Positive measures are uniquely determined by their Laplace transforms [46]; consequently ν is trivial and linear combinations of $y^{1-\epsilon}e^{\epsilon y-ay^2}$ are dense in $C_0(\mathbb{R}^+)$. Now for $g \in C_{y^{1+2\epsilon}e^{-2\epsilon y}}(\mathbb{R}^+)$ linear combinations of $y^{-1-\epsilon}e^{\epsilon y}h(y)$ for different values of a uniformly approximate $y^{-1-\epsilon}e^{\epsilon y}g(y)$ in $C_0(\mathbb{R}^+)$. Equivalently linear combinations of $h(y)$ approximate $g(y)$ with bounds in terms of $y^{1+\epsilon}e^{-\epsilon y}$. The linear combinations of h suitably approximate g and the first equality of tempered distributions consequently follows from Theorem 3.6. The second equality is provided by unfolding. The proof is complete.

3.4. We have in particular found that the zeroth Fourier-Stieljes coefficient of a $C_c(\mathbb{H})$ weak* limit $\lim_j \varphi_j^2 dA$ is the adjoint Radon transform of the $C_c(\mathbb{G})$ weak* limit $\lim_j d\Phi_{\varphi_{j,0}}$. In the general setting the Radon transform is not surjective and hence the adjoint is not injective, [22]. For the sake of independent interest we show that the adjoint is injective for translation invariant positive measures. The first matter is a formula for the Laplace transform of G.

The error function is defined as $\operatorname{Erf}(u) = 2\pi^{-1/2}\int_0^u e^{-v^2}dv$ [13, pg.387]. The Laplace transform of the error function is given as

$$\int_0^\infty e^{-sa}e^{-at^{-2}}\operatorname{Erf}(ia^{1/2}t^{-1})da = its^{-1/2}(t^2s+1)^{-1} \quad \text{for } s,t>0 \quad [13, \text{pg.176(5)}]$$

(the current s, a and t correspond respectively to p, t and $ia^{-1/2}$ in the Bateman manuscript). From the relation $G(u) = \pi^{1/2}(2i)^{-1}ue^{-u^2}\,\mathrm{Erf}(iu)$ the formula for the Laplace transform of G follows

$$\int_0^\infty e^{-sa}a^{-1/2}G(a^{1/2}t^{-1})\,da = \pi^{1/2}(2s^{1/2}(t^2s+1))^{-1} \quad \text{for} \quad t, s > 0. \tag{3.5}$$

We are ready to show that certain positive measures are determined by their G-transforms. Let $\mathcal{M}_{\mathrm{sub1}}((0,\infty))$ be the cone of positive measures ν on $(0,\infty)$ with cumulative distribution functions $\Upsilon(t) = \int_0^t \nu$ being $O(t)$ for t large. The bound (3.2) and considerations in the proof of Theorem 3.6 provide that the particular measures $\sigma = \mu_{\mathrm{limit},0}$ are elements of $\mathcal{M}_{\mathrm{sub1}}((0,\infty))$. An integration by parts and the bound that $G'(u)$ is $O(u)$ for u small are combined to show that the G-transform $\int G(a^{1/2}t^{-1})\nu(t)$ is finite for ν in $\mathcal{M}_{\mathrm{sub1}}((0,\infty))$. A similar analysis provides that the G-transform is $O(a^{1/2})$ for a large. From (3.5) we have a formula for the Laplace transform of $\omega \in \mathcal{M}_{\mathrm{sub1}}((0,\infty))$

$$\int_0^\infty\int_0^\infty e^{-sa}a^{-1/2}G(a^{1/2}t^{-1})\omega(t)\,da = \pi^{1/2}\int_0^\infty (2s^{1/2}(t^2s+1))^{-1}\omega(t) \tag{3.6}$$

convergent for all positive s.

Proposition 3.8 *Notation as above. The elements of $\mathcal{M}_{\mathrm{sub1}}((0,\infty))$ are uniquely determined by their G-transforms.*

Proof. Observe that $s^{1/2}(t^2s+1) = s^{3/2}(\tau + s^{-1})$ for $\tau = t^2$ and thus (3.6) (modulo the factor of $\pi^{1/2}s^{-3/2}$) is the Stieljes transform in s^{-1} of $\omega(\sqrt{\tau})$, [46, Chap.VIII]. Positive measures on $(0,\infty)$ with convergent Stieljes transforms are uniquely determined by their transforms [46, Chap.VIII, Theorem 5b]. The proof is complete.

Widder also provides an inversion formula for the Stieljes transform [46, Chap.VIII, Theorem 10a]. The inversion is given as a limit of indefinite integrals of increasing order differential expressions in the Stieljes transform.

4 The Macdonald-Bessel microlocal lift and the geodesic-indicator measure

4.1. We introduce the formalism for the microlocal lift of the Macdonald-Bessel functions and develop the analysis to consider automorphic eigenfunctions. The foundation for the analysis is the norm of the *raisings* and *lowerings* of the Macdonald-Bessel functions. An exact norm formula is presented in Lemma 4.3 with further bounds presented in Lemma 4.4. The bounds are necessary to analyze the microlocal lift of an automorphic eigenfunction in terms of its Fourier series expansion. In Section 4.3 the bounds are also used to show that the Macdonald-Bessel microlocal lifts are uniformly bounded tempered distributions. We present the Zelditch equation for a general microlocal lift in Lemma 4.7. In Lemma 4.8 and Theorem 4.9 the Zelditch equation, certain bounds and integration by parts are combined to establish the approximation of the Macdonald-Bessel microlocal lifts to the geodesic-indicator on $SL(2;\mathbb{R})$. The approximation is uniform for a parameter range and large eigenvalues. The consideration of automorphic eigenfunctions is taken up in Section 4.4. The main result

$$Q_{\text{limit}} = \frac{\pi}{8}\int_{\mathbb{G}} \Delta_{\widehat{\alpha\beta}}\mu_{\text{limit}}$$

is developed in Theorem 4.11. First consequences are presented in Corollary 4.12 and the connection to coefficient sums S_φ is presented in Corollary 4.13. In the final section we use Zelditch's result to study congruence subgroups. We first determine the spectral average of coefficient sums. Then we compare the result that a full-spectral-density sequence of modular eigenfunctions microlocally converges to a constant [52] and the residue formula for the Rankin-Selberg convolution L-function.

4.2. We review the formalism for $SL(2;\mathbb{R})$, [30]. An element $B \in SL(2;\mathbb{R})$ has the unique Iwasawa decomposition

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

which provides for an equivalence of $SL(2;\mathbb{R})$ with $S^*(\mathbb{H})^{1/2}$ the square root of the unit cotangent bundle to the upper half plane by the rule

$$x + iy = y^{1/2}e^{i\theta}(ai + b), \quad y^{-1/2}e^{i\theta} = d - ic$$

for $z = x + iy \in \mathbb{H}$ and θ the argument for the root cotangent vector measured from the positive vertical. The equivalence will play a basic role throughout the chapter. The bi-invariant volume form (Haar measure) for $SL(2;\mathbb{R})$ is $d\mathcal{V} = y^{-2}dxdyd\theta$. The Lie algebra acts on the right of $SL(2;\mathbb{R})$ with

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } E^{\pm} = H \pm iV.$$

The infinitesimal generator of geodesic-flow is $H = \frac{1}{2}(E^+ + E^-)$; W is the infinitesimal generator of K, the fiber rotations of $S^*(\mathbb{H})^{1/2}$.

A function u on $\mathbb{H}$ satisfying the differential equation $Du + (\frac{1}{4} + r^2)u = 0$ lifts to a K-invariant function on $SL(2;\mathbb{R})$ satisfying $\mathcal{C}u = (2ir+1)(2ir-1)u$ for the Casimir operator $\mathcal{C} = E^-E^+ - W^2 - 2iW$. The Casimir operator is in the center of the enveloping algebra. A ladder of functions, the *raisings* and *lowerings* of u, is determined by the scheme

$$\begin{aligned} u_0 &= u \\ (2ir + 2m + 1)u_{2m+2} &= E^+u_{2m} \\ (2ir - 2m + 1)u_{2m-2} &= E^-u_{2m} \end{aligned} \tag{4.1}$$

for m integral. The function u_{2m} is in the weight $2m$ irreducible representation for K as demonstrated by $Wu_{2m} = i2m\ u_{2m}$. The sum $u^\infty = \sum_m u_{2m}$ is a distribution that is N-invariant and an eigendistribution of H [50, pg.44;][49, Prop. 2.2].

Elements of the Lie algebra $sl(2;\mathbb{R})$ preserve the volume form and can be integrated by parts. In particular consider B in the Lie algebra with corresponding flow on $\Gamma_\infty\backslash SL(2;\mathbb{R})$ either periodic, or with forward and backward flows converging to the one-point-compactification infinity. Examples of such elements are H, V and X. The resulting integral $\int_{\Gamma_\infty\backslash SL(2;\mathbb{R})} B\kappa d\mathcal{V}$ vanishes for smooth functions κ vanishing at the one-point infinity with $|B\kappa|$ integrable. The vanishing of the integral is the consequence of the Fundamental Theorem of Calculus applied along the trajectories of the flow.

We will consider Γ_∞-invariant solutions u, v of the equation $\mathcal{C}u = (2ir+1)(2ir-1)u$ and a smooth function χ. Provided the product $\kappa = u_{2j}\overline{v_{2k}}\chi$ vanishes at infinity and $|E^+(u_{2j}\overline{v_{2k}}\chi)|$ is integrable we have the relation

$$0 = \int_{\Gamma_\infty\backslash SL(2;\mathbb{R})} ((E^+u_{2j})\overline{v_{2k}}\chi + u_{2j}\overline{E^-v_{2k}})\chi + u_{2j}\overline{v_{2k}}E^+\chi d\mathcal{V}. \tag{4.2}$$

In terms of the coordinates (x, y, θ) for $SL(2;\mathbb{R})$ the operator E^+ is simply $E^+ = 4iye^{2i\theta}\frac{\partial}{\partial z} - ie^{2i\theta}\frac{\partial}{\partial\theta}$ and the operator X is simply $y\cos 2\theta\frac{\partial}{\partial x} + y\sin 2\theta\frac{\partial}{\partial y} + y\sin^2\theta\frac{\partial}{\partial\theta}$, [30].

We are ready to study the normalized Macdonald-Bessel solutions for the equation $\mathcal{C}u = (2ir+1)(2ir-1)u$.

Definition 4.1 *For $r \in \mathbb{R},\ n \in \mathbb{Z},\ t = 2\pi nr^{-1}$ and $z = x + iy \in \mathbb{H}$ set*

$$\mathcal{K}(z,t) = (ry\sinh\pi r)^{1/2}K_{ir}(2\pi|n|y)e^{2\pi inx}.$$

To investigate the integrals of $\mathcal{K}$ we introduce a pairing and semi-norm. For suitable functions f and g on $SL(2;\mathbb{R})$ define

$$\langle f, g\rangle_\natural = \int_0^\infty f(z,\theta)\overline{g(z,\theta)}y^{-1}dy \quad\text{and}\quad \|f\|_\natural = \langle f, f\rangle_\natural^{1/2}.$$

As a preparatory matter to considering $\|\mathcal{K}\|_\natural$ we recall that $\left(y\frac{d}{dy}\right)^k(y^{1/2}K_{ir}(y))$ is $O(y^{1/2})$ at zero and $O(e^{(\epsilon-1)y})$ at infinity for each positive ϵ and integer k. To establish the vanishing first note that the product $y^{1/2}K_\nu(y)$ is given as $Re\ y^{1/2+\nu}f(y)$ for an entire function f, [31]. The function y^μ is an eigenfunction of the operator $y\frac{d}{dy}$; the vanishing at the origin follows from the two observations. Now for z large the function $(2z\pi^{-1})^{1/2}K_\nu(z)$ has the asymptotic expansion $e^{-z}(1 + O(|z|^{-1}))$ in a sector about the positive real-axis, [31]. From the Cauchy estimates it follows that $|(y^{1/2}K_\nu(y))^{(j)}|$ is $O(e^{(\epsilon-1)y})$ for each integer j, positive ϵ and appropriate constants. The vanishing of $(y\frac{d}{dy})^k(y^{1/2}K_{ir}(y))$ at infinity is a consequence.

Lemma 4.2 *Notation as above.* $\langle\mathcal{K}_{2m},\mathcal{K}_{2m+2}\rangle_\natural = (2ir-2m-1)^{-1}|t^{-1}r|\pi e^{-2i\theta}$.

Proof. We proceed by induction and first consider $m=0$. For $\mathcal{K}=f(y)e^{2\pi inx}$ then $E^+\mathcal{K}=e^{2i\theta}2iy((2\pi in)f(y)-if'(y))e^{2\pi inx}$ and

$$\langle\mathcal{K},\mathcal{K}_2\rangle_\natural = e^{-2i\theta}(-2ir+1)^{-1}\int_0^\infty((-4\pi n)f^2(y)+\frac{d}{dy}f^2(y))dy$$

with f vanishing at zero and infinity. The first integrand has a tabulated integral which gives the desired expression [12, (49) Sec. 10.3], [15, 4. Sec. 6.576]. The second integrand has a vanishing integral. We next establish a recursion. The product $\mathcal{K}_{2m}\overline{\mathcal{K}_{2m}}$ is a function of y alone and as such E^- acts simply as $2ye^{-2i\theta}\frac{d}{dy}$ and since $\mathcal{K}_{2m}$ vanishes at zero and infinity then the integral $\int_0^\infty y^{-1}E^-(\mathcal{K}_{2m}\overline{\mathcal{K}_{2m}})dy$ vanishes. Now $E^-(\mathcal{K}_{2m}\overline{\mathcal{K}_{2m}}) = (2ir-2m+1)\mathcal{K}_{2m-2}\overline{\mathcal{K}_{2m}}+(-2ir+2m+1)\mathcal{K}_{2m}\overline{\mathcal{K}_{2m+2}}$ and thus $(2ir-2m+1)\langle\mathcal{K}_{2m-2},\mathcal{K}_{2m}\rangle_\natural = (2ir-2m-1)\langle\mathcal{K}_{2m},\mathcal{K}_{2m+2}\rangle_\natural$. The induction step follows. The proof is complete.

Lemma 4.3 *Notation as above.*

$$\|\mathcal{K}_{2m}\|^2_\natural = \sum_{k=1}^{|m|}\frac{2|t^{-1}r|\pi sgn(m)}{(2k-1)^2+4r^2}+\frac{\pi^2}{4}|t^{-1}\tanh\pi r|$$

and in particular for $|r|\geq 1$ *then* $\|\mathcal{K}_{2m}\|^2_\natural\leq 6|t|^{-1}$.

Proof. For $m=0$ the integral is tabulated as already noted. We next derive a recursion relation. From $y^{-1}E^+(e^{-2i\theta}f(y)) = -2y^{-1}f(y)+2\frac{d}{dy}f(y)$ and the vanishing at zero and infinity we have the formula that $\int_0^\infty E^+(\mathcal{K}_{2m}\overline{\mathcal{K}_{2m+2}})y^{-1}dy = -2e^{2i\theta}\langle\mathcal{K}_{2m},\mathcal{K}_{2m+2}\rangle_\natural$. On the other hand expanding the derivative $E^+(\mathcal{K}_{2m}\overline{\mathcal{K}_{2m+2}})$ and substituting gives the relation $2e^{2i\theta}\langle\mathcal{K}_{2m},\mathcal{K}_{2m+2}\rangle_\natural = (2ir+2m+1)(\|\mathcal{K}_{2m}\|^2_\natural-\|\mathcal{K}_{2m+2}\|^2_\natural)$. Lemma 4.2 now combines with the relation to provide the induction to establish the formula for $\|\mathcal{K}_{2m}\|^2_\natural$. The resulting sum is bounded by an integral comparison. The proof is complete.

Our considerations will require further majorants for the integral of $\mathcal{K}_{2m}(z,t)$.

Lemma 4.4 *Notation as above. For* σ *positive and* m *an integer the integral* $\int_{\Gamma_0\backslash SL(2;\mathbb{R})}y^2e^{-\sigma y}\mathcal{K}_{2m}\overline{\mathcal{K}_{2m}}d\mathcal{V}$ *is bounded independent of* t *and* $|r|\geq 1$. *For* σ *negative,* t_0 *positive and* m *an integer the integral* $\int_{\Gamma_0\backslash SL(2;\mathbb{R})}y^2e^{-\sigma y}\mathcal{K}_{2m}\overline{\mathcal{K}_{2m}}d\mathcal{V}$ *is bounded independent of* $|t|\geq t_0$ *and* $|r|$ *large.*

Proof. First consider the cases for σ positive and negative with $m=0$. We start with the basic formula [31, Chap.4 Prob.7] to find

$$2\sinh\pi r\int_0^\infty e^{-\sigma y}yK_{ir}(2\pi|n|y)^2dy = \pi\int_0^\infty\int_0^\infty e^{-\sigma y}yJ_0(4\pi|n|y\sinh\tau/2)\sin r\tau d\tau dy.$$

The right hand side is absolutely convergent and so the Laplace transform [15, pg.712, 6.621,4] can be applied to obtain

$$\pi\int_0^\infty \sigma(\sigma^2+(4\pi n\sinh\tau/2)^2)^{-3/2}\sin r\tau d\tau.$$

Now we integrate by parts with $v=r^{-1}\cos r\tau$ and $u=\sigma(\sigma^2+(4\pi n\sinh\tau/2)^2)^{-3/2}$. Since $u(0)=\sigma^{-2}$ and u vanishes at infinity the result is the quantity $\pi r^{-1}\sigma^{-2}+\pi r^{-1}\int_0^\infty \cos r\tau\frac{du}{d\tau}d\tau$. Since $u(\tau)$ is decreasing and vanishes at infinity the last integral is bounded by $u(0)$. In particular given σ positive the integral $\int e^{-\sigma y}\mathcal{K}\overline{\mathcal{K}}dy$ is bounded independent of t and $r\geq 1$. The integral bound for σ negative will be based on the WKB-asymptotics for the Macdonald-Bessel functions. From Section 2.4 esp. Theorem 2.2 and Lemma 2.3 of [47] we have for $\lambda=\frac{1}{4}+r^2, Y=L\lambda^{-1/2}y, L,y>0$ and $\zeta(Y)$ the analytic solution of $\zeta\left(\frac{d\zeta}{dY}\right)^2=(1-Y^{-2})$ the bound

$$(y\sinh\pi|r|)K_{ir}(Ly)^2\leq CL^{-1}\begin{cases} Y(1-Y)^{-1/2} & , Y\leq 1\\ (1-Y^{-1})^{-1/2}e^{-4/3\,\lambda^{1/2}\zeta^{3/2}} & , Y\geq 1.\end{cases}$$

The bound follows from the cited theorem and the given bounds $|Ai(\mu)|$, $|ME(\mu)|$ and $|\mu^{-1/2}(Ai'(\mu)-Ai'(0))|\leq C|\mu|^{-1/4}$ for $\mu<0$; $|Ai(\mu)|$, $|ME(\mu)|$ and $|\mu^{-1/2}(Ai'(\mu)-Ai'(0))|\leq C|\mu|^{-1/4}e^{-2/3\mu^{3/2}}$ for $\mu>0$ and $|B_0(\mu)|\leq C(1+|\mu|)^{-2}$, [47]. From the reference for Y large positive $\frac{2}{3}(\zeta(Y))^{3/2}=Y-\frac{\pi}{2}+O(Y^{-1})$ with $\zeta(Y)$ increasing and thus given ϵ positive there exists $Y_0>1$ such that $e^{-2/3\lambda^{1/2}\zeta^{3/2}}$ is bounded by unity for $1\leq Y\leq Y_0$ and by $e^{-\lambda^{1/2}(2-\epsilon)Y}$ for $Y\geq Y_0$. We have for $|t|\geq t_0$ that $\mathcal{K}^2dy$ is bounded by $CY(1-Y)^{-1/2}dY$ for $Y\leq Y_0$ and $Ce^{-\lambda^{1/2}(2-\epsilon)Y}dY$ for $Y\geq Y_0$. Since $y=\lambda^{1/2}L^{-1}Y$ with $L=2\pi|n|, \lambda^{1/2}L^{-1}\leq t_0^{-1}$ it follows that $\int e^{-\sigma y}\mathcal{K}^2dy$ is bounded independent of r large for σ negative. The bounds for $m=0$ now follow.

We proceed by induction and assume that $I(m,\sigma)=\int_{\Gamma_0\backslash SL(2;\mathbb{R})}y^2e^{-\sigma y}\mathcal{K}_{2m}\overline{\mathcal{K}_{2m}}d\mathcal{V}$ is suitably bounded for a particular m. From the preparatory discussion for Lemma 4.2 each normalized derivative $\mathcal{K}_{2k}$ is $O(e^{(\epsilon-1)|t|ry})$ at infinity and thus each $I(k,\sigma)$ is convergent for r large. We start with $\int_{\Gamma_0\backslash SL(2;\mathbb{R})}y^2e^{-\sigma y}(E^-E^+\mathcal{K}_{2m})\overline{\mathcal{K}_{2m}}d\mathcal{V}$ and integrate by parts to find the expression

$$2e^{-2i\theta}\int_{\Gamma_0\backslash SL(2;\mathbb{R})}y^2e^{-\sigma y}(y\sigma-2)E^+\mathcal{K}_{2m}\overline{\mathcal{K}_{2m}}d\mathcal{V}-\int_{\Gamma_0\backslash SL(2;\mathbb{R})}y^2e^{-\sigma y}E^+\mathcal{K}_{2m}\overline{E^+\mathcal{K}_{2m}}d\mathcal{V}.$$

The first resulting integral from Hölder's inequality is bounded by $C_\epsilon|2ir+2m+1|(|\sigma|+1)I(m+1,\sigma)^{1/2}I(m,\sigma-\epsilon)^{1/2}$. From (4.1) the second resulting integral is simply $|2ir+2m+1|^2I(m+1,\sigma)$. Now from the Casimir equation we have $(E^-E^++4m^2+4m)\mathcal{K}_{2m}=-(4r^2+1)\mathcal{K}_{2m}$ and thus the integration by parts can be recast as an approximate relation

$|2ir+2m+1|^2|I(m,\sigma)-I(m+1,\sigma)| \leq C_\epsilon|2ir+2m+1|(|\sigma|+1)I(m+1,\sigma)^{1/2}I(m,\sigma-\epsilon)^{1/2}$. It follows that $I(m+1,\sigma)$ is bounded for σ positive independent of t and $|r| \geq 1$ and for σ negative independent of $|t| \geq t_0$ and $|r|$ large. The proof is complete.

4.3. We are ready to prescribe the microlocal lift of the Macdonald-Bessel functions to $SL(2;\mathbb{R})$. The construction is motivated by S. Helgason's eigenfunction representation theorem [21, 35] and is based on the ladder of *raisings* and *lowerings* [50, Sec.1] [52, Sec.1]. The analysis will be based on the double Fourier-Stieljes expansion for quantities on $SL(2;\mathbb{R}) : \Gamma_\infty\backslash N$ will act on the left and K will act on the right (Γ_∞ is the group of integer translations and N the full group of translations).

Definition 4.5 *For $j,k,n \in \mathbb{Z}$, $r \in \mathbb{R}$ and $z \in \mathbb{H}$ let $t = 2\pi nr^{-1}$, $\Delta t = 2\pi|r|^{-1}$ and set $\mathcal{K}(z,t)^\infty = \sum_{m\in\mathbb{Z}} \mathcal{K}(z,t)_{2m}$, $Q_k(t) = \mathcal{K}(z,t+k\Delta t)\overline{\mathcal{K}(z,t)^\infty}$, $Q_k^j(t) = \mathcal{K}(z,t+k\Delta t)\overline{\mathcal{K}(z,t)_{-2j}}$ and $Q(t) = \sum_{k\in\mathbb{Z}} Q_k(t)$.*

Let $C_y^4(\Gamma_\infty\backslash SL(2;\mathbb{R}))$ denote the space of four-times differentiable functions with derivatives bounded by a multiple of y. We are ready to prepare the way for considering limits of the microlocal lift.

Proposition 4.6 *Notation as above. Given t_0 positive for $|t| \geq t_0$ the quantities $Q_k(t)$ and $Q(t)$ are uniformly bounded tempered distributions for $C_y^4(\Gamma_\infty\backslash SL(2;\mathbb{R}))$.*

Proof. A function $\chi \in C_y^4(\Gamma_\infty\backslash SL(2;\mathbb{R}))$ has a $\Gamma_\infty\backslash N \times K$ Fourier expansion $\chi = \sum_{k,m} \chi_{(k,m)}$ with $|\chi_{(k,m)}| \leq Cy((1+|k|)(1+|m|))^{-2}$. The pairings with $Q_k(t)$ and $\sum_k Q_k(t)$ are given respectively as

$$\sum_m \int_{\Gamma_\infty\backslash SL(2;\mathbb{R})} Q_k^m(t)\overline{\chi_{(k,2m)}}d\mathcal{V} \quad \text{and} \quad \sum_{m,k} \int_{\Gamma_\infty\backslash SL(2;\mathbb{R})} Q_k^m(t)\overline{\chi_{(k,2m)}}d\mathcal{V}.$$

The quantities $|\chi_{(k,m)}|d\mathcal{V}$ are bounded by $C((1+|k|)(1+|m|))^{-2}y^{-1}dxdyd\theta$; it follows from Lemma 4.3 that the $(k,m)^{th}$ summands are bounded by a fixed multiple of $|t|^{-1}((1+|k|)(1+|m|))^{-2}$. The distributions are uniformly bounded. The proof is complete.

S. Zelditch discovered that the essential properties of the microlocal lift are given by an exact differential equation. The following is an extension of Zelditch's result [50, pg.44].

Lemma 4.7 *For u and v weight zero eigenfunctions of the Casimir operator with eigenvalue $-(4r^2+1)$ then $(H^2+4X^2+4irH)u\overline{v^\infty} = 0$.*

Proof. We recall from Helgason's eigenfunction representation theorem it follows that $Xv^\infty = 0$ and $Hv^\infty = (2ir-1)v^\infty$ [50, pg.44]. We also note for the Casimir operator $\mathcal{C} = H^2 + (2X-W)^2 - W^2 = H^2 + 4X^2 - 4XW - 2H$ and the eigenequation $\mathcal{C}u =$

$(2ir+1)(2ir-1)u$. Now on K-weight zero vectors $\mathcal{C}$ is equivalent to H^2+4X^2-2H. We are ready to calculate $(H^2+4X^2+4irH)u\overline{v^\infty} = ((H^2u) - 2(2ir+1)Hu + (2ir+1)^2u + 4X^2u + 4irHu - 4ir(2ir+1)u)\overline{v^\infty}$. On substituting $4X^2u = (\mathcal{C} - H^2 + 2H)u$ the desired differential equation results. The proof is complete.

We review the setup from Chapter 2. $\mathbb{G}$ is the space of non vertical geodesics on the upper half plane; a non vertical geodesic is prescribed by coordinates $(\hat{x}_0, t_0)$. Associated to each point of a geodesic are the four square-root unit cotangent vectors based at the point. The association provides a lift of $\widehat{\alpha\beta}$ to $S^*(\mathbb{H})^{1/2} \approx SL(2;\mathbb{R})$ and a lift $\Delta_{\widehat{\alpha\beta}}$ of $\delta_{\widehat{\alpha\beta}}$. The measure $\Delta_{\widehat{\alpha\beta}}$ is the sum over the four lifts of the lifted infinitesimal arc-length elements. The family of measures $\Delta_{\widehat{\alpha\beta}}$ on $SL(2;\mathbb{R})$ is parameterized by the points of $\mathbb{G}$: $\Delta_{\widehat{\alpha\beta}}$ has the tensor-type of a function on $\mathbb{G}$ and line-element on $SL(2;\mathbb{R})$. We will notate the sum $\sum_{\gamma\in\Gamma_\infty} \Delta_{\gamma(\widehat{\alpha\beta})}$ by $\Delta_{\Gamma_\infty(\widehat{\alpha\beta})}$. The geodesic-indicator $\Delta_{\widehat{\alpha\beta}}$ is a section of the trivial bundle $\widehat{\mathbb{G}} \times \mathcal{M}(SL(2;\mathbb{R})) \to \widehat{\mathbb{G}}$.

We are ready to compare the Fourier-Stieljes coefficients of $Q(t)$ and $\Delta_{\Gamma_\infty(\widehat{\alpha\beta})}\, d\mathcal{V}^{-1}$. Let $\Delta^{(k,2m)}_{\Gamma_\infty(\widehat{\alpha\beta})}\, d\mathcal{V}^{-1}$ be the left-$\Gamma_\infty\backslash N$ index k, right-K index $2m$ Fourier-Stieljes coefficient. For $m \geq 2, k$ given, let

$$F = y^m f(y) e^{2\pi ikz+2im\theta} = Q^m_k(t) - \frac{\pi^2}{8}\Delta^{(k,2m)}_{\Gamma_\infty(\widehat{\alpha\beta})} d\mathcal{V}^{-1}$$

be the difference of Fourier-Stieljes coefficients and define

$$G = g(y)e^{-2\pi ikx-2i(m-1)\theta}$$

for $(x+iy,\theta)$ the coordinates for $SL(2;\mathbb{R})$, parameter value a and

$$g(y) = ay^{-m+1}e^{2\pi ky}\int_0^y \tau^m e^{-a\tau^2-2\pi k\tau}d\tau.$$

From the remarks after Definition 4.1 the function F is $O(e^{(\epsilon-2)|rt|y})$ for y large and $O(y)$ for y small. The function G is $O(e^{2\pi ky})$ for y large and $O(y^2)$ for y small. For the test function $h(y) = 2ay^2e^{-ay^2}$ set $\mathbf{h} = h(y)e^{-2\pi ikx-2im\theta}$ and we have the important equations $E^-F = 2y^{m+1}\frac{df}{dy}e^{2\pi ikz+2i(m-1)\theta}$, $E^-G = \mathbf{h}$ and $E^+G = ((4\pi ky + 2(1-m))g(y) + 2y\frac{dg}{dy})e^{-2\pi ikx-2i(m-2)\theta}$. In particular since F and G vanish at infinity we find for $|t| \geq t_0$ that

$$\int_{\Gamma_\infty\backslash SL(2;\mathbb{R})} E^-FGd\mathcal{V} = -\int_{\Gamma_\infty\backslash SL(2;\mathbb{R})} FE^-Gd\mathcal{V} = -\int_{\Gamma_\infty\backslash SL(2;\mathbb{R})} F\mathbf{h}d\mathcal{V}.$$

We begin the comparison of Q_k and $\Delta_{\Gamma_\infty(\widehat{\alpha\beta})}$ for $SL(2;\mathbb{R})$ weights congruent to zero modulo four.

Lemma 4.8 *Notation as above. Given* $0 < t_0 < t_1$ *for integers* m*, nonnegative, even, and* k*, let* $\mathbf{h} = h(y)e^{-2\pi ikx-2im\theta}$*; for* $t = 2\pi nr^{-1}$ *and* $\widehat{\alpha\beta}$ *the geodesic with coordinate* $(0,|t|)$ *the integral*

$$\int_{\Gamma_\infty\backslash SL(2;\mathbb{R})} (Q_k(t) - \frac{\pi^2}{8}\Delta_{\Gamma_\infty(\widehat{\alpha\beta})} d\mathcal{V}^{-1})\,\mathbf{h}d\mathcal{V}$$

is close to zero uniformly for $|r|$ *large,* $t_0 \le |t| \le t_1$ *and* a *bounded.*

Proof. The argument will be by induction on the even integer m. For $m = 0$ the result follows from Proposition 2.1, Theorem 2.4 and the observation that $\Delta_{\widehat{\alpha\beta}} = \delta_{\widehat{\alpha\beta}} \sum_j \delta(\theta - \frac{\theta_0}{2} + j\frac{\pi}{2} + \frac{\pi}{4})$ for θ_0 the radial-angle given in Proposition 2.1. The constant is determined by integration over K. The induction step will involve the generator H of geodesic-flow. We have from the flow equations $(H^2 + 4X^2 + 4irH)Q_k = 0$ and $H(\Delta_{\Gamma_\infty(\widehat{\alpha\beta})} d\mathcal{V}^{-1}) = 0$ the combined equation

$$(4.3)\qquad \begin{aligned} &\int_{\Gamma_\infty\backslash SL(2;\mathbb{R})} 2(E^-F)Gd\mathcal{V} = \\ &\int_{\Gamma_\infty\backslash SL(2;\mathbb{R})} (-2E^+(Q_k - \frac{\pi^2}{8}\Delta_{\Gamma_\infty(\widehat{\alpha\beta})} d\mathcal{V}^{-1}) + ir^{-1}(H^2+4X^2)Q_k)Gd\mathcal{V}. \end{aligned}$$

We shall analyze the magnitude of the second integral. The operator E^+ is weight-graded with weight two; H and X are sums of graded-operators with weights $-2, 0, 2$ and $H^2 + 4X^2$ a sum with even weights $-4\cdots 4$. For the second integral since G has weight $2-2m$ the distribution $Q_k(t)$ can be replaced with the smooth function $\widehat{Q}(t) = \sum_{j=m-3}^{m+1} Q_k^j(t)$ and $\Delta_{\Gamma_\infty(\widehat{\alpha\beta})}$ replaced by $\Delta^{(k,2m-4)}_{\Gamma_\infty(\widehat{\alpha\beta})}$. We wish to integrate by parts. To that end we must determine the order of magnitude for the derivatives of $\widehat{Q}_k$ and G. The derivatives E^+, H and X are each a linear combination of $y\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}$, and $\frac{\partial}{\partial\theta}$ with coefficients depending only on θ. The operator $H^2 + 4X^2$ is likewise quadratic in the operators $y\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial\theta}$. The E^+, H, X and $H^2 + 4X^2$ derivatives of $\widehat{Q}_k(t)$ are each $O(y)$ at zero and $O(e^{(\epsilon-2)r|t|y})$ at infinity. The derivatives $\frac{\partial G}{\partial y}$ and $\frac{\partial^2 G}{\partial y^2}$ have order respectively $O(y)$ and $O(1)$ at zero and $O(e^{(2\pi k+\epsilon)y})$ at infinity. It follows that the derivatives E^+, H, X and $H^2 + 4X^2$ of $\widehat{Q}_k G$ are at least $O(y^3)$ at zero and $O(e^{(\epsilon-2)r|t|y})$ at infinity for r large. It further follows that the operators acting on $\widehat{Q}_k$ can be integrated by parts and likewise for $\Delta^{(k,2m-4)}_{\Gamma_\infty(\widehat{\alpha\beta})}$ giving

$$(4.4)\qquad \int_{\Gamma_\infty\backslash SL(2;\mathbb{R})} 2(\widehat{Q}_k - \frac{\pi^2}{8}\Delta^{(k,2m-4)}_{\Gamma_\infty(\widehat{\alpha\beta})} d\mathcal{V}^{-1})E^+Gd\mathcal{V} - ir^{-1}\int_{\Gamma_\infty\backslash SL(2;\mathbb{R})} \widehat{Q}_k(H^2+4X^2)Gd\mathcal{V}$$

for the second integral of (4.3).

The final matter is to bound the integrals (4.4). Consider the first integral; from the definition $E^+Gd\mathcal{V}$ is bounded for y small and has order $O(e^{2\pi ky})$ for y large. Now let χ be a smooth approximate characteristic function vanishing on $\mathbb{R}^-$ and unity on $\mathbb{R}^+ +1$. From Lemma 4.4 with $-\sigma = 2\pi k+1$ it follows that $\int(\hat{Q}_k - \frac{\pi^2}{8}\Delta^{(k,2m-4)}_{\Gamma_\infty(\widehat{\alpha\beta})}d\mathcal{V}^{-1})\chi(y-b)E^+Gd\mathcal{V}$ is $O(e^{-b})$ uniformly in r for $|t|\geq t_0$. From the considerations of Corollary 3.7 $(1-\chi)E^+Gd\mathcal{V}$ is approximated on $[0,\infty)$ by linear combinations of $hd\mathcal{V}$ with bounds in terms of $y^{1+\epsilon}e^{-\epsilon y}$. It now follows by the induction hypothesis (and Lemma 4.3 to bound the approximation contribution) that the first integral of (4.4) tends to zero as r tends to infinity, uniformly for $t_0 \leq |t| \leq t_1$. For the second integral from the given bounds $(H^2+4X^2)G$ has order $O(y^2)$ at zero and $O(e^{(2\pi k+\epsilon)y})$ at infinity. From Lemma 4.4 the contribution of the second integral is $O(r^{-1})$, uniformly for $t_0 \leq |t| \leq t_1$. In conclusion the right hand side of (4.3) tends to zero as r tends to infinity. As already noted the left hand side is the pairing of F and $\mathbf{h}$. The proof is complete.

We are now ready to present the main result on semi-classical convergence of products of Macdonald-Bessel functions. We write $Q^{\text{symm}}(t)$ for the restriction of the distribution to functions invariant by the right action of $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$, equivalently the restriction to functions with non trivial K components only for weights congruent to zero modulo four.

Theorem 4.9 *Notation as above. Given $0<t_0<t_1$ for $t=2\pi nr^{-1}$ and $\widehat{\alpha\beta}$ the geodesic with coordinate $(0,|t|)$ the $C^4_y(\Gamma_\infty\backslash SL(2;\mathbb{R}))$ tempered distribution $Q^{\text{symm}}(t)d\mathcal{V}$ is close to the positive measure $\frac{\pi^2}{8}\Delta_{\Gamma_\infty(\widehat{\alpha\beta})}$ uniformly for r large and $t_0\leq|t|\leq t_1$. Furthermore for integers m,j and k with $m-j$ even let $\mathbf{h}=h(y)e^{-2\pi ikx-2i(m-j)\theta}$; the integral*

$$\int_{\Gamma_\infty\backslash SL(2;\mathbb{R})}(\mathcal{K}(z,t+k\Delta t)_{2m}\overline{\mathcal{K}(z,t)_{2j}}-\frac{\pi^2}{8}\Delta_{\Gamma_\infty(\widehat{\alpha\beta})}d\mathcal{V}^{-1})\mathbf{h}d\mathcal{V}$$

is close to zero uniformly for r large, $t_0\leq|t|\leq t_1$ and a bounded.

Proof. We first consider the pairings of products with $\mathbf{h}$ and extend Lemma 4.8 to the case of m negative. From the definition (4.1) we have that $\overline{u_{2m}}=(-1)^m(\bar{u})_{-2m}+O(r^{-1}|u_{2m}|)$ and further from Definition 4.1 that $\overline{\mathcal{K}(z,t)}=\mathcal{K}(z,-t)$. In consequence from Definition 4.5 we have that $\overline{Q^m_k(t)}=(-1)^m\mathcal{K}(z,-t-k\Delta t)\overline{\mathcal{K}(z,-t)_{2m}}+O(r^{-1}|Q^m_k|)=(-1)^mQ^{-m}_{-k}(-t)+O(r^{-1}|Q^m_k|)$. The pairing of the remainder with $\mathbf{h}$ is $O(r^{-1})$ by Lemma 4.4. The result of Lemma 4.8 is now extended to the case of m negative by simply considering conjugates. The result further extends to the products $\mathcal{K}(z,t+k\Delta t)_{2m}\overline{\mathcal{K}(z,t)_{2j}}$ by successively applying the relation (4.2) for $\mathcal{K}(z,t+k\Delta t)\overline{\mathcal{K}(z,t)_{2j-2m}}\mathbf{h}$ and using Lemma 4.3 to bound the resulting remainder terms.

We are ready to consider the sums $Q^{\text{symm}}(t)$. Since the sums are uniformly bounded distributions by Proposition 4.6 weak* convergence is provided for by the termwise convergence of Fourier-Stieljes expansions. We will follow the argument for Lemma 4.8.

In particular consider $g(y)$ in $C_y(\mathbb{R}^+)$ and χ a smooth approximate characteristic function vanishing on $\mathbb{R}^-$ and unity on $\mathbb{R}^+ + 1$. From Lemma 4.4 the integral $\int (Q_k^m(t) - \frac{\pi^2}{8}\Delta^{(k,2m)}_{\Gamma_\infty(\widehat{\alpha\beta})} d\mathcal{V}^{-1})\chi(y-b)g(y)d\mathcal{V}$ is $O(e^{-b})$ uniformly in r for $|t| \geq t_0$. From the considerations of Corollary 3.7 $(1-\chi)g(y)$ is approximated on $[0,\infty)$ by linear combinations of $hd\mathcal{V}$ with bounds in terms of $y^{1+\epsilon}e^{-\epsilon y}$. The Fourier-Stieljes coefficients of $Q^{\mathrm{symm}}(t)$ paired with the combinations of h converge by the extension of Lemma 4.8 and the approximation contribution is bounded by Lemma 4.3. The proof is complete.

4.4. We are ready to study the semi-classical limits of microlocal lifts of automorphic eigenfunctions. Again let $\Gamma \subset SL(2;\mathbb{R})$ be a cofinite group with a width-one cusp and φ an $L^2(\Gamma\backslash\mathbb{H})$ eigenfunction with unit-norm. The function φ lifts to a K-invariant function on $SL(2;\mathbb{R})$ satisfying $\mathcal{C}\varphi = (2ir+1)(2ir-1)\varphi$. We will consider the ladder $\{\varphi_{2m}\}$ of *raising* and *lowerings*, as well as the distribution $\varphi^\infty = \sum_m \varphi_{2m}$. The ladder $\{\varphi_{2m}\}$ is an orthogonal basis for an irreducible *principal continuous series* representation of $SL(2;\mathbb{R})$, [30]. For the $L^2(\Gamma\backslash SL(2;\mathbb{R}))$ Hermitian product $\langle\varphi_{2m},\varphi_{2m}\rangle = 2\pi$ is satisfied and from integration by parts $\langle E^+\varphi_{2j},\varphi_{2k}\chi\rangle + \langle\varphi_{2j},E^-(\varphi_{2k}\chi)\rangle = 0$ for a Γ-invariant test function χ.

Definition 4.10 *Set* $Q(\varphi) = \varphi\overline{\varphi^\infty}$.

The microlocal lift $Q(\varphi)$ is a basic quantity of the ΨDO-calculus based on Helgason's Fourier transform [50, Section 1];[52, Section 1]. The quantity $Q(\varphi)$ gives a Γ-invariant $C^2(\Gamma\backslash SL(2;\mathbb{R}))$ distribution, bounded independent of φ. We are ready to extend the considerations of Theorem 3.6 to the context of $SL(2;\mathbb{R})$ and present the main result for automorphic eigenfunctions. Let $\{\varphi_j\}$ be a $*$-convergent sequence (see Section 3.2) with limiting measure $\mu_{\mathrm{limit}} = \lim_j \mu_{\varphi_j}$ on the space of non vertical geodesics $\mathbb{G}$.

Theorem 4.11 *Notation as above. For ϵ positive the sequence $Q(\varphi_j)d\mathcal{V}$ converges with limit*

$$Q_{\mathrm{limit}} = \frac{\pi}{8}\int_{\mathbb{G}} \Delta_{\widehat{\alpha\beta}}\mu_{\mathrm{limit}}$$

in the sense of tempered positive distributions relative to $C^\infty_{y^{1+\epsilon}e^{-\epsilon y}}(\Gamma_\infty\backslash SL(2;\mathbb{R}))$.

Proof. First we show that the terms $\varphi\overline{\varphi_{2m}}$ with m odd do not contribute to the limit. From (4.1) we have that $\overline{u_{-2m}} = (-1)^m(\bar{u})_{2m} + O(r^{-1}|u_{2m}|)$. Since φ is real we have that $\varphi\overline{\varphi_{-2m}} = -\varphi\varphi_{2m} + O(r^{-1})$ in the sense of tempered distributions. Now by a repeated application of (4.2) we have for $m = 2q+1$ that $4i\varphi\varphi_{2m} = (-1)^q r^{-1}E^+((\varphi_{2q})^2) + O(r^{-1})$. The leading-term $E^+((\varphi_{2q})^2)$ is itself a bounded distribution; for m odd $\varphi\varphi_{2m}$ and $\varphi\overline{\varphi_{-2m}}$ have magnitude $O(r^{-1})$ and thus do not contribute to the limit.

We next show that Q_{limit} is a positive distribution in order to simplify a later argument. From integration by parts we have the relation $\varphi_{2j+2}\overline{\varphi_{2k}} = \varphi_{2j}\overline{\varphi_{2k-2}} + O(r^{-1})$

in the sense of tempered distributions. For M a positive integer we introduce the Fejér sum $Q_M(\varphi) = (2M+1)^{-1}\big|\sum_{m=-M}^{M}\varphi_{4m}\big|^2$. From the relation we find that $\lim_j Q_M(\varphi_j) = \lim_j \sum_{m=-2M}^{2M}(1-\frac{|m|}{2M+1})\varphi_j\overline{\varphi_{j,4m}}$ and consequently that $\lim_M \lim_j Q_M(\varphi_j)d\mathcal{V} = Q_{\text{limit}}^{\text{symm}} = Q_{\text{limit}}$ is a positive distribution.

We are ready to consider the general situation. Since the distributions $Q(\varphi_j)$ are uniformly bounded it is enough to examine the limit of their $\Gamma_\infty\backslash N\times K$ Fourier-Stieljes coefficients. To this purpose we will establish the convergence of the index-k $\Gamma_\infty\backslash N$ Fourier-Stieljes coefficient of $\varphi_j\overline{\varphi_{j,2m}}d\mathcal{V}$. To this purpose for $\mathbf{h}=h(y)e^{-2\pi ikx+2im\theta}$ we first consider the convergence of

$$\begin{aligned}&\int_{\Gamma_\infty\backslash SL(2;\mathbb{R})}\varphi_j\overline{\varphi_{j,2m}}\mathbf{h}d\mathcal{V}=\\ &2\pi e^{2im\theta}\int_{\Gamma_\infty\backslash\mathbb{H}}\sum_n a_{n+k}\overline{a_n}r^{-1}\mathcal{K}(z,t+k\Delta t)\overline{\mathcal{K}(z,t)_{2m}}h(y)dA\end{aligned}\tag{4.5}$$

for $t=2\pi nr^{-1}$ and the Fourier expansion (3.1). The n-sum will be considered in three ranges in terms of the parameter t. We first show that given $0<t_0<t_1$ the combined consideration from terms with $|t|\le t_0$ or $|t|\ge t_1$ is bounded by $o(1)$ for t_0 small and t_1 large. For terms with $|t|\le t_0$ we first note from Lemma 4.4 and Hölder's inequality that the integrals $\int|\mathcal{K}\mathcal{K}_{2m}|hdA$ are uniformly bounded in t and r. The total contribution for $|t|\le t_0$ is bounded by a multiple of $\sum_{n=1}^{rt_0/2\pi+k}|a_n|^2r^{-1}$. The result on $\lim_j d\Phi_{j,0}$ from Theorem 3.6 provides that the sum is $o(1)$, as desired. Now for $|t|\ge t_1$, from Lemma 2.5 then $\int|\mathcal{K}|^2hd\mathcal{V}$ is $O(|t|^{-2})$ and since $h(y)$ is dominated by y then $\int|\mathcal{K}_{2m}|^2hd\mathcal{V}$ is $O(|t|^{-1})$ by Lemma 4.3. It follows from Hölder's inequality that the n-sum for terms $|t|\ge t_1$ is bounded by $r^{1/2}\sum_{n=rt_1/2\pi}|a_n|^2n^{-3/2}$. This last sum is evaluated by parts for $u(t)=\sum_{n=rt_1/2\pi}^{rt/2\pi}|a_n|^2r^{-1}$ and from (3.2) is found to be $O(t_1^{-1/2})$, a suitable bound.

We are ready to compare (4.5) to

$$\frac{\pi}{8}\int_{\Gamma_\infty\backslash SL(2;\mathbb{R})\times\Gamma_\infty\backslash\mathbb{G}}\mathbf{h}\Delta_{\Gamma_\infty(\widehat{\alpha\beta})}\mu_{\text{limit}}$$

The Fourier-Stieljes coefficient of $\Delta_{\Gamma_\infty(\widehat{\alpha\beta})}$ are dominated by the $m=0,k=0$ coefficient; it follows from Theorem 3.6, (3.2) and (2.2) that the contribution to the integral from the $\Gamma_\infty\backslash\mathbb{G}-$region $|t|\le t_0, t_1\le|t|$ is $o(1)$. For the principal range $t_0\le|t|\le t_1$ the convergence of terms of (4.5) for m even to the indicated integral is provided by Theorem 4.9 and the bound (3.2). The equality of integrals is established for the test function $\mathbf{h}$.

We are ready to extend the result to a larger class of test-functions. From the approximation considerations of Corollary 3.7 for $g \in C_{y^{1+2\epsilon}e^{-2\epsilon y}}(\mathbb{R}^+)$ linear combinations of $h(y)$ approximate $g(y)$ with bounds in terms of $y^{1+\epsilon}e^{-\epsilon y}$. Now the positivity of Q_{limit} and the bound $\int_{y_0 \leq y} Q_{\text{limit}} \leq C(y_0^{-1}+1)$ from the proof of Theorem 3.5 can be combined with integration by parts to give that Q_{limit} is a tempered distribution for $C_{y^{1+\epsilon}e^{-\epsilon y}}(\Gamma_\infty \backslash SL(2;\mathbb{R}))$. Similarly (2.2) can be combined with (3.2) to give that $\int \Delta_{\widehat{\alpha\beta}}\mu_{\text{limit}}$, a positive distribution, is tempered for $C_{y^{1+\epsilon}e^{-\epsilon y}}(\Gamma_\infty \backslash SL(2;\mathbb{R}))$. It follows that linear combinations of the pairings of Q_{limit} and $\int \Delta_{\widehat{\alpha\beta}}\mu_{\text{limit}}$ with $\mathbf{h}$ approximate the pairings with g. The formula is established in the sense of tempered positive distributions relative to $C_{y^{1+\epsilon}e^{-\epsilon y}}(\Gamma_\infty \backslash SL(2;\mathbb{R}))$. The proof is complete.

Theorem 4.11 has immediate applications for semi-classical limits.

Corollary 4.12 *Notation as above. The vertical geodesics form a null set for Q_{limit} and μ_{limit} is Γ-invariant.*

Proof. We begin with simple observations. Geodesic flow provides a fibration by trajectories $SL(2;\mathbb{R}) \to SL(2;\mathbb{R})/\{e^{tH} \mid t \in \mathbb{R}\}$. A geodesic has four lifts to $SL(2;\mathbb{R})$ and thus the space of geodesic-flow trajectories is a four-fold cover of $\widehat{\mathbb{G}}$. Correspondingly a measure κ on $\mathbb{G}$ corresponds to a geodesic-flow invariant measure on $SL(2;\mathbb{R})$ by $\tilde{\kappa} d\ell = \int_{\mathbb{G}} \Delta_{\widehat{\alpha\beta}}\kappa$ for $d\ell$ the infinitesimal flow time. We trivially extend μ_{limit} to a measure on $\widehat{\mathbb{G}}$; the formula of Theorem 4.11 remains valid. It follows that the vertical geodesics form a null set for Q_{limit}. The Γ-invariance of μ_{limit} follows from the Γ-invariance of Q_{limit} and the formula for $\tilde{\kappa} d\ell$. The proof is complete.

The next result gives the equivalence between convergence of microlocal lifts and convergence of coefficient sums.

Corollary 4.13 *Notation as above. Let $\{\varphi_j\}$ be a sequence of eigenfunctions with bounded L^2-norms: φ_j with Fourier coefficients $\{a_n\}$ and microlocal lift $Q(\varphi_j)$. The following conditions are equivalent:*

i) the microlocal lifts $Q(\varphi_j)$ converge $C_c^\infty(\Gamma \backslash SL(2;\mathbb{R}))$ weak to unity;*

ii) the coefficient distributions μ_{φ_j} converge $C_c^2(\widehat{\mathbb{G}})$ weak to $8\pi^{-1}\omega$ for ω the natural measure on $\widehat{\mathbb{G}}$;*

iii) $|S_{\varphi_j}(t,\hat{x})|^2$ converges to $4\pi^{-1}t$ weak in $\hat{x}$ for each positive t for $(t,\hat{x})$ coordinates for $\mathbb{R}^+ \times \mathbb{R}$.*

Proof. It will suffice to consider subsequences since in each instance the actual limits are predetermined. Consider a $*$-convergent subsequence with limits Q_{limit} on $SL(2;\mathbb{R})$

and μ_{limit} on $\widehat{\mathbb{G}}$. The first equivalence follows from Theorem 4.11 and the relation $d\mathcal{V} = \int_{\mathbb{G}} \Delta_{\widehat{\alpha\beta}}\omega$. We are ready to consider the equivalence of the second and third conditions. From the bounds following Definition 3.1 the weak* limit μ_{limit} has Fourier-Stieljes coefficients $\lim_j d\Phi_{\varphi_j,k}$. Since $2\omega = dtd\hat{x}$ the second condition specifies $\lim_j d\Phi_{j,k}$ as $4\pi^{-1}$ for k zero and as zero for k nonzero. Furthermore from Theorem 3.6 each limit $\lim_j \Phi_{\varphi_j,k}$ is $o(1)$ for t small. It follows that the second condition is equivalent to the pointwise convergence in t of $\lim_j \Phi_{\varphi_j,k}$ as $4\pi^{-1}t$ for k zero and as zero for k nonzero. From Section 3.2 for each t positive $|S_{\varphi_j}(t,\hat{x})|^2$ is positive with a uniformly bounded integral in $\hat{x}$. In this setting convergence of Fourier-Stieljes coefficients is equivalent to weak* convergence. The proof is complete.

4.5. We consider two applications for $SL(2;\mathbb{Z})$. A. Selberg showed for congruence subgroups that the cumulative spectral distribution is in effect given by the counting of point-spectrum, [19, 45]. In particular for $\{(\varphi_j, \lambda_j)\}$ an orthonormal basis for $L^2(\Gamma\backslash\mathbb{H})$-eigenpairs then $N_\Gamma(R) = \#\{j \mid \lambda_j \leq R^2\} = Area(\Gamma\backslash\mathbb{H})R^2/4\pi + O(R\log R)$, [19, 45]. S. Zelditch showed for R tending to infinity that

$$R^{-2}\sum_{0\leq r_j\leq R}\left|Q(\varphi_j) - \frac{2\pi}{Area(\Gamma\backslash\mathbb{H})}\right| \text{ converges to zero} \tag{4.6}$$

for $Q(\varphi_j)$ and unity denoting the associated $C_c^2(SL(2;\mathbb{R}))$ functionals given by integration [52, Theorem 5.1]. An application of the result and the current considerations is the spectral average of coefficient sums.

Corollary 4.14 *Notation as above. For a congruence subgroup and an orthonormal basis of eigenfunctions* $R^{-2}\sum_{0\leq r_j\leq R}\left||S_{\varphi_j}(t,\hat{x})|^2 - \frac{8t}{Area(\Gamma\backslash\mathbb{H})}\right|$ *converges to zero weak* in $\hat{x}$ for each positive t as R becomes large.*

Proof. We first observe that the association $\chi \to \int_{SL(2;\mathbb{R})}\chi\Delta_{\widehat{\alpha\beta}}$ defines a surjective mapping from $C_c^2(SL(2;\mathbb{R}))$ to $C_c^2(\widehat{\mathbb{G}})$. It follows from Theorem 4.11 that $\frac{\pi}{8}\int_{\widehat{\mathbb{G}}}\Delta_{\widehat{\alpha\beta}}\mu_{\varphi_j}$ can be substituted in (4.6) for $Q(\varphi_j)$ to obtain that

$$R^{-2}\sum_{0\leq r_j\leq R}\left|\mu_{\varphi_j} - \frac{16\omega}{Area(\Gamma\backslash\mathbb{H})}\right| \text{ converges to zero}$$

for R large and μ_{φ_j}, ω denoting the associated $C_c^2(\widehat{\mathbb{G}})$ distributions. The desired conclusion now follows from the considerations of Corollary 4.13. The proof is complete.

Zelditch further found for a congruence subgroup that there is a subsequence of unit-norm eigenfunctions $\{\varphi_{j_k}\}$ with full spectral density ($\#\{j_k \mid \lambda_{j_k}^{1/2} \leq R\} \sim N_\Gamma(R)$)

such that $\varphi_{j_k}^2$ converges to $((Area(\Gamma\backslash\mathbb{H}))^{-1}$ relative to $C_c(\Gamma\backslash\mathbb{H})$, [52]. We apply our considerations for the special sequence $\{\varphi_{j_k}\}$ to find the mean-square Fourier coefficient. From Corollary 4.13 the coefficient partial-sums $\Phi_{k,0}(t) = \pi r_{j_k}^{-1} \sum_{|n|\le r_{j_k} t(2\pi)^{-1}} |a_n(\varphi_{j_k})|^2$ converge pointwise in t to $4(\pi(Area(\Gamma\backslash\mathbb{H}))^{-1}$ for $\lambda_{j_k} = \frac{1}{4} + r_{j_k}^2$. It follows for sums of length proportional to the square root of the eigenvalue that the mean-square Fourier coefficient (as given in (3.1)) converges to $4(\pi(Area(\Gamma\backslash\mathbb{H}))^{-1}$.

A heuristic calculation with the Rankin-Selberg convolution L-function for a congruence subgroup will give the same value. The Eisenstein series $\mathcal{E}(z;s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\operatorname{Im}\gamma(z))^s$ is convergent for $Re\ s > 1$ and has a meromorphic continuation with a simple pole at $s = 1$ with residue $((Area(\Gamma\backslash\mathbb{H}))^{-1}$, [7]. Thus for φ a Γ eigenfunction the integral $I(s) = \int_{\Gamma\backslash\mathbb{H}} \varphi^2(z)\mathcal{E}(z;s)dA$ has a simple pole at $s = 1$ with residue $((Area(\Gamma\backslash\mathbb{H}))^{-1}$. For $Re\ s > 1$ we can unfold the Eisenstein sum to find

$$I(s) = 2\sinh\pi r \sum_{n>0} \frac{|a_n|^2}{(2\pi n)^s} I_{MB}(s;r)$$

where $I_{MB}(s;r) = \int_0^\infty u^{s-1}(K_{ir}(u))^2du = 2^{s-3}\Gamma(\frac{s}{2}+ir)\Gamma(\frac{s}{2}-ir)\Gamma(\frac{s}{2})^2\Gamma(s)^{-1}$ for $Re\ s > 0$ and the Euler Γ-function [15, Sec.6.576, 4.]; for $s = 1$ then $I_{MB}(1,r) = \pi^2 4^{-1}\operatorname{sech}\pi r$. At this point we proceed only formally: for r large, s close to unity, replace the sum $I(s)$ with $\pi 4^{-1}\,\mathrm{mean}(|a_n(\varphi)|^2)\sum_{n>0} n^{-s}$. Since the residue of the Riemann zeta function is unity we formally find that $\mathrm{mean}(|a_n(\varphi)|^2) = 4(\pi(Area(\Gamma\backslash\mathbb{H}))^{-1}$ in agreement with the above result.

5 Applications to coefficient sums

5.1. We begin with considerations of the modular Eisenstein series. The analysis of the previous chapters will be extended to again analyze the microlocal lift in terms of a measure constructed from Fourier coefficients. The Maass-Selberg relation is employed in Section 5.2 to bound the square-coefficient sum as well as the microlocal lift of the Eisenstein series. Then the approach of Chapter 3 is used to obtain a short-range coefficient sum bound. In Section 5.3 we combine our analysis to establish the analog of Theorem 4.11 for the Eisenstein series; the limit of the Eisenstein microlocal lifts is given as an integral of the limit of the coefficient measures. The representation has immediate consequences. The Luo-Sarnak and Jakobson results are shown to be equivalent to a limit-sum formula for the elementary summatory function.

The second topic concerns the normalization of the quantities Q_{limit} and μ_{limit} of Theorem 4.11. In fact a semi-classical limit is trivial if the entire L^2-mass *escapes to infinity*. By comparison a non trivial limit is necessarily obtained if the micolocal lifts are renormalized to have unit mass on a prescribed compact set. For the renormalization the L^2-norms can be tending to infinity and the analysis of the preceeding chapters does not directly apply. In Section 5.4 we modify the considerations and develop an analog of Theorem 4.11 for the index zero Fourier-Stieljes coefficients. The focus is a refined analysis of the mass distribution of the square of the Macdonald-Bessel functions. The resulting formula is presented in Proposition 5.13. In Theorem 5.14 we show that the mapping from eigenfunctions to coefficient sums with an additive character is a uniform quasi-isometry relative to L^2-norms respectively on a suitable compact set and the unit circle. The proof entails a consequence for coefficient sums of the transitivity of the geodesic flow.

5.2. We wish to study the modular Eisenstein series, [7]. We first consider the bounds for the norm and the square-coefficient sum. The Eisenstein series for $SL(2;\mathbb{Z})$ is defined by

$$\mathcal{E}(z;s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\mathrm{Im}\gamma(z))^s = \frac{1}{2}\sum_{(c,d)=1} \frac{y^s}{|cz+d|^{2s}} \tag{5.1}$$

for $Re\, s > 1$ and $z = x+iy \in \mathbb{H}; \mathcal{E}(z;s)$ admits an entire meromorphic continuation in s. The Fourier expansion is

$$\mathcal{E}(z;s) = y^s + \varphi(s)y^{1-s} + \sum_{n\neq 0} \varphi_n(s) y^{1/2} K_{s-\frac{1}{2}}(2\pi|n|y)e^{2\pi i n x} \tag{5.2}$$

where

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s),\ \varphi(s) = \frac{\xi(2s-1)}{\xi(2s)}$$

and

$$\varphi_n(s) = \frac{2|n|^{s-1/2}\sigma_{1-2s}(|n|)}{\xi(2s)}$$

with $\sigma_\nu(m) = \sum_{d|m} d^\nu$ for m positive.

The first matter is to apply the Maass-Selberg relation to find an analog for the Eisenstein series of the coefficient bound (3.2). Recall the definition of the truncated Eisenstein series for $SL(2;\mathbb{Z})$: $\mathcal{E}^Y(z,s)$ is the $SL(2;\mathbb{Z})$ invariant function given in the fundamental domain $\mathcal{F} = \{z \mid |\mathrm{Re}\, z| \le 1/2,\ |z| \ge 1\}$ by

$$\mathcal{E}^Y(z;s) = \begin{cases} \mathcal{E}(z;s) \text{ for } z\in\mathcal{F},\ \mathrm{Im}\, z \le Y \\ \mathcal{E}(z;s) - y^s - \varphi(s)y^{1-s} \quad \text{for} \quad z\in\mathcal{F},\ \mathrm{Im}\, z > Y. \end{cases}$$

The Maass-Selberg relation [7, 45] gives the norm of $\mathcal{E}^Y$, for r real

$$\int_{\mathcal{F}} |\mathcal{E}^Y(z;\tfrac{1}{2}+ir)|^2 dA = 2\log Y + i\tfrac{d}{dr}\log\varphi(\tfrac{1}{2}+ir)$$

$$-r^{-1}\,\mathrm{Im}(\varphi(\tfrac{1}{2}+ir)Y^{-2ir}).$$

From the ζ-functional equation φ is given as

$$\varphi(\tfrac{1}{2}+ir) = \pi^{2ir}\frac{\Gamma(\frac{1}{2}-ir)\zeta(1-2ir)}{\Gamma(\frac{1}{2}+ir)\zeta(1+2ir)},\ [7,\ 19]$$

where from the Weyl-Hadamard-de la Vallée Poussin bound we note that $\frac{d}{dr}\log\zeta(1+2ir)$ is $O((\log|r|)/\log\log|r|)$, [43] and from the Stirling approximation $\frac{d}{dr}\log\Gamma(\frac{1}{2}+ir) = i\log|r|+O(1)$, [31]. On combining the considerations we find the standard expansion for the Eisenstein norm $\|\mathcal{E}^Y(z;\frac{1}{2}+ir)\|_2^2 = 2\log Y|r| + O((\log|r|)/\log\log|r|)$.

Proposition 5.1 *Notation as above. The Fourier coefficients of the modular Eisenstein series satisfy*

$$\sum_{n=1}^{N}|\varphi_n(\tfrac{1}{2}+ir)|^2 \leq C\sinh\pi|r|(N+|r|)(\log N + O((\log|r|)/\log\log|r|))$$

for $|r|$ large.

Proof. We set $Y = 4\pi N|r|^{-1}$. Observe that since the $SL(2;\mathbb{Z})$ images of $\{\operatorname{Im} z \geq Y\}$ are contained in $\{\operatorname{Im} z \leq Y^{-1}\}$ then $\mathcal{E}^Y(z;\frac{1}{2}+ir) = \mathcal{E}(z;\frac{1}{2}+ir)$ on $\{Y^{-1} \leq \operatorname{Im} z \leq Y\}$ and $\mathcal{E}^Y(z;\frac{1}{2}+ir) = \mathcal{E}(z;\frac{1}{2}+ir) - y^{1/2+ir} - \varphi(\frac{1}{2}+ir)y^{1/2-ir}$ on $\{\operatorname{Im} z > Y\}$. In particular on the entire horoball $\{\operatorname{Im} z \geq Y^{-1}\}$ the Fourier expansions relative to Γ_∞ of $\mathcal{E}$ and $\mathcal{E}^Y$ agree except for the zeroth Fourier coefficient. We can apply the arguments of [47, Lemmas 6.1 and 6.2] to conclude that $\sum_{n=1}^{N}|\varphi_n(\frac{1}{2}+ir)|^2 \leq C\sinh\pi|r|\|\mathcal{E}^Y(z;\frac{1}{2}+ir)\|_2^2(N+|r|)$. The conclusion now follows from the above norm expansion. The proof is complete.

We are now ready to consider the raisings and lowerings of the Eisenstein series as prescribed by (4.1). For m an integer from $E^m y^s = 2^{|m|}e^{2im\theta}s^{|m|}y^s$ (here $E^m = (E^\epsilon)^{|m|}, \epsilon = sgn(m)$) we observe that $(y^{1/2+ir})_{2m} = \prod_{k=0}^{|m|-1}(2ir+2k+1)^{-1}(2ir+1)^{|m|}e^{2im\theta}y^{1/2+ir} = (1+O(|r|^{-1}))e^{2im\theta}y^{1/2+ir}$. The present raisings and lowerings $\mathcal{E}_{2m}$ of the Eisenstein series satisfy $\mathcal{E}_{2m} = (1+O(|r|^{-1}))\mathcal{E}_{\text{classical},-2k}$ where $\mathcal{E}_{\text{classical},-2k}$ is the *classical weight $-2k$ Eisenstein series*, [27, 52]. We introduce the truncation $\mathcal{E}^Y_{2m}$ as the $SL(2;\mathbb{Z})$ invariant function given in the fundamental domain by

$$\mathcal{E}^Y_{2m} = \begin{cases} \mathcal{E}_{2m} & \text{for } z\in\mathcal{F},\ \operatorname{Im} z \leq Y \\ (\mathcal{E}^Y)_{2m} & \text{for } z\in\mathcal{F},\ \operatorname{Im} z > Y. \end{cases}$$

A recursion formula for the norm of the truncation is obtained from integration by parts. For $s = \frac{1}{2}+ir$ we find

$$\langle\mathcal{E}^Y_{2m+2}(z;s),\mathcal{E}^Y_{2m+2}(z;s)\rangle =$$

$$2(2ir+2m+1)^{-1}e^{2i\theta}\int_{S^1}((Y^s)_{2m}+\varphi(s)(Y^{1-s})_{2m})(\overline{(Y^s)_{2m+2}+\varphi(s)(Y^{1-s})_{2m+2}}) \\ Y^{-1}d\theta + \langle\mathcal{E}^Y_{2m}(z;s),\mathcal{E}^Y_{2m}(z;s)\rangle. \tag{5.3}$$

Since $|(Y^s)_{2k}+\varphi(s)(Y^{1-s})_{2k}| \leq 2Y^{1/2}$ for k an integer and $s=\frac{1}{2}+ir$, it follows for each m an integer that $|\|\mathcal{E}^Y_{2m+2}\|_2^2 - \|\mathcal{E}^Y_{2m}\|_2^2| \leq 16\pi|2m+1|^{-1}$. We now proceed with the approach of Chapter 4 (see also [27]).

Definition 5.2 *For r real set* $Q_{\mathcal{E}}(r) = \mathcal{E}(z; \frac{1}{2} + ir)\overline{\mathcal{E}(z; \frac{1}{2} + ir)^{\infty}}$.

Proposition 5.3 *Notation as above. For each positive ϵ the quantities* $(\log|r|)^{-1}Q_{\mathcal{E}}(r)$ *are uniformly bounded tempered distributions for* $C^3_{y^{1+\epsilon}e^{-\epsilon y}}(\Gamma_\infty\backslash SL(2;\mathbb{R}))$.

Proof. The matter is to understand the norm of the Eisenstein series. The expansion for the norm and the above recursion give the bound $\|\mathcal{E}^Y_{2m}\|_2^2 \leq 3\log Y|r| + 16\pi(2 + \log|m|) \leq c_1\log(Y|rm|) + c_2$ for $Y \geq 1$. We use this estimate to bound the integral $\mathcal{I}(Y) = \int_1^Y\int_0^1 |\mathcal{E}\overline{\mathcal{E}_{2m}}|dA$. In fact $\mathcal{I}(Y)$ is bounded by $\int_1^Y\int_0^1 |\mathcal{E}^M\overline{\mathcal{E}^M_{2m}}|dA$, $M = \max\{Y, Y^{-1}\}$, since $\mathcal{E} = \mathcal{E}^M$ for $\{M^{-1} \leq \operatorname{Im} z \leq M\}$. The integral $\int_1^Y\int_0^1 |\mathcal{E}^M\overline{\mathcal{E}^M_{2m}}|dA$ is bounded by the product of $\|\mathcal{E}^M\|_2\ \|\mathcal{E}^M_{2m}\|_2$ and the count of fundamental domains intersecting the integration region. Combining considerations we find the overall bound $\mathcal{I}(Y) \leq (Y^{-1} + 1)(c_1\log((Y^{-1} + Y)|rm|) + c_2)$ for Y positive and generic positive constants given the bound [47] for counting fundamental domains.

We are ready to bound the pairing with a test function. We integrate by parts to find $\int_{\Gamma_\infty\backslash\mathbb{H}} |\mathcal{E}\overline{\mathcal{E}_{2m}}|\chi dA = -\int_0^\infty \mathcal{I}(y)\frac{d}{dy}\chi dy$ for $\chi \in C^1_{y^{1+\epsilon}e^{-\epsilon y}}((0,\infty))$ (the bounds for $\mathcal{I}$ and χ provide for the vanishing of the boundary terms). A test function $\eta \in C^3_{y^{1+\epsilon}e^{-\epsilon y}}(\Gamma_\infty\backslash SL(2;\mathbb{R}))$ has a K-Fourier expansion $\eta = \sum_k \eta_k$ with $\int_0^\infty (y^{-1} + 1)\log(y^{-1} + y)|\frac{d\eta_k}{dy}|dy \leq c_\eta(1 + |k|)^{-2}$. It now follows from the bound for $\mathcal{I}(y)$ that the pairing $(Q_{\mathcal{E}}(r), \eta)$ is $O(\log|r|)$. The proof is complete.

We are ready to consider the large-r limit of the Eisenstein coefficient sum. Akin to Definition 3.1 we define a measure from the Fourier coefficients of $\mathcal{E}(z; \frac{1}{2} + ir)$; $-(\frac{1}{4} + r^2)$ is the eigenvalue for the Laplace-Beltrami operator and t is the variable for the measure.

Definition 5.4 *For t positive set*

$$\Phi_r(t) = (|r|\log|r|)^{-1}\pi\operatorname{csch}\pi|r| \sum_{1\leq|n|\leq|r|t/2\pi} |\varphi_n(\tfrac{1}{2} + ir)|^2.$$

We have the initial bound for the square-coefficient sum from Proposition 5.1 $\Phi_r(t) \leq C((\log|r|t)/\log|r| + O((\log\log|r|)^{-1}))(t + 1)$. The sums $\Phi_r(t)$ are bounded for t bounded and thus for a sequence of r-values tending to infinity there is a subsequence with the Lebesgue-Stieljes derivatives $d\Phi_*$ converging weak* relative to $C_c([0,\infty))$. Let σ (a nonnegative tempered distribution on $[0,\infty))$ denote the limit; σ satisfies the bound $\sigma([0,t)) \leq C(t + 1)$.

Proposition 5.5 *Notation as above. The limit satisfies* $\sigma(\{0\}) = 0$.

Proof. First we wish to establish the analog of Theorem 3.5 for ν a $C^3_{y^{1+\epsilon}e^{-\epsilon y}}(\Gamma_\infty\backslash\mathbb{H})$ weak* limit of $(\log|r|)^{-1}|\mathcal{E}|^2dA$ and σ a limit of $d\Phi_r$. By hypothesis $(\log|r|)^{-1}\int|\mathcal{E}|^2hdA$ converges to $\int h\nu$ while the bound and pointwise convergence of Φ_r provide for the convergence of $\int Gd\Phi_r$ to $\int G\sigma$. We are ready to consider the integral

$$(\log|r|)^{-1}\int_{\Gamma_\infty\backslash\mathbb{H}}|\mathcal{E}|^2hdA =$$

$$2\pi\sum_{n>0}(|r|\log|r|)^{-1}\operatorname{csch}\pi|r|\;|\varphi_n(\tfrac{1}{2}+ir)|^2\mathcal{I}((2\pi n)^2/4a,|r|)$$

$$+(\log|r|)^{-1}\int_0^\infty|y^{1/2+ir}+\varphi(\tfrac{1}{2}+ir)y^{1/2-ir}|^2h(y)y^{-2}dy$$

(using the definition of $\mathcal{I}$ from Section 2.4). We again have from Lemma 2.5 for $t=2\pi|n|/|r|$, $A=(2\pi n)^2/4a$ that $\mathcal{I}((2\pi n)^2/4a,|r|)=G(a^{1/2}t^{-1})+\mathcal{R}(n,|r|,a)$ where the remainder is $O_a(t)$ for $t\le a^{1/2}$ and given t_0 positive the remainder is $O_{a,t_0}(n^{-2})$ for $t\ge t_0$. The convergence argument from Theorem 3.5 can now be applied to establish the desired formula

$$\lim_{r_*}(\log|r|)^{-1}\int_{\Gamma_\infty\backslash\mathbb{H}}|\mathcal{E}(z;\tfrac{1}{2}+ir)|^2h(\operatorname{Im}z)dA=\int_0^\infty G(a^{1/2}t^{-1})\sigma(t).$$

The final matter is to show that $\int_0^\infty h\nu$ tends to zero with a and consequently that $\sigma(\{0\})=0$, since $2\int_0^{t_0}G(a^{1/2}t^{-1})\sigma(t)=\sigma(\{0\})+O(t_0)$. We will consider H_0 the incomplete theta series of h. By Minkowski's inequality we have that $\int_{\mathcal{R}}|\mathcal{E}|^2H_0dA\le 2\int_{\mathcal{R}}|\mathcal{E}^Y|^2H_0dA+2\int_{\mathcal{R}\cap\{\operatorname{Im}z\ge Y\}}|y^{1/2+ir}+\varphi(\tfrac{1}{2}+ir)y^{1/2-ir}|^2H_0dA$ for $\mathcal{R}$ a region contained in $\mathcal{F}$, the $SL(2;\mathbb{Z})$ fundamental domain. Since the integral of $(\log|r|)^{-1}|\mathcal{E}^Y|^2$ over $\mathcal{F}$ is bounded it now follows that $\overline{\lim_{r_*}}(\log|r|)^{-1}\int_{\mathcal{F}\cap\{\operatorname{Im}z\ge Y\}}|\mathcal{E}|^2H_0dA$ tends to zero as Y tends to infinity (since the limit of $(\log|r|)^{-1}|\mathcal{E}^Y|^2dA$ is a regular Borel measure). Since the function H_0 is bounded independent of a and tends to zero on compact sets with a (see Section 3.3) it further follows that $\lim_{r_*}(\log|r|)^{-1}\int_{SL(2;\mathbb{Z})\backslash\mathbb{H}}|\mathcal{E}|^2H_0dA$ and $\int_0^\infty h\nu$ tend to zero with a. It now follows that σ has no mass at the origin. The proof is complete.

5.3. We are ready to consider the analog of Theorem 4.11 for the Eisenstein series. The first matter is the analog of $*$-convergence. Introduce from Definition 3.1 the tempered distributions $\Phi_{\mathcal{E},k}$ and $\mu_{\mathcal{E}}$ for the generalized eigenfunction $\mathcal{E}(z;\frac{1}{2}+ir)$. Now by a diagonalization argument given a sequence of r-values tending to infinity there is a subsequence $\{r_j\}$ such that: $d\Phi_{\mathcal{E},k}$ converges weak* relative to the continuous functions for each closed subinterval of $[0,\infty)$; the distributions $\mu_{\mathcal{E}}$ converge weak* and (from Proposition 5.3) the tempered distributions $(\log|r|)^{-1}Q_{\mathcal{E}}(r)$ converge. We will denote limits as follows $Q_{\mathcal{E},\text{limit}} = \lim_j (\log|r_j|)^{-1}Q_{\mathcal{E}}(r_j)d\mathcal{V}$ and $\mu_{\mathcal{E},\text{limit}} = \lim_j \mu_{\mathcal{E}}$. We further write $Q^{\text{symm}}_{\mathcal{E},\text{limit}}$ for the restriction of the distribution to functions invariant by the right action of $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, equivalently the restriction to functions with nontrivial K components only for weights congruent to zero modulo four. $Q^{\text{symm}}_{\mathcal{E},\text{limit}}$ descends to a geodesic flow invariant, time-reversal invariant, distribution on the unit cotangent bundle of $\mathbb{H}$.

Theorem 5.6 *Notation as above. For ϵ positive*

$$Q^{\text{symm}}_{\mathcal{E},\text{limit}} = \frac{\pi}{8}\int_{\mathbb{G}} \Delta_{\widehat{\alpha\beta}}\mu_{\mathcal{E},\text{limit}}$$

in the sense of tempered positive distributions relative to $C^{\infty}_{y^{1+\epsilon}e^{-\epsilon y}}(\Gamma_\infty\backslash SL(2;\mathbb{R}))$.

Proof. The argument for Theorem 4.11 can be adapted to the current situation. We first show that $Q^{\text{symm}}_{\mathcal{E},\text{limit}}$ is a positive distribution. From integration by parts and Proposition 5.3 we have that $\mathcal{E}_{2j+2}\overline{\mathcal{E}_{2k}} = \mathcal{E}_{2j}\overline{\mathcal{E}_{2k-2}} + O(r^{-1}\log|r|)$ in the sense of tempered distributions. For M a positive integer we introduce the Fejér sum $Q_{\mathcal{E},M}(r) = (2M+1)^{-1}(\log|r|)^{-1}\Big|\sum_{m=-M}^{M}\mathcal{E}_{4m}\Big|^2$. From the above relation $\lim_M \lim_j Q_{\mathcal{E},M}(r_j)d\mathcal{V} = Q_{\mathcal{E},\text{limit}}$ is a positive distribution.

The tempered distributions $(\log|r|)^{-1}Q_{\mathcal{E}}$ on $C^{\infty}_{y^{1+\epsilon}e^{-\epsilon y}}(\Gamma_\infty\backslash SL(2;\mathbb{R}))$ are uniformly bounded; to establish the formula it is enough to establish the convergence of the $\Gamma_\infty\backslash N \times K$ Fourier-Stieljes coefficients. In particular for $\mathbf{h} = h(y)e^{-2\pi ikx+2im\theta}$ consider

$$\begin{aligned}&\int_{\Gamma_\infty\backslash SL(2;\mathbb{R})} \mathcal{E}\overline{\mathcal{E}_{2m}}\mathbf{h}d\mathcal{V} \\ &= 2\pi e^{2im\theta}\int_{\Gamma_\infty\backslash\mathbb{H}} \sum_n a_{n+k}\overline{a_n}r^{-1}\mathcal{K}(z,t+k\Delta t)\overline{\mathcal{K}(z,t)_{2m}}h(y)dA\end{aligned} \tag{5.4}$$

for $t = 2\pi nr^{-1}$ and the Fourier expansion (5.2) with $a_n = (\operatorname{csch}\pi r)^{1/2}\varphi_n(\frac{1}{2}+ir)$ (the special terms $n = 0, -k$ have a modified form). The n-sum will be considered in three ranges in terms of the parameter t. For terms with $|t| \le t_0$ the integrals $\int |\mathcal{K}\mathcal{K}_{2m}|hdA$ are uniformly bounded in t and r. For $k = 0$ the special term $n = 0$ is bounded in terms of $\int y^{1/2}|(y^{1/2+ir})_{2m}|hdA$ which is bounded. For k nonzero the special terms $n = 0, -k$ are bounded in terms of $\int |(y^{1/2+ir})_{2m}\mathcal{K}|hdA$ and $\int y^{1/2}|\mathcal{K}_{2m}|hdA$, which are bounded by $(\int |\mathcal{K}_*|^2 hdA)^{1/2}(\int yhdA)^{1/2}$, which by Lemma 4.4 are also uniformly bounded. The total contribution for $|t| \le t_0$ is bounded by a multiple of $(\log|r|)^{-1}\sum_{n=1}^{rt_0/2\pi+k}|\varphi_n|^2 r^{-1}$. Proposition 5.5 provides the bound that the sum is $o(1)$ as desired. Now we consider the terms with $|t| \ge t_1$. We already have from Lemma 2.5 that $\int |\mathcal{K}|^2 hd\mathcal{V}$ is $O(|t|^{-2})$ and from Lemma 4.3 that $\int |\mathcal{K}_{2m}|hd\mathcal{V}$ is $O(|t|^{-1})$. These bounds are combined with Proposition 5.1 and summation by parts to bound the tail segment of the right hand side of (5.4) by $O((1+\log t_1)t_1^{-1/2})$, a suitable bound.

We are now ready to compare (5.4) to

$$\frac{\pi}{8}\int_{\Gamma_\infty\backslash SL(2;\mathbb{R})\times\Gamma_\infty\backslash\mathbb{G}}\mathbf{h}\Delta_{\Gamma_\infty(\widehat{\alpha\beta})}\mu_{\mathcal{E},\mathrm{limit}}.$$

The first matter is the contribution of the zeroth coefficient of $\mathcal{E}$; from (5.1) for $s=\frac{1}{2}+ir$ the zeroth coefficient is bounded by $y^{1/2}$ and so will not contribute to the limit $Q_{\mathcal{E},\mathrm{limit}} = \lim(\log|r|)^{-1}Q_{\mathcal{E}}d\mathcal{V}$. Now from the argument of Theorem 4.11 given Theorem 3.6, Proposition 5.1 and (2.2) the contribution to the integral from the $\Gamma_\infty\backslash\mathbb{G}$-region $|t| \le t_0$ and $t_1 \le |t|$ is $o(1)$ for t_0 small, t_1 large. For the principal range $t_0 \le |t| \le t_1$ the convergence of terms of (5.4) to the indicated integral is provided by Theorem 4.9 and Proposition 5.1. The desired equality of integrals is established for the test function $\mathbf{h}$.

Now given the positivity of $\mu_{\mathcal{E},\mathrm{limit}}$ (established from the considerations of Section 3.2), the positivity of $Q^{\mathrm{symm}}_{\mathcal{E},\mathrm{limit}}$, Propositions 5.1 and 5.3 the argument from Theorem 4.11 can be applied to give equality in the sense of tempered distributions relative to $C_{y^{1+\epsilon}e^{-\epsilon y}}(\Gamma_\infty\backslash SL(2;\mathbb{R}))$. The proof is complete.

Corollary 5.7 *Notation as above. The vertical geodesics form a null set for $Q^{\mathrm{symm}}_{\mathcal{E},\mathrm{limit}}$ and $\mu_{\mathcal{E},\mathrm{limit}}$ is $SL(2;\mathbb{Z})$-invariant.*

The limit of the microlocal lift $Q_{\mathcal{E}}(r)$ was analyzed in the work of W. Luo-P. Sarnak [32] and D. Jakobson [27]. The authors used the explicit analysis available for modular functions to obtain the result that $(\log|r|)^{-1}Q_{\mathcal{E}}(r)$ converges to $48\pi^{-1}$ weak* relative to $C_c(SL(2;\mathbb{Z})\backslash SL(2;\mathbb{R}))$ [32, Theorem 1.1], [27, Proposition 4.4].

We now present a reinterpretation of the Luo-Sarnak and Jakobson result.

Corollary 5.8 *Notation as above. The convergence of $(\log|r|)^{-1}Q_{\mathcal{E}}(r)$ to $48\pi^{-1}$ in the sense of distributions is equivalent to the convergence of*

$$\left|S_{\mathcal{E}}(t,\nu)\right|^2 = (|\zeta(1+2ir)|^2|r|\log|r|)^{-1}\Big|\sum_{1\le n\le rt}\sigma_{2ir}(n)n^{-ir}e^{in\nu}\Big|^2 \quad \text{to} \quad 48\pi^{-2}t$$

weak in ν for each positive t for r tending to infinity and (t,ν) coordinates for $\mathbb{R}^+\times\mathbb{R}$.*

Proof. The considerations will involve a number of reductions. It will suffice to consider subsequences since the actual limits are predetermined. Furthermore since each limit $Q_{\mathcal{E},\text{limit}}$ is $SL(2;\mathbb{Z})$-invariant, it will suffice to consider pairings with elements of $C^{\infty}_{y^{1+\epsilon}e^{-\epsilon y}}(\Gamma_{\infty}\backslash SL(2;\mathbb{R}))$. We first apply Theorem 5.6 and the relation $d\mathcal{V}=\int_{\mathbb{G}}\Delta_{\widehat{\alpha\beta}}\omega$ to find that $(\log|r|)^{-1}Q^{\text{symm}}_{\mathcal{E}}(r)$ converging to $48\pi^{-1}$ is equivalent to $\mu_{\mathcal{E},\text{limit}}=384\pi^{-2}\omega$. We will consider the measure $\mu_{\mathcal{E},\text{limit}}$ which from the given bounds has Fourier-Stieljes coefficients $\lim_j d\Phi_{\mathcal{E},k}$. In comparison the square sum $|S_{\mathcal{E}}(t,\nu)|^2$ has index k Fourier-Stieljes coefficient $(8\pi)^{-1}\Phi_{\mathcal{E},k}(2\pi(t-|k|\max\{0,k\Delta t\}))$, where the terms arise from different parameterizations. From Proposition 5.5 the limits $\lim_j \Phi_{\mathcal{E},k}$ are also $o(1)$ for small t and hence the equality $\mu_{\mathcal{E},\text{limit}}=384\pi^{-2}\omega$ is equivalent for each positive t to the convergence of each Fourier-Stieljes coefficient in ν of $|S_{\mathcal{E}}(t,\nu)|^2$ to $48\pi^{-2}t$. For each positive t the square sum $|S_{\mathcal{E}}(t,\nu)|^2$ is positive with a uniformly bounded integral in ν by Proposition 5.1. In this setting convergence of Fourier-Stieljes coefficients is equivalent to weak* convergence, the stated condition. In summary $Q^{\text{symm}}_{\mathcal{E},\text{limit}}=48\pi^{-1}$ is equivalent to the prescribed convergence of $|S_{\mathcal{E}}|^2$.

It only remains to show that the prescribed convergence of $|S_{\mathcal{E}}|^2$ implies the equality $Q^{\text{symm}}_{\mathcal{E},\text{limit}}=Q_{\mathcal{E},\text{limit}}$. We do not have an explicit general formula in terms of $\mu_{\mathcal{E},\text{limit}}$ for the component of $Q_{\mathcal{E},\text{limit}}$ with $SL(2;\mathbb{R})$ weights congruent to two modulo four. An argument for the equality will be made using several general properties of the limit. We begin and recall from Theorem 4.11 that $\overline{u_{2m}}=(-1)^m(\bar{u})_{-2m}+O(r^{-1}|u_{2m}|)$. In combination with (4.2) we have that $\mathcal{K}(z,t)\overline{\mathcal{K}(z,t)_{2m}}=(-1)^m\mathcal{K}(z,t)\mathcal{K}(z,-t)_{-2m}+O(r^{-1})=(-1)^m\mathcal{K}(z,-t)\overline{\mathcal{K}(z,-t)_{2m}}+O(r^{-1})$ relative to $C^{\infty}_{y^{1+\epsilon}e^{-\epsilon y}}(\Gamma_{\infty}\backslash SL(2;\mathbb{R}))$ from the bounds of Lemmas 4.3 and 4.4. Now the modular group is normalized by the transformation $z\rightarrow-\bar{z}$ and in consequence the modular Eisenstein series has Fourier coefficients $\varphi_n(s)=\varphi_{-n}(s)$. Referring to (5.4) we find that the $\Gamma_{\infty}\backslash N\times K$ index $(k,-2m)$ contribution is a sum of terms $|a_n|^2(\mathcal{K}(z,t)\overline{\mathcal{K}(z,t)_{2m}}+\mathcal{K}(z,-t)\overline{\mathcal{K}(z,-t)_{2m}})$. For m odd from the above relation the contribution has magnitude $O(|a_n|^2r^{-1})$. It follows from the approach of Theorem 5.6 for m odd that the index $(0,-2m)$ terms of $Q_{\mathcal{E},\text{limit}}$ are trivial.

We show as the final step that the $\Gamma_{\infty}\backslash N\times K$ index $(k,2m)$ with k nonzero contribution of $Q_{\mathcal{E},\text{limit}}$ is an integral of a distribution continuous in the parameter t and the Lebesgue-Stieljes measure $\lim_j d\Phi_{\mathcal{E},k}(t)$; the contribution to $Q_{\mathcal{E},\text{limit}}$ is trivial provided the measure is trivial. Given the approach of Theorem 5.6 it suffices to show that a limit of $Q^m_k(t)$ (see Definition 4.5) is continuous in the parameter t. We begin and consider

$$\begin{aligned}Q^m_k(t')-Q^m_k(t)=&\\&(\mathcal{K}(z,t'+k\Delta t)-\mathcal{K}(z,t+k\Delta t))\overline{\mathcal{K}(z,t')_{-2m}}\\&+\mathcal{K}(z,t+k\Delta t)(\overline{\mathcal{K}(z,t')_{-2m}}-\overline{\mathcal{K}(z,t)_{-2m}}).\end{aligned}$$

For the test function $\mathbf{h}=h(y)e^{-2\pi ikx-2im\theta}$ the pairings with the terms of the right hand side are bounded from Lemmas 4.3 and 4.4. The derivatives of $\mathbf{h}$ are suitably bounded and thus for the second summand we can integrate by parts. Now from Hölder's inequality we have that $Q^m_k(t')-Q^m_k(t)$ can be bounded in terms of integrals $\int_{\mathbb{R}^+}|\mathcal{K}(z,t'')-$

$\mathcal{K}(z,t''')|^2h(y)y^{-2}dy$ with $|t''-t'''| = |t-t'|$. From Theorem 2.4 such integrals are bounded in terms of $|t-t'|$ and r^{-1}. We have in summary that a weak* limit of $Q_k^m(t)$ is continuous in the parameter t. The proof is complete.

5.4. We now consider the semi-classical limit for L^2 automorphic eigenfunctions normalized on a large compact set. Let $\Gamma \subset SL(2;\mathbb{R})$ be a general cofinite group with Γ_∞ the stabilizer of a width-one cusp at infinity. We first prescribe a compact set in the space of geodesics that contains a representative of each complete geodesic on $\Gamma\backslash\mathbb{H}$. The prescription follows the *highest point* condition for specifying a fundamental domain.

Proposition 5.9 *Notation as above. Given Γ there exists a compact set $\mathcal{R} \subset \mathbb{H}$ such that each complete geodesic on $\mathbb{H}$ has a Γ-translate a non vertical geodesic with highest point in $\mathcal{R}$.*

Proof. We begin with considerations for a fundamental domain. A *standard cusp* is a region isometric to the quotient $\Gamma_\infty\backslash\{z|Im\, z > 1\}$. Mutually disjoint standard cusps are associated to the distinct cusps of $\Gamma\backslash\mathbb{H}$. Accordingly a fundamental domain can be decomposed into a *cuspidal* region and its complement the *body*. A particular fundamental domain $\mathcal{F}$ is determined by selecting from each Γ-orbit a representative z_0 with $0 \leq Re\, z_0 < 1$ and $Im\, z_0$ maximal (if the maximum is not uniquely assumed the point with minimal real part can be chosen). The fundamental domain $\mathcal{F}$ is a disjoint union of its cuspidal region $\mathcal{F}_c$ and its relatively compact body $\mathcal{F}_b$. The body plays a specific role for geodesics since a complete geodesic cannot be entirely contained in a standard cusp region. In particular each complete geodesic on $\mathbb{H}$ has (at least one) translate intersecting $\mathcal{F}_b$. There is accordingly a trichotomy since $\mathcal{F}$ contains the standard cusp at infinity: either a geodesic is vertical, or has a translate with highest point in the closure of $\mathcal{F}_b$, or has a translate with highest point greater than unity.

We will transform the vertical geodesics and the geodesics with Euclidean radius greater than unity. Choose $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in Γ with $B^{-1}(\infty)$ negative; the product cd is positive. For a geodesic which is vertical or has radius greater than unity translate by an element of Γ_∞ to obtain a representative γ_0 connecting points p and q with $0 \leq p < 1$ and $2 < q \leq \infty$. The quantity $(B(q) - B(p)) = (q-p)(cq+d)^{-1}(cp+d)^{-1}$ has q-partial derivative $(cq+d)^{-2}$ which is nonnegative. In particular for p fixed then $(B(q) - B(p))$ is increasing in q and thus $(2c+d)^{-2} \leq (B(2) - B(p)) \leq (B(q) - B(p)) \leq (B(\infty) - B(p)) \leq (cd)^{-1}$. It follows that a Γ_∞-translate of $B\gamma_0$ has highest point z_1 with $0 \leq Re\, z_1 < 1$ and $(2c+d)^{-2} \leq 2Im\, z_1 \leq (cd)^{-1}$. The desired compact set is now provided by the union of the closure of $\mathcal{F}_b$ and the prescribed region. The proof is complete.

The next item, a refinement of Theorem 2.4, Lemma 2.5 and Lemma 4.4, concerns the mass distribution for the square of the Macdonald-Bessel functions. Set for μ positive

$$\mathbf{A}(\mu) = \operatorname{arccosh}(\max\{1, \mu^{-1}\});$$

recall for $\tau \geq 1$ that $\operatorname{arccosh}\tau = \log(\tau + (\tau^2-1)^{1/2})$; $\mathbf{A}(\mu)$ is strictly decreasing for μ on the interval $(0,1]$ and near the origin is asymptotic to $\log 2\mu$; further set for L, α, and β positive, $\alpha < \beta$

$$\mathcal{I}(\alpha,\beta) = r\sinh\pi r\int_\alpha^\beta K_{ir}(Ly)^2y^{-1}dy.$$

Proposition 5.10 *Notation as above. For L, α, β and r positive with $\alpha < \beta$ and r large then $\mathcal{I}(\alpha,\beta) = \pi(\mathbf{A}(\alpha Lr^{-1}) - \mathbf{A}(\beta Lr^{-1})) + O(r^{-1/6})$ with a uniform constant for $L \geq 2\pi$. Furthermore there exists a strictly increasing positive analytic function η on $(0,\infty)$ such that given $Y_0 > 1$ there exist positive constants c_0, c_1 with $c_0 \leq \big(rY_1^2e^{r\,\eta(Y_1)}\big)\,\mathcal{I}(L^{-1}\lambda^{1/2}Y_1,\infty) \leq c_1$ for $Y_1 \geq Y_0$.*

Proof. We again refer to the WKB-asymptotics from Section 2.4 esp. Theorem 2.2 and Lemma 2.3 of [47]. We have for $\lambda = \frac14 + r^2, Y = L\lambda^{-1/2}y, L, y > 0$ and $\zeta(Y)$ the analytic solution of $\zeta(\frac{d\zeta}{dY})^2 = (1 - Y^{-2})$ the expansion

$$2r\sinh\pi rK_{ir}(Ly)^2 = \pi(1-Y^2)^{-1/2}(1+\cos\Theta + O(r^{-1/2})) \tag{5.5}$$

for $Y \leq 1 - \lambda^{-1/6}$, $\Theta = \frac43\lambda^{1/2}|\zeta|^{3/2} - \frac{\pi}{2}$ and $\frac{d\Theta}{dY} = -2\lambda^{1/2}(Y^{-2}-1)^{1/2}$; we also have the upper bound

$$r\sinh\pi rK_{ir}(Ly)^2 \leq C\begin{cases}(1-Y^2)^{-1/2} & ,\ Y\leq 1\\ (Y^2-1)^{-1/2}e^{-4/3\,\lambda^{1/2}\zeta^{3/2}} & ,\ Y \geq 1\end{cases}$$

and for a positive constant depending on $Y_0 > 1$ the lower bound

$$r\sinh\pi rK_{ir}(Ly)^2 \geq C'(Y^2-1)^{-1/2}e^{-4/3\,\lambda^{1/2}\zeta^{3/2}},\ Y \geq Y_0.$$

The bound follows from the cited theorem, the expansion $Ai(\mu) = |\mu|^{-1/4}(\cos(\frac23|\mu|^{3/2} - \frac{\pi}{4}) + O(|\mu|^{-3/2}))$ for μ negative, the asymptotic $Ai(\mu) \sim \mu^{-1/4}e^{-2/3\,\mu^{3/2}}$ for μ large positive, and the given bounds $|Ai(\mu)|$, $|ME(\mu)|$ and $|\mu^{-1/2}(Ai'(\mu) - Ai'(0))| \leq C|\mu|^{-1/4}$ for $\mu < 0$; $|Ai(\mu)|$, $|ME(\mu)|$ and $|\mu^{-1/2}(Ai'(\mu)-Ai'(0))| \leq C|\mu|^{-1/4}e^{-2/3\mu^{3/2}}$ for $\mu > 0$ and $|B_0(\mu)| \leq C(1+|\mu|)^{-2}$, [47].

We begin and bound the contribution to the integral $\mathcal{I} = r\sinh\pi r\int K_{ir}(Ly)^2Y^{-1}dY$ from the interval $[1, Y_0]$. For $1 \leq Y \leq Y_0$ then ζ dominates a positive multiple of $(Y-1)$ and for $\tau = (Y-1)$ the integral is dominated by $\int_0^\infty e^{-\lambda^{1/2}\tau^{3/2}}\tau^{-1/2}d\tau$, which is $O(r^{-1})$ by scaling considerations. For the interval $[Y_1,\infty)$, $Y_1 \geq Y_0$, the integrand is bounded above and below by positive multiples of $(Y^2-1)^{-1}e^{-4/3\,\lambda^{1/2}\zeta^{3/2}}\zeta^{1/2}d\zeta$, which can be integrated by parts twice with $u = \lambda^{-1/2}e^{-4/3\,\lambda^{1/2}\zeta^{3/2}}$ to obtain upper and lower bounds by positive multiples of $\lambda^{-1/2}Y_1^{-2}e^{-4/3\,\lambda^{1/2}\zeta(Y_1)^{3/2}}$, as claimed on setting $\eta = 4/3\,\zeta^{3/2}$. We note in particular that the contribution from any interval $[1, \beta Lr^{-1}]$ is uniformly $O(r^{-1})$.

The next step is to use the expansion (5.5) to find the contribution to $\mathcal{I}$ from the range $Y \leq 1 - \lambda^{-1/6}$. The leading term contributes the inverse hyperbolic cosine.

The contribution from $\cos\Theta$ can be evaluated by parts with $u = \sin\Theta$ and $v = ((1-Y^2)^{1/2}Y\frac{d\Theta}{dY})^{-1} = (-2\lambda^{1/2}(1-Y^2))^{-1}$. Given the bound $Y \le 1-\lambda^{-1/6}$ the contribution is $O(\lambda^{-1/3})$ a suitable bound. The remainder contribution from (5.5) is bounded by $O(\lambda^{-1/12})$. For the final range $1-\lambda^{-1/6} \le Y \le 1$ we simply use the majorant $(1-Y^2)^{-1/2}$ for the quantity $r\sinh\pi r K_{ir}(Ly)^2$ to find that the contribution is also $O(\lambda^{-1/12})$. All possible ranges have been considered. The proof is complete.

We are ready to compare the $L^2(\Gamma\backslash\mathbb{H})$-norm for an automorphic eigenfunction to its L^2-norm relative to a large compact set. Provided $\Gamma\backslash\mathbb{H}$ has p inequivalent cusps, let $\mathcal{F}_j$, $j = 1,\dots,p$, be the *highest point* fundamental domains for the conjugates $B_j^{-1}\Gamma B_j$ obtained by successively representing the inequivalent cusps by width-one cusps at infinity. For $\tau > 1$ let $\mathcal{H}_\tau$ denote the quotient horocycle region $\Gamma_\infty\backslash\{z|\tau^{-1} < \operatorname{Im} z < \tau\}$. Choose a positive value τ_0 to provide that $\{z|\tau_0^{-1} < \operatorname{Im} z < \tau_0\} \subset \mathbb{H}$ contains the body regions $B_j\mathcal{F}_{j,b}$ (see the proof of Proposition 5.9) as relatively compact subsets. For φ an $L^2(\Gamma\backslash\mathbb{H})$ eigenfunction we now denote by $\|\varphi\|_\tau$ the $L^2(\mathcal{H}_\tau)$-norm of φ; we continue to denote the $L^2(\Gamma\backslash\mathbb{H})$-norm by $\|\varphi\|$.

Proposition 5.11 *Notation as above. There exist positive constants c_0, c_1 such that for an $L^2(\Gamma\backslash\mathbb{H})$ eigenfunction φ with eigenvalue $\frac{1}{4}+r^2$, $r \ge 1$, then $c_0\|\varphi\|^2_{\tau_0} \le \|\varphi\|^2 \le c_1\log r\,\|\varphi\|^2_{\tau_0}$. For each $\tau > \tau_0$ there exists a positive constant c_τ such that $\|\varphi\|_\tau \le c_\tau\|\varphi\|_{\tau_0}$.*

Proof. The first inequality is straightforward. For the second inequality the essential matter is to bound the integral of φ^2 in each cusp. Since for each cusp $\mathcal{H}_{\tau_0}$ contains the region between two horocycles it is enough for $y_* < y_{**}$ to establish the basic inequality $\int_{\Gamma_\infty\backslash\{y_{**}<\operatorname{Im} z<y_{***}\}} \varphi^2 dA \le c'\log\min\{y_{***}, r\}\int_{\Gamma_\infty\backslash\{y_*<\operatorname{Im} z<y_{**}\}} \varphi^2 dA$ for $y_{***} > y_{**}$. Given Parseval's formula it is enough to establish the inequality $\mathcal{I}(y_{**}, y_{***}) \le c'\log\min\{y_{***}, r\}\mathcal{I}(y_*, y_{**})$ for the Macdonald-Bessel functions for all $L \ge 2\pi$ and a constant depending on y_* and y_{**}. To this purpose we consider the two cases for $Y_{**} = L\lambda^{-1/2}y_{**}$ in comparison to $\rho = (y_{**}y_*^{-1})^{1/3}$. For $Y_{**} \le \rho^2$ then $L\lambda^{-1/2}y_* \le \rho^{-1} < 1$ and the formula of Proposition 5.10 provides that $\mathcal{I}(y_*, y_{**})$ is bounded below by a positive constant for all appropriate L, a suitable bound. We next consider $\mathcal{I}(y_{**}, y_{***})$. For $y_{***} \le ry_{**}$ Proposition 5.10 and the observation that $\mathbf{A}(y_{**}Lr^{-1}) - \mathbf{A}(y_{***}Lr^{-1})$ is bounded by $\log y_{***}y_{**}^{-1}$ provide that $\mathcal{I}(y_{**}, y_{***})$ is bounded by a positive multiple of $\log y_{***}$; for $y_{***} > ry_{**}$ observe that $L\lambda^{-1/2}y_{**} \ge 2\pi\lambda^{-1/2}y_{**}$ and the formula of Proposition 5.10 provides that the majorant $\mathcal{I}(y_{**}, \infty)$ is bounded by $\log r$. The inequality for $\mathcal{I}$ is established for $Y_{**} \le \rho^2$. For $Y_{**} \ge \rho^2$ and the choice $Y_0 = \rho$ the inequality from Proposition 5.10 provides that $\mathcal{I}(y_{**}, \infty)$ is already exponentially small in comparison to $\mathcal{I}(\rho^{-1}y_{**}, y_{**})$, a suitable bound. The integral $\mathcal{I}$ is suitably bounded. The proof is complete.

We are ready to consider limits of eigenfunctions normalized by their L^2-norm on a compact set. The convergence considerations will differ from those of Chapter 4 since possibly the $L^2(\Gamma\backslash\mathbb{H})$-norms are tending to infinity. At the center of the considerations is the Fejér sum $Q_M(\varphi) = (2M+1)^{-1}|\sum_{m=-M}^{M}\varphi_{4m}|^2$ for an eigenfunction φ. We will employ the lifts $\widetilde{\mathcal{H}_\tau}$ to $\Gamma_\infty\backslash SL(2;\mathbb{R})$ of the quotient horocycle regions $\mathcal{H}_\tau \subset \Gamma_\infty\backslash\mathbb{H}$. From Proposition 5.11 we can now renormalize each $L^2(\Gamma\backslash\mathbb{H})$ eigenfunction to have $L^2(\mathcal{H}_{\tau_0})$-norm $(2\pi)^{-1}$.

Proposition 5.12 *Notation as above. For a sequence of $L^2(\Gamma\backslash\mathbb{H})$ eigenfunctions, normalized on $\mathcal{H}_{\tau_0}$, with eigenvalues tending to infinity there exists a subsequence $\{\varphi_j\}$ such that the limit $\mu_{\text{ren}} = \lim_M \lim_j Q_M(\varphi_j)$ exists in the sense of σ-finite measures on $\Gamma_\infty\backslash SL(2;\mathbb{R})$. The positive measure μ_{ren} has unit mass on $\widetilde{\mathcal{H}_{\tau_0}}$, is Γ-invariant, right $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ invariant and geodesic-flow invariant.*

Proof. The first considerations are the consequences of integration by parts. From (4.1) and (4.2) we find $\varphi_{2k+2}\overline{\varphi_{2m}} = \varphi_{2k}\overline{\varphi_{2m-2}} + O(r^{-1}(\|\varphi_{2k+2}\|+\|\varphi_{2k}\|)(\|\varphi_{2m}\|+\|\varphi_{2m-2}\|))$. The basic bound that $\varphi_{2k+2}\overline{\varphi_{2m}} - \varphi_{2k}\overline{\varphi_{2m-2}}$ is $O(r^{-1}\log r)$ as a distribution for $C^1_c(SL(2;\mathbb{R}))$ follows now from Proposition 5.11. Furthermore by Proposition 5.11 the sequence of eigenfunction squares can be considered as a family of positive measures uniformly bounded on compact sets. Select a subsequence $\{\varphi_j^2\}$ that converges on each compact set. The products $\varphi_{j,2k}\overline{\varphi_{j,2k}}$ are positive, thus also determine measures, and from the basic bound the sequences φ_j^2 and $\varphi_{j,2k}\overline{\varphi_{j,2k}}$ have a common limit. There are additional consequences. In particular for each k, m the products $\varphi_{j,2k}\overline{\varphi_{j,2m}}$ determine uniformly bounded σ-finite measures. Select a subsequence (to simplify notation we continue to write $\{\varphi_j\}$ for the subsequence) to provide that for each k, m the products $\varphi_{j,2k}\overline{\varphi_{j,2m}}$ converge in the sense of σ-finite measures. From the basic bound the limit $\lim_j \varphi_{j,2k}\overline{\varphi_{j,2m}}$ depends only on the difference $k-m$.

We are ready to consider the limit $\mu_M = \lim_j Q_M(\varphi_j)$. The index zero K Fourier-Stieljes coefficient of μ_M is $\lim_j (2M+1)^{-1}\sum_{m=-M}^{M}\varphi_{j,4m}\overline{\varphi_{j,4m}}$ which as already noted is the σ-finite positive measure $\lim_j \varphi_j^2$. Given the normalization for the eigenfunctions it follows that each μ_M is a probability measure for $\widetilde{\mathcal{H}_{\tau_0}}$ and by Proposition 5.11 that $\{\mu_M\}$ is a sequence of σ-finite measures uniformly bounded on compact sets. For a convergent subsequence of $\{\mu_M\}$ the index $4m$ K Fourier-Stieljes coefficient of the limit is simply $\lim_j \varphi_j\overline{\varphi_{j,-4m}}$, independent of the choice of subsequence; the sequence $\{\mu_M\}$ converges to a probability measure on $\widetilde{\mathcal{H}_{\tau_0}}$.

We set $\mu_{\text{ren}} = \lim_M \lim_j Q_M(\varphi_j)$ and note that μ_{ren} has K Fourier-Stieljes coefficients $\mu_{\text{ren}}^{(4m)} = \lim_j \varphi_j\overline{\varphi_{j,-4m}}$ with $|\mu_{\text{ren}}^{(4m)}| \le \mu_{\text{ren}}^{(0)}$. We consider the invariance properties. The Fejér sum $Q_M(\varphi)$ is Γ-invariant with non trivial K components only for weights congruent to zero modulo four. The measure μ_{ren} is consequently Γ-invariant and right $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ invariant. Geodesic-flow invariance is equivalent to the triviality of the

$C_c^1(SL(2;\mathbb{R}))$ distribution $H\mu_{\text{ren}}$. It will suffice to establish triviality relative to the dense subspace $C_c^\infty(SL(2;\mathbb{R}))$. We will show for $\chi \in C_c^\infty(SL(2;\mathbb{R}))$ that the pairing of $H\chi$ and μ_{ren} is arbitrarily small by approximating μ_{ren} by $Q_P(\varphi_j) = \sum_{k=-2P}^{2P} \varphi_j\overline{\varphi_{j,-2k}}$ and then using Zelditch's relation to show that $HQ_P(\varphi_j)$ is small. The test function χ has first and second derivatives (with respect to elements of the enveloping algebra) with K Fourier coefficients bounded by multiples of $(1+|k|)^{-2}$; for example $H\chi$ has K expansion $\sum_k (H\chi)_{(k)}$ with $|(H\chi)_{(k)}| \le C(1+|k|)^{-2}$. We begin the approximation given ϵ positive by observing for P sufficiently large that $\int_{SL(2;\mathbb{R})} (H\chi)(\mu_{\text{ren}} - \sum_{k=-P}^{P} \mu_{\text{ren}}^{(4k)})\, d\mathcal{V}$ is bounded by ϵ given the bounds for the expansion of $H\chi$ and that $|\mu_{\text{ren}}^{(4k)}| \le \mu_{\text{ren}}^{(0)}$. We next recall from the proof of Theorem 4.11 for $m = 2q+1$ the relation $4i\varphi\varphi_{2m} = (-1)^q r^{-1} E^+((\varphi_{2q})^2) + O(r^{-1}\|\varphi\|_*^2)$ for $\|\varphi\|_*$ the sum of the $L^2(\Gamma\backslash\mathbb{H})$-norms of $\varphi, \dots, \varphi_{4q}$. We apply the basic bound, Proposition 5.11 for the norm $\|\varphi\|_*$, and the convergence of $\lim_j \varphi_j\overline{\varphi_{j,-2k}}$ to uniformly bounded σ-finite measures to find for j sufficiently large that $\int_{SL(2;\mathbb{R})} (H\chi)(\sum_{k=-P}^{P} \mu_{\text{ren}}^{(4k)} - Q_P(\varphi_j))\, d\mathcal{V}$ is also bounded by ϵ. Furthermore since for P fixed $Q_P(\varphi_j)$ is a uniformly bounded distribution for $Z = H + (4ir)^{-1}(H^2 + 4X^2)$, the operator from Lemma 4.7, and j sufficiently large the difference $((H-Z)\chi)\, Q_P(\varphi_j)$ is also bounded by ϵ. Since χ has compact support we can integrate by parts and consider $-\chi Z Q_P(\varphi_j)$ in place of $(Z\chi)\, Q_P(\varphi_j)$; we have in summary that $|\int_{SL(2;\mathbb{R})} (H\chi)\mu_{\text{ren}} + \chi Z Q_p(\varphi_j)\, d\mathcal{V}| < 3\epsilon$ for each P and $j \ge j_0(P)$ sufficiently large.

The last step is provided by Lemma 4.7. As already noted H and X are sums of graded-operators with weights $-2, 0, 2$ and thus Z is a sum $\sum_{m=-2}^{2} Z_{2m}$ of graded-operators with even weights $-4 \dots 4$. Zelditch's relation is equivalent to a system of relations, one for each K weight; for the complete microlocal lift $Q(\varphi)$ (see Definition 4.10) the weight $2k$ term of $ZQ(\varphi)$ is $\sum_{m=-2}^{2} Z_{2m}\varphi\overline{\varphi_{-2k+2m}} = 0$. For the truncated sum $Q_P(\varphi)$ the individual weight relations provide vanishing for the terms of $ZQ_P(\varphi)$, except at the *ends* of the sum. There are a fixed number of terms not canceled by the relation; the terms have the form $Z_{2m}(\varphi\overline{\varphi_{4\epsilon P-2\epsilon k}})$ for $m = -2, \dots, 2$, $\epsilon = \pm 1$ and $k = 0, \dots, 3$. Again bounds for $\chi_{(k)}$ and $|\mu_{\text{ren}}^{(2k)}| \le \mu_{\text{ren}}^{(0)}$ provide that for P and j sufficiently large the combined contribution of the cited terms is bounded by ϵ. We have in summary that $H\mu_{\text{ren}}$ is the trivial distribution for $C_c^\infty(SL(2;\mathbb{R}))$. The proof is complete.

We are ready to consider the analog of Theorem 3.5 for eigenfunctions normalized on the compact set $\mathcal{H}_{\tau_0}$. For a normalized φ with eigenvalue $\lambda = \frac{1}{4} + r^2$, Fourier expansion (3.1), and t positive set

$$\Phi_\varphi(t) = \pi \sum_{|n|\leq rt(2\pi)^{-1}} |a_n|^2 r^{-1}.$$

For a sequence of normalized eigenfunctions with squares $C_c(\Gamma\backslash\mathbb{H})$ weak* convergent let ν_{ren} be the index zero left-$\Gamma_\infty\backslash N$ Fourier-Stieljes coefficient of the lift to $\Gamma_\infty\backslash\mathbb{H}$ of the limit $\lim_j \varphi_j^2 dA$. We wish to relate the large-r limit of $\Phi_\varphi(t)$ to ν_{ren}. For this purpose consider for a σ-finite positive measure ω on $[0,\infty)$ the integral transform

$$\mathbb{A}[\omega](y) = \frac{1}{2}\int_0^\infty (1-t^2y^2)^{-1/2}\omega(t);$$

by an application of the Tonelli Theorem the integral is convergent almost everywhere in y and the result is a locally-integrable function; $\mathbb{A}$ is the Abel transform of $\omega(\sqrt{\tau})$ for $\tau = t^2$, [44]. The Abel transform is injective for σ-finite measures.

Proposition 5.13 *Notation as above. There exists a positive constant c_0 such that for $t > 1$ then $\Phi_\varphi(t) \leq c_0\|\varphi\|_{4t}^2$. The sequence of Lebesgue-Stieljes derivatives $\{d\Phi_{\varphi_j}\}$ has a $C_c([0,\infty))$ weak* limit σ_{ren} with $\nu_{\text{ren}}(y) = (\mathbb{A}[\sigma_{\text{ren}}](y))y^{-1}dy$.*

Proof. The first matter is to provide uniform bounds for the sums Φ_φ. Observe that the expression from Proposition 5.10 $\mathbf{A}(t(4t')^{-1}) - \mathbf{A}(t(2t')^{-1})$, $0 < t \leq t'$, is bounded below by a positive constant independent of t'. The simple bound $\Phi_\varphi(t') \leq c'\|\varphi\|_{4t'}^2$ for $t' > 1$ now follows from Parseval's formula and Proposition 5.10. The Lebesgue-Stieljes derivatives $\{d\Phi_{\varphi_j}\}$ are consequently uniformly bounded; we can consider $C_c([0,\infty))$ weak* convergent subsequences. Let σ_* be the limit of a convergent subsequence. We will show that $\nu_{\text{ren}} = \mathbb{A}[\sigma_*]y^{-1}dy$.

We consider the analog of Theorem 2.4 (the test function $h(y)$ will now be replaced by $\chi \in C_c^1((0,\infty))$ and the range will be expanded to all t). Consider the quantity $r\sinh\pi r\int_0^\infty K_{ir}(2\pi|n|y)^2\chi(y)y^{-1}dy$ for $\chi \in C_c^1((0,\infty))$. We can integrate by parts for $dv = K_{ir}(2\pi|n|y)^2y^{-1}dy$ to obtain the integral $-\int_0^\infty \mathcal{I}(1,y)\chi'(y)dy$. Proposition 5.10 provides for the uniform convergence of the integrand and the integral; integration of the resulting expression by parts provides the desired result. In summary for $\chi \in C_c^1((0,\infty))$ the integral $r\sinh\pi r\int_0^\infty K_{ir}(2\pi|n|y)^2\chi(y)y^{-1}dy$ converges to the integral $\frac{\pi}{2}\int_0^{t^{-1}}(1 - t^2y^2)^{-1/2}\chi(y)y^{-1}dy$ uniformly in $t = 2\pi|n|r^{-1}$.

We are ready to consider sums and the convergence for the sequence $\{\varphi_j\}$. For $\chi \in C_c((0,\infty))$ associate the function on the upper half plane with values $\chi(y)$ for $z = x+iy \in \mathbb{H}$ and consider

$$\int_{\Gamma_\infty\backslash\mathbb{H}} \varphi_j^2\chi dA = \sum_n |a_n|^2\sinh\pi r\int_0^\infty K_{ir}(2\pi|n|y)^2\chi(y)y^{-1}dy.$$

Choose α positive such that the support of χ is contained in the interval (α, ∞). We first show that the contribution to the n-sum from terms with $|t| \geq \alpha^{-1}$ is $O(r^{-1}\log r)$ as j tends to infinity. For $|t| \geq \alpha^{-1}$ then $Y \geq Y_0 > 1$ on the support of χ and the above integrals are bounded by a multiple of n^{-2} from the bounding factor $r^{-2}Y_1^{-2}$ from Proposition 5.10. By (3.2) the sum $\Phi_{\varphi_j}(t)$ is bounded by a multiple of $\|\varphi_j\|^2(t+1)$ and since the φ_j are normalized on $\mathcal{H}_{\tau_0}$ we have by Proposition 5.11 that $\|\varphi_j\|^2$ is bounded by a multiple of $\log r$. The total contribution from the range $|t| \geq \alpha^{-1}$ is bounded by a multiple of $\log r \sum_{n=r(2\pi\alpha)^{-1}} |a_n|^2 n^{-2}$; from summation by parts the n-sum is $O(r^{-1}\log r)$, a suitable bound. For the principal range $|t| \leq \alpha^{-1}$ the uniform convergence of terms and convergence of the selected subsequence of $\{d\Phi_{\varphi_j}\}$ to σ_* provides the result $\int_0^\infty \mathbb{A}[\sigma_*](y)\chi(y)y^{-1}dy$. Finally since the Abel transform is injective the original sequence $\{d\Phi_{\varphi_j}\}$ also converges to σ_*. The proof is complete.

We consider again the geodesic-indicator measure. We wish to parameterize the index zero left-$\Gamma_\infty\backslash N$ Fourier-Stieljes component of $\delta_{\widehat{\alpha\beta}}$ on $\widehat{\mathbb{G}}$, the space of all complete geodesics (including the vertical geodesics). The non vertical geodesics can alternately be parameterized by the ordered pair $(\rho, t)_*$ for t the reciprocal radius and $\rho = \hat{x} + t^{-1}$ the abscissa of the right end point. The parameters $(\rho, t)_*$ extend to continuous coordinates for $\widehat{\mathbb{G}}$; the vertical geodesics are the locus $\{t = 0\}$. By Proposition 2.1 for a non vertical geodesic $\widehat{\alpha\beta}$ with coordinate $(\hat{x}, t)$ the index zero Fourier-Stieljes component of $\delta_{\widehat{\alpha\beta}}$ is $(1-t^2y^2)^{-1/2}y^{-1}dy$; for a vertical geodesic the index zero component is $y^{-1}dy$. In summary the index zero component of $\delta_{\widehat{\alpha\beta}}$, a σ-finite measure on $\mathbb{R}^+$, is continuously parameterized on $\widehat{\mathbb{G}}$ by the values $0 \leq t < \infty$.

Given the parameter value τ_0 of Proposition 5.11 we choose $\tau_1 \geq \tau_0$ to provide that each complete geodesic on $\Gamma\backslash\mathbb{H}$ has a lift with Euclidean radius in the range (τ_1^{-1}, τ_1); a choice is guaranteed by Proposition 5.9. We now use the value τ_1 to prescribe a linear map from the direct sum of eigenspaces of the hyperbolic Laplacian to the space of continuous functions on the unit circle. In particular for an $L^2(\Gamma\backslash\mathbb{H})$ eigenfunction φ with eigenvalue $\lambda = \frac{1}{4} + r^2$ and Fourier expansion (3.1) define

$$\mathbb{S}[\varphi] = r^{-1/2} \sum_{\tau_1^{-1} \leq 2\pi n r^{-1} \leq \tau_1} a_n e^{in\theta}.$$

Further for an eigenvalue λ let V_λ be the associated eigenspace endowed with the $L^2(\mathcal{H}_{\tau_1})$ inner product.

Theorem 5.14 *Notation as above. For all large eigenvalues λ the linear map $\mathbb{S}$ is a uniform quasi-isometry from V_λ into $L^2(S^1)$. In particular there exist positive constants c_0, c_1 and c_2 such that for φ with large eigenvalue $\lambda = \frac{1}{4} + r^2$ and Fourier coefficients $\{a_n\}$ then $c_0(\log r)^{-1}\|\varphi\|^2 \leq c_1\|\varphi\|^2_{\tau_1} \leq r^{-1} \sum_{\tau_1^{-1} \leq 2\pi n r^{-1} \leq \tau_1} |a_n|^2 \leq c_2\|\varphi\|^2_{\tau_1}$. Furthermore the* *measure* $\sigma_{\rm ren}$ *is the index zero* $\Gamma_\infty\backslash N$ *Fourier-Stieljes coefficient of a* Γ*-invariant measure on* $\widehat{\mathbb{G}}$.

Proof. The first step is to show that $\|\varphi\|^2_{\tau_1}$ is uniformly comparable to $\Phi_\varphi(\tau_1)$ for large eigenvalues. The considerations of the proof of Proposition 5.13 provide the expansion

$$\int_{\Gamma_\infty\backslash\mathbb{H}}\varphi^2\chi dA=\frac{1}{2}\int_0^\infty \mathbb{A}[d\Phi_\varphi]\chi y^{-1}dy\;+\;O(\Phi_\varphi(\alpha^{-1})r^{-1/6}+\|\varphi\|^2_{\tau_0}r^{-1}\log r)$$

for $\chi\in C^1_c((0,\infty))$ with $\mathrm{supp}(\chi)\subset(\alpha,\infty)$. From (3.2) and Proposition 5.11 we find that $\Phi_\varphi(\alpha^{-1})$ is bounded by $C_\chi\|\varphi\|^2_{\tau_0}\log r$, a suitable quantity. We further observe in particular for $\chi\ge 0$, somewhere positive, with support $[\beta,\beta']$ then $\int_0^\infty(1-t^2y^2)^{-1/2}\chi y^{-1}dy$ has support $[0,\beta^{-1}]$ and is bounded below by a positive constant on each proper subinterval $[0,\eta]$ of $[0,\beta^{-1}]$. Now choose a first test function χ to have support (τ_2^{-1},τ_2) for $\tau_2<\tau_1$. It follows that $\int_0^\infty\mathbb{A}[d\Phi_\varphi]\chi y^{-1}dy$ is bounded below by a positive multiple of $\Phi_\varphi(\tau_1)$ and that $\int_{\Gamma_\infty\backslash\mathbb{H}}\varphi^2\chi dA$ is bounded above by a multiple of $\|\varphi\|^2_{\tau_2}$, which by Proposition 5.11 is bounded by a multiple of $\|\varphi\|^2_{\tau_1}$. Choose a second test function χ identically unity on (τ_0^{-1},τ_0) and with support contained in (τ_1^{-1},τ_1). From the first property for the support we have that $\|\varphi\|^2_{\tau_0}\le\int_{\Gamma_\infty\backslash\mathbb{H}}\varphi^2\chi dA$ and from the second property we have that $\int_0^\infty\mathbb{A}[d\Phi_\varphi]\chi y^{-1}dy$ is bounded above by a multiple of $\Phi_\varphi(\tau_1)$. It now follows for large eigenvalues that the sum $\Phi_\varphi(\tau_1)$ is uniformly comparable to the norm squared $\|\varphi\|^2_{\tau_1}$.

We next in effect use Proposition 5.9 to show that the sum $\Phi_\varphi(\tau_1)$ is bounded above by a positive multiple of $\Phi_\varphi(\tau_1)-\Phi_\varphi(\tau_1^{-1})$. We proceed by contradiction. Consider then that there exists a sequence of eigenfunctions $\{\varphi_k\}$, normalized on $\mathcal{H}_{\tau_1}$, with the associated sequence of Lebesgue-Stieljes derivatives $\{d\Phi_{\varphi_k}\}$ converging to a measure $\sigma_{\rm ren}$ that is trivial on the interval $(\tau_1^{-1},\tau_1]$. The bound $\Phi_{\varphi_k}(\tau_1)\ge c\|\varphi_k\|^2_{\tau_1}$ and normalization provide that $\sigma_{\rm ren}$ is not the trivial measure. By Proposition 5.12 we can further assume that the quantities $Q_M(\varphi_k)$ on $SL(2;\mathbb{R})$ converge to an invariant measure $\mu_{\rm ren}$ with index $(0,0)$ $\Gamma_\infty\backslash N\times K$ Fourier-Stieljes coefficient $\nu_{\rm ren}$. A (right $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ invariant) geodesic-flow invariant measure μ on $SL(2;\mathbb{R})$ has a unique representation as a geodesic-indicator integral of a measure κ on $\widehat{\mathbb{G}}$; from the preparatory discussion the index $(0,0)$ coefficient of μ is given by the transform $2\mathbb{A}[\sigma]$ for σ the index zero $\Gamma_\infty\backslash N$ Fourier-Stieljes coefficient of κ. Since the $\mathbb{A}$-transform is injective it follows for the present circumstance that $\sigma_{\rm ren}$ is the index zero $\Gamma_\infty\backslash N$ Fourier-Stieljes coefficient of a Γ-invariant measure on $\widehat{\mathbb{G}}$. The vanishing of the non trivial measure $\sigma_{\rm ren}$ on the region $\tau_1^{-1}<t\le\tau_1$ contradicts Proposition 5.9. The proof is complete.

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