A research brief on the Weil-Petersson metric

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A. Weil introduced a Kähler metric for the Teichmüller space $T_{g,n}$, the space of homotopy marked Riemann surfaces of genus $g$ with $n$ punctures and negative Euler characteristic,[1]. The cotangent space at a marked Riemann surface $\{R\}$ the space $\mathcal{Q}(R)$ of holomorphic quadratic differentials on $R$ is considered with the Petersson hermitian pairing. The Weil-Petersson metric calibrates the variations of the complex structure of $\{R\}$. For a surface of negative Euler characteristic by the Uniformization Theorem two determinations are equivalent: a complex structure and a complete hyperbolic metric. Accordingly the Weil-Petersson metric has been studied through quasiconformal maps, solutions of the inhomogeneous $\overline{\partial}$-equation, the prescribed curvature equation and global analysis, [1, 7, 11].

The quotient of the Teichmüller space $T_{g,n}$ by the action of the mapping class group is the moduli space of Riemann surfaces $\mathcal{M}_{g,n}$; the Weil-Petersson metric is mapping class group invariant and descends to $\mathcal{M}_{g,n}$. $\overline{\mathcal{M}_{g,n}}$ the stable-curve compactification of $\mathcal{M}_{g,n}$ is a projective variety with $\mathcal{D}_{g,n} = \overline{\mathcal{M}_{g,n}} - \mathcal{M}_{g,n}$ the divisor of noded stable-curves i.e. the Riemann surfaces “with disjoint simple loops collapsed to points” and each component of the nodal-complement having negative Euler characteristic. Expansions for the Weil-Petersson metric in a neighborhood of $\mathcal{D}_{g,n}$ provide that the metric on $\mathcal{M}_{g,n}$ is not complete and that there is a distance completion separating points on $\overline{\mathcal{M}_{g,n}}$, [5].

The Weil-Petersson metric has negative sectional curvature, [10, 14]. The behavior near $\mathcal{D}_{g,n}$ provides that the sectional curvature has infimum negative infinity and supremum zero. The holomorphic sectional, Ricci and scalar curvatures are each bounded above by genus dependent negative constants. A modification of the metric introduced by C. T. McMullen is Kähler-hyperbolic in the sense of M. Gromov, has positive first eigenvalue, and pro-
vides that the sign of the \( \mathcal{M}_{g,n} \) orbifold Euler characteristic is given by the parity of the dimension, \([6]\).

The Weil-Petersson Kähler form \( \omega_{WP} \) appears in several contexts. L. A. Takhtajan and P. G. Zograf considered the local index theorem for families of \( \bar{\partial} \)-operators and calculated the first Chern form of the determinant line bundle \( \text{det ind} \bar{\partial} \) with D. G. Quillen’s construction of a metric based on the hyperbolic metric; the Chern form is \( \frac{1}{12\pi} \omega_{WP} \), \([9]\). The “universal curve” is the fibration \( \mathcal{C}_{g,n} \) over \( \mathcal{T}_{g,n} \) with fibre \( R \) above the class \( \{R\} \). The Uniformization Theorem provides a metric for the vertical line bundle \( \mathcal{V}_{g,n} \) of the fibration. The setup extends to the compactification: the pushdown of the square of the first Chern form of \( \mathcal{V}_{g,n} \) for the hyperbolic metric is the current class of \( \frac{1}{2\pi} \omega_{WP} \), \([16]\). The result is the basis for a proof of the projectivity of \( \overline{\mathcal{M}}_{g,n} \), \([12]\). The Weil-Petersson volume element appears in the calculation by E. D’Hoker and D. H. Phong of the partition function integrand for A. M. Polyakov’s String Theory, \([3]\). Generating functions have also been developed for the volumes of moduli spaces, \([4, 17]\).

Fenchel-Nielsen presented “twist-length” coordinates for \( \mathcal{T}_{g,n} \) as the parameters \( \{(\tau_j, \ell_j)\} \) for assembling “pairs of pants”, three-holed spheres with hyperbolic metric and geodesic boundaries, to form hyperbolic surfaces. The Kähler form has a simple expression in terms of the coordinates \( \omega_{WP} = \sum_j d\ell_j \wedge d\tau_j \), \([13]\). Each geodesic length function \( \ell_* \) is convex along Weil-Petersson geodesics, \([15]\). In consequence \( \mathcal{T}_{g,n} \) has an exhaustion by compact Weil-Petersson convex sets, \([15]\).

A. Verjovsky and S. Nag considered the Weil-Petersson geometry for the infinite dimensional universal Teichmüller space and found that the form \( \omega_{WP} \) coincides with the Kirillov-Kostant symplectic structure coming from \( \text{Diff}^+(S^1)/\text{Mob}(S^1) \), \([8]\). I. Biswas and S. Nag showed that the analog of the Takhtajan-Zograf result is valid for the universal moduli space obtained from the inductive limit of Teichmüller spaces for characteristic coverings, \([2]\).

References


