A research brief on the Weil-Petersson metric

Scott A. Wolpert

March 15, 2000

A. Weil introduced a Kähler metric for the Teichmüller space $T_{g,n}$, the space of homotopy marked Riemann surfaces of genus g with n punctures and negative Euler characteristic,[1]. The cotangent space at a marked Riemann surface $\{R\}$ the space $Q(R)$ of holomorphic quadratic differentials on R is considered with the Petersson hermitian pairing. The Weil-Petersson metric calibrates the variations of the complex structure of $\{R\}$. For a surface of negative Euler characteristic by the Uniformization Theorem two determinations are equivalent: a complex structure and a complete hyperbolic metric. Accordingly the Weil-Petersson metric has been studied through quasiconformal maps, solutions of the inhomogeneous $\bar{\partial}$-equation, the prescribed curvature equation and global analysis, [1, 7, 11].

The quotient of the Teichmüller space $\mathcal{T}_{g,n}$ by the action of the mapping class group is the moduli space of Riemann surfaces $\mathcal{M}_{g,n}$; the Weil-Petersson metric is mapping class group invariant and descends to $\mathcal{M}_{g,n}$. $\overline{\mathcal{M}_{g,n}}$ the stable-curve compactification of $\mathcal{M}_{g,n}$ is a projective variety with $\mathcal{D}_{g,n} = \overline{\mathcal{M}_{g,n}} - \mathcal{M}_{g,n}$ the divisor of noded stable-curves i.e. the Riemann surfaces "with disjoint simple loops collapsed to points" and each component of the nodal-complement having negative Euler characteristic. Expansions for the Weil-Petersson metric in a neighborhood of $\mathcal{D}_{g,n}$ provide that the metric on $\mathcal{M}_{g,n}$ is not complete and that there is a distance completion separating points on $\overline{\mathcal{M}_{g,n}}$, [5].

The Weil-Petersson metric has negative sectional curvature, [10, 14]. The behavior near $\mathcal{D}_{g,n}$ provides that the sectional curvature has infimum negative infinity and supremum zero. The holomorphic sectional, Ricci and scalar curvatures are each bounded above by genus dependent negative constants. A modification of the metric introduced by C. T. McMullen is Kähler-hyperbolic in the sense of M. Gromov, has positive first eigenvalue, and provides that the sign of the $\mathcal{M}_{g,n}$ orbifold Euler characteristic is given by the parity of the dimension, [6].

The Weil-Petersson Kähler form ω_{WP} appears in several contexts. L. A. Takhtajan and P. G. Zograf considered the local index theorem for families of $\bar{\partial}$-operators and calculated the first Chern form of the determinant line bundle $det\, ind\, \bar{\partial}$ with D. G. Quillen's construction of a metric based on the hyperbolic metric; the Chern form is $\frac{1}{12\pi^2}\omega_{WP}$, [9]. The "universal curve" is the fibration $\mathcal{C}_{g,n}$ over $\mathcal{T}_{g,n}$ with fibre R above the class $\{R\}$. The Uniformization Theorem provides a metric for the vertical line bundle $\mathcal{V}_{g,n}$ of the fibration. The setup extends to the compactification: the pushdown of the square of the first Chern form of $\overline{\mathcal{V}_{g,n}}$ for the *hyperbolic metric* is the current class of $\frac{1}{2\pi^2}\omega_{WP}$,[16]. The result is the basis for a proof of the projectivity of $\overline{\mathcal{M}_{g,n}}$,[12]. The Weil-Petersson volume element appears in the calculation by E. D'Hoker and D. H. Phong of the partition function integrand for A. M. Polyakov's String Theory, [3]. Generating functions have also been developed for the volumes of moduli spaces, [4, 17].

Fenchel-Nielsen presented "twist-length" coordinates for $\mathcal{T}_{g,n}$ as the parameters $\{(\tau_j, \ell_j)\}$ for assembling "pairs of pants", three-holed spheres with hyperbolic metric and geodesic boundaries, to form hyperbolic surfaces. The Kähler form has a simple expression in terms of the coordinates $\omega_{WP} = \Sigma_j\, d\ell_j \wedge d\tau_j$, [13]. Each geodesic length function ℓ_* is convex along Weil-Petersson geodesics, [15]. In consequence $\mathcal{T}_{g,n}$ has an exhaustion by compact Weil-Petersson convex sets, [15].

A. Verjovsky and S. Nag considered the Weil-Petersson geometry for the infinite dimensional universal Teichmüller space and found that the form ω_{WP} coincides with the Kirillov-Kostant symplectic structure coming from $Diff^+(\mathbb{S}^1)/Mob(\mathbb{S}^1)$, [8]. I. Biswas and S. Nag showed that the analog of the Takhtajan-Zograf result is valid for the universal moduli space obtained from the inductive limit of Teichmüller spaces for characteristic coverings, [2].

References

[1] Lars V. Ahlfors. Some remarks on Teichmüller's space of Riemann surfaces. *Ann. of Math. (2)*, 74:171–191, 1961.

[2] Indranil Biswas and Subhashis Nag. Weil-Petersson geometry and determinant bundles on inductive limits of moduli spaces. In *Lipa's legacy (New York, 1995)*, pages 51–80. Amer. Math. Soc., Providence, RI, 1997.

[3] Eric D'Hoker and D. H. Phong. Multiloop amplitudes for the bosonic Polyakov string. *Nuclear Phys. B*, 269(1):205–234, 1986.

[4] R. Kaufmann, Yu. Manin, and D. Zagier. Higher Weil-Petersson volumes of moduli spaces of stable n-pointed curves. *Comm. Math. Phys.*, 181(3):763–787, 1996.

[5] Howard Masur. Extension of the Weil-Petersson metric to the boundary of Teichmuller space. *Duke Math. J.*, 43(3):623–635, 1976.

[6] Curtis T. McMullen. The moduli space of Riemann surfaces is Kähler hyperbolic. preprint, 1998.

[7] Subhashis Nag. *The complex analytic theory of Teichmüller spaces*. John Wiley & Sons Inc., New York, 1988. A Wiley-Interscience Publication.

[8] Subhashis Nag and Alberto Verjovsky. Diff(S^1) and the Teichmüller spaces. *Comm. Math. Phys.*, 130(1):123–138, 1990.

[9] L. A. Takhtajan and P. G. Zograf. A local index theorem for families of $\overline{\partial}$-operators on punctured Riemann surfaces and a new Kähler metric on their moduli spaces. *Comm. Math. Phys.*, 137(2):399–426, 1991.

[10] A. J. Tromba. On a natural algebraic affine connection on the space of almost complex structures and the curvature of Teichmüller space with respect to its Weil-Petersson metric. *Manuscripta Math.*, 56(4):475–497, 1986.

[11] Anthony J. Tromba. *Teichmüller theory in Riemannian geometry.* Birkhäuser Verlag, Basel, 1992. Lecture notes prepared by Jochen Denzler.

[12] Scott A. Wolpert. On obtaining a positive line bundle from the Weil-Petersson class. *Amer. J. Math.*, 107(6):1485–1507 (1986), 1985.

[13] Scott A. Wolpert. On the Weil-Petersson geometry of the moduli space of curves. *Amer. J. Math.*, 107(4):969–997, 1985.

[14] Scott A. Wolpert. Chern forms and the Riemann tensor for the moduli space of curves. *Invent. Math.*, 85(1):119–145, 1986.

[15] Scott A. Wolpert. Geodesic length functions and the Nielsen problem. *J. Differential Geom.*, 25(2):275–296, 1987.

[16] Scott A. Wolpert. The hyperbolic metric and the geometry of the universal curve. *J. Differential Geom.*, 31(2):417–472, 1990.

[17] Peter Zograf. The Weil-Petersson volume of the moduli space of punctured spheres. In *Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991)*, pages 367–372. Amer. Math. Soc., Providence, RI, 1993.