Weil-Petersson sampler

A highlight of themes, current understanding and research

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Table of contents

Abstract

In 1958, Weil introduced a Hermitian metric for the deformation space of Riemann surfaces based on the Petersson pairing for automorphic forms. Investigations include the symplectic geometry, topology of moduli space of Riemann surfaces, metric space geometry, as well as the analytic/algebraic geometry of curvature, characteristic classes and local index formulas.
Applications: proofs of projectivity of compactified moduli, Nielsen realization, description of cycles/cocycles on moduli space, positivity of line bundles, local index formulas for $\overline{\partial}$, geometry of quasi Fuchsian manifolds, rigidity of the mapping class group, a Witten-Kontsevich solution, arithmetic intersection theory, dynamical and statistical quantities for surfaces, and description of the complete Kähler-Einstein metric for moduli space.

References

This presentation is an effort to give highlights of understanding of Weil-Petersson geometry. Material is presented without references and not especially following the order of discovery. For recent results begin by Googling for webpages, using Arxiv.org or clicking on the names - Jeffrey Brock, Zeno Huang, K. Liu & X. Sun & S. T. Yau, Howard Masur, Curt McMullen, Maryam Mirzakhani, Robert Penner, Leon Takhtajan, Richard Wentworth, Michael Wolf, Peter Zograf, and the author. The introduction of the *following* contains an overview current up until 2001

Scott A. Wolpert, Geometry of the Weil-Petersson completion of Teichmüller space. In *Surveys in Differential Geometry VIII: Papers in Honor of Calabi, Lawson, Siu and Uhlenbeck*, pages 357–393. Intl. Press, Cambridge, MA, 2003.

References

and for background reading

W. J. Harvey, editor. *Discrete groups and automorphic functions*. Academic Press [Harcourt Brace Jovanovich Publishers], London, 1977.

John Hamal Hubbard. *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1*.

Athanase Papadopoulos, editor. *Handbook of Teichmüller theory. Vol. I*, volume 11 of *IRMA Lectures in Mathematics and Theoretical Physics*. European Mathematical Society (EMS), Zürich, 2007.

Conformal structure & variation of a Riemann surface

A conformal structure is defined by an atlas of charts $\{z_\alpha\}$.
A variation of conformal structure is defined by:

- varying chart overlaps, or
- gluing in an elementary family, or
- in general by a vector field on the universal cover.

Examples of elementary families come from lower genus or the collar plumbing family $\{zw = t\} \to \{t\}$ - the family of complex hyperbolas over the t-disc.
Interchanging primitives, all variations can be described by Beltrami differentials - tensors μ of type $\frac{\partial}{\partial z} \otimes \overline{dz}$, $\|\mu\|_\infty < 1$, prescribing new charts $\{w_\alpha(z_\alpha)\}$, with $dw_\alpha = (w_\alpha)_{z_\alpha}(dz_\alpha + \mu\overline{dz_\alpha})$ and the charts local solutions of $w_{\overline{z}} = \mu w_z$.

Variation of hyperbolic structure

The corresponding variation of hyperbolic structure is determined:

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- for finite area hyperbolic metrics, Wolf describes: for D the hyperbolic Laplacian (spectrum ≤ 0), $\mu = \overline{\phi}(ds^2)^{-1}$ with $\phi \in Q$, a holomorphic quadratic differential, the hyperbolic metric is

$$d\sigma^2 = \phi + (\mathcal{H} + \mathcal{L})ds^2 + \overline{\phi}$$

for the solution of $D \log \mathcal{H} = 2\mathcal{H} - 2\mathcal{L} - 2$ and $\mathcal{H}\mathcal{L} = |\phi|^2$.

General setting

R a Riemann surface of genus g with n punctures with hyperbolic metric ds^2. $\mathcal{T}$ the Teichmüller space of homotopy marked conformal or equivalently hyperbolic structures. $\mathcal{T}$ is a $\mathbb{C}$ manifold - with Bers embedding realize as a domain in $\mathbb{C}^{3g-3+n}$.

For R with canonical bundle κ_R, the space of infinitesimal deformations is the Čech group $\check{H}^1(\mathcal{O}(\kappa_R^{-1}))$ with dual space $Q(R)$ - the R-holomorphic quadratic differentials.

Definition

Weil introduced the Hermitian cotangent pairing

$$\langle \phi, \psi \rangle = \int_R \phi\overline{\psi}\,(ds^2)^{-1} \quad \text{for } \phi,\ \psi \in Q$$

with dual the Weil-Petersson (WP) metric.

WP basic properties

- Metric is Kähler, incomplete, negative sectional curvature with sup $= 0$ (except in dim $= 1$) and inf $= \infty$.

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- Metric in higher genus limits under degeneration (pinching short geodesics) of Riemann surfaces to lower genera metrics.
- Metric is natural with respect to unbranched surface covers.
- Multiple connections and interplays between WP geometry and 2-d & 3-d hyperbolic geometry.

Geodesic-length functions: a fundamental tool

For a closed curve α on surface - length ℓ_α of unique R geodesic in free homotopy class defines a function on $\mathcal{T}$. For the pairing of WP gradients have

$$\langle \operatorname{grad} \ell_\alpha, \operatorname{grad} \ell_\beta \rangle = \frac{2}{\pi}\ell_\alpha \delta_{\alpha\beta} + O(\ell_\alpha^2 \ell_\beta^2) \text{ for } \alpha,\ \beta \text{ simple disjoint.}$$

Riera gives an exact pairing formula as infinite sum of lengths of geodesics connecting α & β. For the second derivative have

$$2\ell_\alpha \ddot{\ell}_\alpha[\mu,\mu] - \dot{\ell_\alpha}^2[\mu] - 3\dot{\ell_\alpha}^2[i\mu] \geq 0, \text{ and is } O(\ell_\alpha^3 \|\mu\|_{WP}).$$

Applications. ℓ_α, $\ell_\alpha^{1/2}$ are convex along WP geodesics. With negative curvature, $\mathcal{T}$ is a length space - each pair of points is connected by unique length minimizing path.

Fenchel-Nielsen (FN) coordinates

A *pair of pants* is a $g = 0$ hyperbolic surface with three geodesic circle boundaries. Same length pants' boundaries can be abutted to form larger hyperbolic surfaces. The abutting parameters are: ℓ_α the boundary length; τ_α the relative displacement between circles.

Theorem (Fenchel-Nielsen coordinates)

For a maximal set of disjoint simple closed geodesics (a pants decomposition) the parameters $\{(\ell_\alpha, \tau_\alpha)\}_{\alpha=1}^{3g-3+n}$ *provide a real analytic equivalence of* $\mathcal{T}$ *to* $(\mathbb{R}_+ \times \mathbb{R})^{3g-3+n}$

The subset $\{0 < \ell_\alpha < c, 0 < \tau_\alpha \leq \ell_\alpha\}$ is a *MCG* rough fundamental domain. The *augmented Teichmüller space* $\overline{\mathcal{T}}$ is defined: for all possible partitions extend parameter ranges with $\ell_\alpha = 0$ (with τ_α undefined) specifying a pair of hyperbolic cusps.

Augmented Teichmüller space properties

- $\overline{\mathcal{T}}$ is non locally compact; $\overline{\mathcal{T}}/MCG$ is topologically the Deligne-Mumford compactification.

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- For J the complex structure and $g_{WP;\mathcal{T}(\sigma)}$ the WP metric on $\mathcal{T}(\sigma)$, then near the stratum, in terms of $\{\operatorname{grad}\ell_*\}$

$$g_{WP} = 2\pi\sum_{\alpha\in\sigma}(d\ell_\alpha^{1/2})^2+(d\ell_\alpha^{1/2}\circ J)^2 + g_{WP;\mathcal{T}(\sigma)} + O(d_{\mathcal{T}(\sigma)}).$$

Symplectic geometry, twists & lengths

For simple closed geodesics - variation of the displacement parameter τ_α defines the twist vector field t_α on $\mathcal{T}$. The fundamental twist-length Kähler form duality

$$d\ell_\alpha = 2\omega_{WP}(\ ,t_\alpha)$$

presents ℓ_α as a Hamiltonian. Calculation in universal covers gives

$$t_\alpha \ell_\beta = 2\omega_{WP}(t_\alpha, t_\beta) = \sum_{p\in\alpha\cap\beta} \cos\theta_p \quad \text{and}$$

$$2t_\alpha^2 \ell_\beta = \coth \ell_\beta/2\,(\sin\theta)^2 \quad \text{for a single intersection.}$$

Positivity of the second derivative is an instance of earthquake convexity. Kerckhoff uses convexity for Nielsen realization.

$d\ell \wedge d\tau$ and $dL \wedge dL$

Twist-length duality and symmetries provide for FN coordinates

$$2\omega_{WP} = \sum_{\alpha \in \sigma} d\ell_\alpha \wedge d\tau_\alpha$$

an essential formula for Mirzakhani's integrations. A surface with cusps is also a union of ideal triangles. Logarithms of edge lengths between horocycles provide global coordinates for $\mathcal{T}$. Penner provides the lambda-length formula

$$2\omega_{WP} = -\sum_{triangles} dL_0 \wedge dL_1 + dL_1 \wedge dL_2 + dL_2 \wedge dL_0$$

and uses for integrating and describing a Poincaré dual to ω_{WP}.

Mirzakhani's investigations

Mirzakhani generalizes McShane's identity

$$\sum_{simple\ \ geodesics\ \ \alpha} \frac{1}{e^{\ell_\alpha}+1} = \frac{1}{2}, \quad \text{for } g=1,\ n=1,$$

describes a recursive integration scheme, and applies symplectic reduction for - intersection calculations on $\mathcal{M}$, and a solution of Witten-Kontsevich. Her recursion for moduli space volumes for surfaces with prescribed boundary lengths provides for $g=1$,

$$V_L = \frac{\pi^2}{12}+\frac{L^2}{48},\ \ V_{L_1,L_2} = \frac{\pi^4}{4}+\frac{\pi^2(L_1^2+L_2^2)}{12}+\frac{L_1^2L_2^2}{96}+\frac{(L_1^4+L_2^4)}{192}.$$

And in general the number of simple closed geodesics of length at most Λ is asymptotic to Λ^{6g-6+n}.

Sectional curvatures

Calculation of curvature begins with the perturbation expansion

$$dA_\epsilon \;=\; (1-\epsilon^2(1+2(D-2)^{-1})\mu\overline{\mu}+O(\epsilon^3))\,dA$$

for $\mu=\overline{\phi}(ds^2)^{-1}$. WP curvatures satisfy: sectional <0; holomorphic sectional $<-1/(2\pi(g-1))$; and the dual metric is Nakano positive. Current investigations involve specific sectional curvatures. The Hermitian curvature tensor satisfies for gradient fields $\lambda_*=\operatorname{grad}\ell_*^{1/2}$,

$$R(\lambda_\alpha,\lambda_\alpha,\lambda_\alpha,\lambda_\alpha)=\frac{3}{16\pi^3\ell_\alpha}+O(\ell_\alpha)$$

and for disjoint simple geodesics not all the same

$$R(\lambda_\alpha,\lambda_\beta,\lambda_\gamma,\lambda_\delta)=O((\ell_\alpha,\ell_\beta,\ell_\gamma,\ell_\delta)^{1/2}).$$

Analytic geometry

Liu-Sun-Yau use the negative Ricci form as a reference metric for studying the complete Kähler-Einstein metric on $\mathcal{T}$ and show

Theorem (Liu, Sun & Yau)

The Teichmüller-Kobayashi, Carathéodory, Bergman, Kähler-Einstein, McMullen, asymptotic Poincaré, and negative Ricci are all comparable metrics.

Comparability brings together properties of the classical domain metrics. A range of applications follows.

Takhtajan-Zograf investigate the Quillen metric for determinant index bundles for $\overline{\partial}$ acting on smooth $dz^{\otimes k}$-tensors, $k > 1$, with the Selberg zeta function $Z(s)$ for the determinant of the Laplacian.

Takhtajan-Zograf index formulas & Selberg zeta values

For the Hermitian connection curvature $\Theta^{(k)}$ for the Petersson pairing on holomorphic $dz^{\otimes k}$-tensors then

$$c_1(\det \operatorname{ind} \overline{\partial}_k) = \frac{i}{2\pi}(\Theta^{(k)} - \overline{\partial}\partial \log Z(k)) = \frac{6k^2 - 6k + 1}{12\pi^2}\omega_{WP}.$$

A counterpart formula is developed for surfaces with cusps and holomorphic factorizations are given for $\overline{\partial}\partial$-primitives for ω_{WP}. Freixas i Montplet studies arithmetic intersection pairings and analytic hyperbolic geometry to find for three-pointed spheres

$$\log Z'(1) = 4\zeta(-1) + \log 2\pi + \frac{10}{9}\log 2.$$

The augmented Teichmüller space $CAT(0)$ geometry

- $\overline{\mathcal{T}}$ is $CAT(0)$, a simply connected, complete generalized non positively curved length space. Distances between sides of a geodesic triangle are bounded by corresponding distances for a Euclidean triangle with corresponding side lengths.

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- $\overline{\mathcal{T}}$ is a stratified metric space with geodesically convex strata. Geodesics at most change strata at endpoints.
- $\overline{\mathcal{T}}$ is closed convex-hull of maximally pinched surfaces (unions of thrice-punctured spheres). Geodesics to maximal surfaces are dense. $\overline{\mathcal{T}}$ is infinite polyhedron. **Masur-Wolf Theorem:** MCG is the full WP isometry group.

Geodesics beginning at the augmentation

Geodesics on $\overline{\mathcal{T}}$ have well defined endpoints. Geodesic-length functions are identically zero or strictly increasing on geodesics beginning at the augmentation. For unit-speed beginning geodesics the Alexandrov angle is defined by limiting for the Law of Cosines. The Alexandrov tangent cone AC_p is the set of geodesics beginning at p modulo the relation of same speed and zero angle. AC_p has the structure of a cone in an inner product space. For a suitable collection of geodesic-lengths $(\ell_\alpha^{1/2}, \ell_\beta^{1/2})$, initial derivatives provide a mapping Λ from AC_p to $\mathbb{R}_{\geq 0}^{|\sigma|} \times T_p\mathcal{T}(\sigma)$, for $\sigma = \{\alpha\}$.

Theorem: The mapping Λ from AC_p to $\mathbb{R}_{\geq 0}^{|\sigma|} \times T_p\mathcal{T}(\sigma)$ is an isometry of cones with restrictions of inner products. A geodesic with initial derivative $\dot{\ell_\alpha}^{1/2}(0)$ vanishing is contained in the stratum $\{\ell_\alpha = 0\}$.

More behavior of geodesics

The augmentation set has the local structure of an orthogonal product of geodesic hyperplanes. Yamada introduces formal reflections in the proper $\overline{\mathcal{T}}$ strata, and constructs an infinite development $\mathcal{D}(\overline{\mathcal{T}})$ (a Coxeter complex) - a simply connected $CAT(0)$ space, now with geodesics having infinite prolongations. Yamada shows the space is finite rank in sense of Korevaar-Schoen and applies their harmonic map existence to give a proof of **Daskalopoulos-Wentworth Theorem:** For an isometric action on $\overline{\mathcal{T}}$ of the fundamental group of a compact Riemannian manifold, either there is an equivariant harmonic map of universal covers or an equivalence class of rays is fixed by the action.

Also, dynamical properties of infinite geodesics are being investigated. There are geodesics dense in the tangent bundle.

Uniformly distributed geodesics and length variation

On a hyperbolic surface R, the closed geodesics $\{\gamma \mid \ell_\gamma \leq L\}$ become uniformly distributed as L becomes large. The unit-tangents limit to the uniform distribution in the unit-tangent bundle. The first derivative of length $\dot{\ell}_\gamma[\mu]$ for $\mu = \overline{\phi}(ds^2)^{-1}$ is given by an integral of ϕ along γ; the first derivative of log length of a uniformly distributed sequence is given by the mean of ϕ on the unit-tangent bundle; the mean is zero. Thurston combines these observations and geodesic-length convexity to find for a uniformly distributed sequence the second derivative $\lim_{\{\gamma\}} D^2 \log \ell_\gamma[\mu, \mu]$ converges and defines a Riemannian metric for $\mathcal{T}$ - the random geodesic metric.

Theorem: $g_{random\ geodesic} = 4/(3area(R))\, g_{WP}$.

quasi Fuchsian groups

A pair $(R,S) \in \mathcal{T} \times \mathcal{T}$ determines a hyperbolic 3-manifold $Q(R,S)$ with $R \sqcup S$ its conformal boundary at infinity. For $Q(R,S) = \mathbb{H}^3/\Gamma$ - the group Γ acts as $PSL(\mathbb{C})$ transformations on $\hat{\mathbb{C}}$, the boundary of hyperbolic space, with limit set $\Lambda(\Gamma)$ a topological circle of Hausdorff dimension Hdim(Γ). The convex core $C(R,S)$ is the smallest convex subset of $Q(R,S)$ carrying the fundamental group.
Brock's volume comparison: $d_{WP}(R,S)$ and $vol(C(R,S))$ are comparable in the sense of quasi-isometries.
The Liouville action functional $F[\varphi] = \int_R e^{\varphi} + |\varphi_z|^2 dzd\overline{z}$ is a tool for understanding the $PSL(\mathbb{C})$-structure uniformization of R.
Takhtajan-Teo holography: For a test function φ, the Liouville action $F[\varphi]$ gives the φ-renormalized volume of $Q(R,S)$.
McMullen-Bridgeman-Taylor Theorem: The initial second derivative satisfies $D^2\,\mathrm{Hdim}(\Gamma)[\mu,\mu] = g_{WP}/(3area(R))$.