Geometry of the Weil-Petersson completion of Teichmüller space

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1 Introduction

Let R be a Riemann surface of genus g with n punctures, $3g-3+n>0$, and $\mathcal{T}$ the Teichmüller space of R. The Weil-Petersson (WP) metric for $\mathcal{T}$ is a Kähler metric with negative sectional curvature [4, 35, 36, 41]. With the WP metric $\mathcal{T}$ is a unique geodesic space [42]: for each pair of points there is a unique distance-realizing joining curve. The *augmented Teichmüller space* $\overline{\mathcal{T}}$, a stratified non locally compact space, is the space of marked noded Riemann surfaces and is a bordification of $\mathcal{T}$ in the style of Baily-Borel, [2, 5]. For $(g,n)=(1,1)$, $\overline{\mathcal{T}}$ is the bordification $\mathbb{H}\cup\mathbb{Q}$ of the upper half-plane with the horoball-neighborhood topology. The augmented Teichmüller space is in fact the WP metric completion of the Teichmüller space [30]. The strata of $\overline{\mathcal{T}}$ are lower-dimensional Teichmüller spaces; each stratum with its natural WP metric isometrically embeds into the completion $\overline{\mathcal{T}}$.

Our purpose is to present a view of the current understanding of the geometry of the WP geodesics on $\overline{\mathcal{T}}$. The behavior of geodesics in-the-large has significant consequences for the action of the mapping class group; see [7, 13, 31, 42, 47] and Section 7 below. The behavior of geodesics is also an important consideration for the harmonic map problem, as well as the study of rigidity of homomorphisms of lattices in Lie groups to the mapping class group [11, 12, 13, 18, 26, 47]. Furthermore, the behavior of geodesics is a consideration for the *rank* of $\mathcal{T}$ [9]. We begin by mentioning a collection of recent results [8, 7, 9, 13, 31, 32, 47]. Recall that for a hyperbolic surface, the length of the unique geodesic in a prescribed free homotopy class provides a

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function, the *geodesic-length*, on $\overline{\mathcal{T}}$ valued in $[0, \infty)$. A general fact is that geodesic-length functions are strictly convex along WP geodesics [42].

The work of C. McMullen provides a prelude [32]. Recall that a Bers embedding $\beta_S : \mathcal{T} \to \mathbf{T}^*_S\mathcal{T}$ is a biholomorphic map of the Teichmüller space to a domain in a cotangent space; from the Nehari estimate the image is bounded independent of S in terms of the Teichmüller and the WP co-metrics. Observe for S_0 fixed, $-\beta_S(S_0)$ is a section of the cotangent bundle $\mathbf{T}^*\mathcal{T}$, a differential 1-form $\theta_{WP}(S)$ on $\mathcal{T}$. McMullen showed [32, Thrm. 1.5] that $d(i\theta_{WP}) = \omega_{WP}$ is the WP symplectic form. An application is a positive lower bound for the WP Rayleigh-Ritz quotient. He then introduced a smooth modification of the WP metric by including the complex Hessians of the small-valued geodesic-length functions. He combined the above and estimates for geodesic-length derivatives to show that the modification is a *Kähler hyperbolic metric* for the moduli space of Riemann surfaces that is comparable to the Teichmüller metric [32, Thrm. 1.1]. As applications McMullen found: a positive lower bound for the Teichmüller Rayleigh-Ritz quotient, a complex submanifold isoperimetric inequality, and the alternating sign of the orbifold Euler characteristic for the moduli spaces [32].

J. Brock has established important results on the large-scale behavior of WP distance,[8]. Brock considered the metric space, the *pants graph* $C_{\mathbf{P}}(F)$, having vertices the distinct pants decompositions of F and joining edges of unit-length for pants decompositions differing by a single elementary move,[8]. He showed that the 0-skeleton of $C_{\mathbf{P}}(F)$ is quasi-isometric to $\mathcal{T}$ with the WP metric. In particular by an observation of L. Bers there is a constant L such that each hyperbolic surface has a pants decomposition by geodesics of length at most L. For a pants decomposition $\mathcal{P}$, denote by $V(\mathcal{P}) \subset \mathcal{T}$ the subset of surfaces with the designated decomposition. The union $\cup_{\mathcal{P}} V(\mathcal{P})$ provides an open cover for $\mathcal{T}$. Brock found that WP distance records the configuration of the open sets $V(\mathcal{P})$ with the 0-skeleton of $C_{\mathbf{P}}(F)$ as the metric model. An important consequence of Brock's result is the correspondence between *quasi-geodesics* (quasi length-minimizing paths) on $\mathcal{T}$ and *quasi-geodesics* on $C_{\mathbf{P}}(F)$. He further showed for $p, q \in \mathcal{T}$ that the corresponding quasifuchsian hyperbolic three-manifold has convex-core volume comparable to $d_{WP}(p, q)$. At large-scale WP distance and convex-core volume are approximately combinatorially determined. He also showed that the first eigenvalue of the hyperbolic manifold and corresponding Hausdorff dimension of the limit set are estimated in terms of WP distance.

J. Brock and B. Farb used the correspondence to study the *rank* of $\mathcal{T}$ in the sense of M. Gromov [9]. A notion for the rank of a metric space is the maximal dimension of a quasi-flat, a quasi-isometric embedding of a Euclidean space. Brock and Farb found that $C_{\mathbf{P}}(F)$ contains quasi-flats of dimension $g-1+\lfloor\frac{g+n}{2}\rfloor$. It follows from application of Brock's quasi-isometry that the WP rank is likewise bounded. Gromov-hyperbolic metric spaces have rank one and thus the bound provides for $\dim\mathcal{T}>2$, that $\mathcal{T}$ is not Gromov-hyperbolic [9, Thrm. 1.1]. The authors further found for $\dim\mathcal{T}\leq 2$ that $C_{\mathbf{P}}(F)$ and thus $\mathcal{T}$ are Gromov-hyperbolic [9, Thrm. 5.1]. Yamada and M. Bestvina had also considered the maximal dimension of a flat, [46]. Z. Huang has discovered further new asymptotic flatness [19]. Variation of independent plumbing parameters t prescribes planes with WP curvature $O((-\log|t|)^{-1})$.

W. Ballman and P. Eberlein posed a group-theoretic notion of the rank [21]. For discrete cofinite isometry groups of complete simply connected Riemannian manifolds with non positive curvature bounded from below the Ballman-Eberlein notion coincides with the geometrically defined rank. N. Ivanov has shown that mapping class groups have rank one [21]. N. Ivanov and independently B. Farb, A. Lubotzky and Y. Minsky further proved that any infinite-order element in the mapping class group has linear growth in the word metric; at least $O(n)$ generators of the group are required to write the n^{th} iterate of an element of infinite order [14, 23]. Rank-1 lattices in simple Lie groups have the $O(n)$ writing-property, while higher-rank lattices do not have the property.

An important discovery of Sumio Yamada was the non refraction of WP geodesics: a geodesic on $\overline{\mathcal{T}}$ at most changes strata at its endpoints; see [47, Thrm. 2], [13, Lemma 3.6] and Propositions 11 and 12 below. A second important observation was that the strata of $\overline{\mathcal{T}}$ are geodesically convex. Yamada refined the original WP expansion of Masur [30] to present a third-order remainder expansion of the metric in the C^1-category. A key ingredient was the use of an improved estimate for degenerating families of hyperbolic metrics. The considerations were based on the relatively technical work of Wolf [37] and Wolf-Wolpert [38]. Yamada used the expansion to study the behavior of geodesics in neighborhoods of the bordification. He considered the WP Levi-Civita connection and one-dimensional harmonic maps to investigate the non refraction. Yamada then used the convexity of geodesic-length functions and the negative WP curvature to find that $\overline{\mathcal{T}}$ is a $CAT(0)$ space; see [13], the attribution to B. Farb in [31] and Theorem 14 below. He further noted that geodesic convexity of strata is an immediate consequence of the convexity of geodesic-length functions [47, Thrm. 1]. He applied the statements to give consideration of fixed-points and realizing translation lengths for mapping classes. Yamada also presented that irreducible elements of the mapping class group have positive translation length and a unique axis. The work has served as an inspiration for the work of Daskalopoulos and Wentworth [13], as well as the author.

The geometry of $CAT(0)$ spaces is developed in Bridson-Haefliger [6]. A *geodesic triangle* is prescribed by a triple of points and a triple of joining length-minimizing curves. A characterization of curvature for metric spaces is provided in terms of distance-comparisons for geodesic triangles. In a $CAT(0)$ space the distance and angle measurements for a triangle are bounded by the corresponding measurements for a Euclidean triangle with the corresponding edge-lengths [6, Chap. II.1, Prop. 1.7].

G. Daskalopoulos and R. Wentworth gave an independent treatment of the WP expansion, the non refraction, the $CAT(0)$ result and a more extensive consideration of actions of mapping classes [13]. The authors obtained a C^0-category expansion by applying the cut-and-paste based estimates for degenerating families of hyperbolic metrics from [44]. Scaling considerations were used for the energy of a parameterized curve to establish non refraction. The authors proved that irreducible mapping classes have positive translation length and a unique axis [13, Thrm. 1.1]. Previously G. Daskalopoulos, L. Katzarkov and R. Wentworth studied the finite energy equivariant harmonic map problem for the target $\mathcal{T}$, [12]. In general a condition on an isometric action is required for the existence of an energy minimizing equivariant map. In the case of a symmetric space target the action should be *reductive*. For $\mathcal{T}$, the authors [12, 13] propose *sufficiently large* as the counterpart of the reductive hypothesis. A subgroup of the mapping class group is sufficiently large provided it contains two irreducible mapping classes acting with distinct fixed points on the space of projective measured foliations. Daskalopoulos and Wentworth established [13, Thrm. 6.2] divergence of the axes for two as above independent irreducible mapping classes. The authors applied their considerations and studied equivariant maps from universal covers of finite volume complete Riemannian manifolds with finitely generated fundamental groups. They showed that if there is a finite energy map with sufficiently large image of the fundamental group, then there is a finite energy equivariant harmonic map [13, Cor. 1.3].

B. Farb and H. Masur established general higher rank superrigidity for the mapping class group as image. For an irreducible lattice in a semisimple Lie group of $\mathbb{R}$-rank at least two, a homomorphism to the mapping class group has finite image [15, Thrm. 1.1]. The authors also considered homomorphisms from $SL_n(\mathbb{Z})$ to the group of homeomorphism of a surface. They showed that all homomorphisms are trivial for n greater than an explicit bound in the genus.

H. Masur and M. Wolf established the WP-analogue of H. Royden's celebrated result: for $3g-3+n>1$ and $(g,n)\neq(1,2)$, every WP isometry of $\mathcal{T}$ is induced by an element of the extended mapping class group. They considered the asymptotic WP geometry to reduce the matter to considering the restriction of an isometry to $\overline{\mathcal{T}}-\mathcal{T}$. In particular an isometry of $\mathcal{T}$ extends to the completion $\overline{\mathcal{T}}$; an isometry of $\overline{\mathcal{T}}$ preserves the strata structure and following an approach of N. Ivanov agrees with a mapping class on the maximally noded surfaces. They then established that the set of maximally noded surfaces forms a uniqueness set for WP isometries [31].

Brock has also studied the family of WP geodesic rays based at a point, the WP visual sphere, [7]. Rays are considered with the topology of convergence of initial segments. He established that the action of the mapping class group does not extend continuously to an action on the WP visual spheres, and that the rays to noded surfaces are dense in the visual spheres. An additional discovery was that convergence of initial segments in general does not provide for convergence of entire rays; see [7] and Section 7 below.

The purpose of this paper is to continue the study in detail of the geometry of WP geodesics on $\overline{\mathcal{T}}$. We provide an independent treatment of the WP expansion based on the less technical approach of [44]. We then use the opportunity to give a range of new applications including: a thorough treatment of the strata structure, a classification of locally Euclidean subspaces of $\overline{\mathcal{T}}$, for the Masur-Wolf theorem a new proof based on a convex hull property, and a classification of limits of WP geodesics.

We find that $\overline{\mathcal{T}}$ is a *stratified* unique geodesic space with the strata intrinsically characterized by the metric geometry (see Theorem 13), [47]. For a reference surface F and $C(F)$, the partially ordered set *the complex of curves*, consider Λ the natural labeling function from $\overline{\mathcal{T}}$ to $C(F)\cup\{\emptyset\}$. For a marked noded Riemann surface (R,f) with $f:F\rightarrow R$, the labeling $\Lambda((R,f))$ is the simplex of free homotopy classes on F mapped to the nodes on R. The level sets of Λ are the strata of $\overline{\mathcal{T}}$. The unique WP geodesic $\widehat{pq}$ connecting $p,q\in\overline{\mathcal{T}}$ is contained in the closure of the stratum with label $\Lambda(p)\cap\Lambda(q)$ (see Theorem 13). The open segment $\widehat{pq}-\{p,q\}$ is a solution of the WP geodesic differential equation on the stratum with label $\Lambda(p)\cap\Lambda(q)$. For a point p, the stratum with label $\Lambda(p)$ is the union of the open geodesic segments containing the point (see Theorem 13).

The central consideration is the expansion of the WP metric in a neighborhood of a point of a positive codimension m stratum $\mathcal{S}$. For s a general multi-index local coordinate for $\mathcal{S}$ and t a plumbing construction multi-index parameter for the transverse to $\mathcal{S}$, we show for the multi-index parameter $r=(-\log|t|)^{-1/2}$ the following expansion for the *metric symmetric-tensor*

(see Corollary 4)

$$dg_{WP}^2(s,t) \;=\; \left(dg_{WP}^2(s,0) \;+\; \pi^3\sum_{k=1}^{m}(4dr_k^2 + r_k^6 d\arg^2 t_k)\right)(1 + O(\|r\|^3)).$$

In particular along $\mathcal{S}$ the WP metric to third-order remainder is a *product-metric* of the WP metric of $\mathcal{S}$ and metrics $(4dr^2 + r^6 d\arg^2 t)$, one for each t parameter. The product-structure with higher-order remainder suggests the isometric embedding of $\mathcal{S}$ into $\overline{\mathcal{T}}$. In the transverse direction to $\mathcal{S}$ the metric is *modeled* by the surface of revolution about the x-axis of $y = (x/2)^3$. The third-order remainder suggests *higher-order flatness* for the normal along $\mathcal{S}$. We combine the above expansion, the rescaling argument for metric spaces and an elementary quadratic inequality to establish the non refraction of geodesics (see Propositions 11 and 12).

Beyond $CAT(0)$, there are important applications for the above expansion. We are able to combine the *flat triangle lemma* of A. D. Alexandrov [6] and Theorem 13 to study the locally Euclidean isometric subspaces (flats) of $\overline{\mathcal{T}}$. A classification is established, and it is found that the maximal dimensional flats are submanifolds of: a product of Teichmüller spaces of g once-punctured tori and $\lfloor\frac{g+n}{2}\rfloor - 1$ four-punctured spheres (see Proposition 16). The result is consistent with the conjecture of Brock-Farb regarding the rank (the maximal dimension of a quasi-isometric embedding of a Euclidean space) of the WP metric, [9]. Following a suggestion of Brock, the considerations also provide that for $\dim\mathcal{T} > 2$ the WP metric is not *Gromov-hyperbolic*. Flat geodesic triangles in $\overline{\mathcal{T}} - \mathcal{T}$ are uniformly approximated by geodesic triangles in $\mathcal{T}$.

We also investigate applications of the Brock result [7] that the geodesic rays from a point of $\mathcal{T}$ to the noded Riemann surfaces have initial tangents dense in the initial tangent space. We generalize the result and show that the geodesics connecting maximally noded Riemann surfaces have tangents dense in the tangent bundle of $\mathcal{T}$ (see Corollary 18). An immediate consequence is that $\overline{\mathcal{T}}$ is the closed WP convex hull of the subset of maximally noded Riemann surfaces (see Corollary 19). The maximally noded Riemann surfaces play a basic role for the WP $CAT(0)$ geometry. In Theorem 20 we combine the convex hull property, the intrinsic nature of the strata structure and the classification of simplicial automorphisms of $C_{\mathbf{P}}(F)$ to study WP isometries. A new proof of the Masur-Wolf result is provided: for $3g-3+n > 1$ and $(g,n) \neq (1,2)$, every WP isometry of $\mathcal{T}$ is induced by an element of the extended mapping class group.

The WP metric is mapping class group invariant. H. Masur found that the Deligne-Mumford moduli space of stable curves $\overline{\mathcal{M}}$ is the WP quotient-metric completion of the moduli space of Riemann surfaces [30]. We note that the WP metric for $\overline{\mathcal{M}}$ is not locally uniquely geodesic near the compactification divisor of noded Riemann surfaces (see Proposition 15). A complete, convex subset of a $CAT(0)$ space is the base for an *orthogonal projection*, [6, Chap. II.2]. The closure of a stratum is complete and convex. We show that the distance to a stratum $\mathcal{S}$ has an expansion in terms of the defining *geodesic-length functions*. For a positive codimension m stratum $\mathcal{S}$, defined by the vanishing of the geodesic-length sum $\ell = \ell_1 + \cdots + \ell_m$, the distance to the stratum has the simple expansion $d(\cdot, \mathcal{S}) = (2\pi\ell)^{1/2} + O(\ell^2)$ (see Corollary 21). Furthermore the vector fields $\{\operatorname{grad}(2\pi\ell_j)^{1/2}\}$ are close to orthonormal near $\mathcal{S}$.

Our final application concerns limits of sequences of geodesics. We consider the classification problem (see Proposition 23). We might expect the compactness of $\overline{\mathcal{M}}$ to be manifested in the sequential compactness of the space of geodesics. But Brock already found that convergence of initial segments in general does not provide for convergence of entire rays. In fact for each sequence of bounded length geodesics there is a subsequence of mapping class group translates that converges geometrically (sequences of products of Dehn twists are applied to subsegments of the geodesics) to a polygonal path, a curve piecewise consisting of geodesics connecting different strata (see Proposition 23). Polygonal paths were first considered by Brock in his investigation of the WP visual sphere and the action of the mapping class group, [7, esp. Secs. 4, 5]. We find that the limit polygonal path is unique length-minimizing amongst paths joining prescribed strata. A simple example of a polygonal path is presented in the opening of Section 7. We apply the considerations and show that a mapping class acting on $\overline{\mathcal{T}}$ either: has a fixed-point, or positive translation length realized on a closed convex set, possibly contained in $\overline{\mathcal{T}} - \mathcal{T}$ (see Theorem 25). For *irreducible* mapping classes, the positive translation length is realized on a unique geodesic within $\mathcal{T}$, [13, 47].

We begin our detailed considerations in the next section with a summary of the notions associated with lengths of curves in metric spaces, [6]. We also review the local deformation theory of noded Riemann surfaces, as well as the specification of Fenchel-Nielsen coordinates and the construction of the augmented Teichmüller space. In the third section we provide the WP expansion. We begin considerations with the exact expansion of the hyperbolic metrics for the model case $zw = t$. Then we consider in detail families of noded Riemann surfaces and their hyperbolic metrics. Beginning with Masur's description of families of holomorphic 2-differentials, we give a simple and self-contained development of the tangent-cotangent coordinate frame pairing for the local deformation space and the desired WP expansion. In the fourth section we develop the length-minimizing properties of the solutions of the WP geodesic differential equation on $\mathcal{T}$. The considerations extend the earlier treatment [42]. In the fifth section we develop the length-minimizing properties of curves on $\overline{\mathcal{T}}$, including the non refraction results and the main theorems. The labeling function Λ serves an important role. WP length-minimizing curves can be analyzed in terms of their strata-behavior and geodesics within strata. WP convexity of the geodesic-length functions also serves an important role. WP geodesics are confined by the level sets and sublevel sets of geodesic-length functions. Non refraction is established by a local rescaling of the metric, and an application of the strict inequality $((a+b)^2+c^2)^{1/2} < (a^2+c^2)^{1/2}+b$ for positive values. In the sixth section first we examine the circumstance for the WP distance between corresponding points of a pair of geodesics not strictly convex. Then we consider the locally Euclidean isometric subspaces of $\overline{\mathcal{T}}$. We also consider the distance to a stratum. In the final section we consider sequences of geodesics and establish the sequential compactness, as well as a general classification for geodesic limits. The results are applied to study the existence of axes for mapping classes.

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2 Preliminaries

We begin with a summary of the notions associated with lengths of curves in a metric space. We closely follow the exposition of Bridson-Haefliger [6] and commend their treatment to the reader. For a metric space (M,d) the *length* of a curve $\gamma : [a,b] \to M$ is

$$L(\gamma) = \sup_{a=t_0\le t_1\le\cdots\le t_n=b} \sum_{j=0}^{n-1} d(\gamma(t_j),\gamma(t_{j+1}))$$

where the supremum is over all possible partitions with no bound on n. A curve is *rectifiable* provided its length is finite. The basic properties of length are provided in [6, Prop. 1.20]. Length is lower semi continuous for a sequence of rectifiable curves converging uniformly to a rectifiable curve. A curve $\gamma : [a,b] \to M$ is *parameterized proportional to arc-length* provided the length of γ restricted to subintervals $[a,t] \subset [a,b]$ is a linear function of t, [6, Defn. 1.21]. A space (M,d) is a *length space* provided the distance between each pair of points is equal to the infimum of the length of rectifiable curves joining the points. It is an observation that the completion of a length space is again a length space [6, Exer. 3.6 (3)]. A curve $\gamma : [a, b] \to M$ is *length-minimizing* provided for all $a \leq t \leq t' \leq b$ that $L(\gamma|_{[t,t']}) = d(\gamma(t), \gamma(t'))$; we initially reserve the word *geodesic* for curves which are solutions of the geodesic differential equation on a Riemannian manifold. A space with every pair of points having a (unique) length-minimizing joining curve is a *(unique) geodesic space*. In a metric space a *geodesic triangle* is prescribed by a triple of points and a triple of joining length-minimizing curves. A geodesic triangle can be compared to a triangle in a constant-curvature space with the corresponding sides having equal lengths [6, Chap. II.1]. A characterization of curvature for metric spaces is provided in terms of distance-comparisons for comparison triangles [6, Chap. II.1].

Consider R a Riemann surface with complete hyperbolic metric having finite area. The homeomorphism type of R is given by its genus and number of punctures. Relative to a reference topological surface F, the surface R is *marked* by an orientation-preserving homeomorphism $f : F \to R$. Marked surfaces (R, f) and (R', f') are *equivalent* provided for $h : R \to R', h$ a conformal homeomorphism, $h \circ f$ is homotopic rel boundary to f'. The set of equivalence classes of the F-marked Riemann surfaces is the Teichmüller space $\mathcal{T}$, [20]. A neighborhood of the marked surface (R, f) is given by first specifying smooth Beltrami differentials $\nu_1, \ldots, \nu_m$ spanning the Dolbeault group $H^{0,1}_{\bar{\partial}}(\bar{R}, \mathcal{E}((\kappa p_1 \cdots p_n)^{-1}))$ for κ the canonical bundle of the compactification $\bar{R}$ and $p_1, \ldots, p_n$ the point line bundles for the punctures, [25]. For $s \in \mathbb{C}^m$ set $\nu(s) = \sum_j s_j \nu_j$; for s small there is a Riemann surface $R^{\nu(s)}$ and a diffeomorphism $\zeta^{\nu(s)} : R \to R^{\nu(s)}$ satisfying $\bar{\partial}\zeta^{\nu(s)} = \nu(s)\partial\zeta^{\nu(s)}$. The parameterization of marked surfaces $s \to (R^{\nu(s)}, \zeta^{\nu(s)} \circ f)$ is a holomorphic local coordinate for the Teichmüller space $\mathcal{T}$.

The *mapping class group* $Mod = Homeo^+(F)/Homeo_0(F)$ is the quotient of the group of orientation-preserving homeomorphisms of F fixing the punctures by the subgroup of homeomorphisms isotopic to the identity. The *extended mapping class group* is the quotient $Mod^* = Homeo(F)/Homeo_0(F)$. A mapping class $[h]$ acts on equivalence classes of marked surfaces by taking $\{(R, f)\}$ to $\{(R, f \circ h^{-1})\}$. The action of Mod on $\mathcal{T}$ is by biholomorphic maps; the quotient $\mathcal{M}$ is the *moduli space* of Riemann surfaces. The holomorphic cotangent space of $\mathcal{T}$ at the marked surface (R, f) is $Q(R) \backsimeq \check{H}^0(\bar{R}, \mathcal{O}(\kappa^2 p_1 \cdots p_n))$, the space of integrable holomorphic quadratic differentials. A co-metric for the cotangent spaces of Teichmüller space is prescribed by the Petersson Hermitian pairing $\int_R \varphi\bar{\psi}(dh^2)^{-1}$ for $\varphi, \psi \in Q(R)$ and dh^2 the R-hyperbolic metric, [4]. The dual metric is the Weil-Petersson

(WP) metric. The (extended) mapping classes act on $\mathcal{T}$ as WP isometries; the WP metric projects to $\mathcal{M}$. The WP metric is Kähler with negative sectional curvature and holomorphic sectional curvature bounded away from zero, [35, 36, 41]. Masur estimated the metric near the compactification divisor $\mathcal{D}$ of the moduli space, [30]. His preliminary expansion can be used for after-the-fact insights: the metric is not complete, [39]; there is an almost-product structure at infinity, [47]; and there are submanifolds of $\mathcal{T}$ that approximate Euclidean space (see the present Section 6). The expansion provides that the WP diameter and volume of $\mathcal{M}$ are finite. In [44] an improved analysis was presented for the extension of the WP Kähler form considered *in the sense of currents*. The $\frac{1}{2\pi^2}$ multiple of the WP Kähler form is the pushdown of the square of the curvature of the hyperbolic metric considered on the vertical line bundle for the fibration of the universal curve $\overline{\mathcal{C}}$ over $\overline{\mathcal{M}}$. The multiple of the Kähler form is a nonsmooth characteristic class representative of the Mumford class κ_1, [44].

The *complex of curves* $C(F)$ is defined as follows. The vertices of $C(F)$ are (free) homotopy classes of homotopically nontrivial, nonperipheral, simple closed curves on F. An edge of the complex consists of a pair of homotopy classes of disjoint simple closed curves. A k-simplex consists of $k+1$ homotopy classes of mutually disjoint simple closed curves. A maximal set of mutually disjoint simple closed curves, a *pants decomposition*, has $3g-3+n$ elements. Brock has described the large-scale WP geometry of Teichmüller space in terms of the *pants graph* $C_{\boldsymbol{P}}(F)$, a complex whose vertices are the distinct pants decompositions, [8]. The mapping class group Mod acts on curve complexes and in particular on $C(F)$.

A free homotopy class α of a closed curve on F determines a geodesic-length function ℓ_α on $\mathcal{T}$. For a marked surface (R, f), ℓ_α is the length of the R-hyperbolic metric geodesic homotopic to $f(\alpha)$. Geodesic-length functions provide parameters for the Teichmüller space. Suitable collections provide local coordinates, [20]. A collection of free homotopy classes $\{\alpha_1, \dots, \alpha_q\}$ is *filling* provided for a set of representatives with minimal number of self and mutual intersections that $F - \cup_j \alpha_j$ is a union of topological discs and punctured discs. A filling geodesic-length sum $\mathcal{L} = \sum_j \ell_{\alpha_j}$ is a proper function on the Teichmüller space. The differential and the WP gradient of an ℓ_α are given by the classical Petersson theta-series for the geodesic. In [42] we established that the WP Hessian of ℓ_α is positive-definite: geodesic-length functions are strictly convex along WP geodesics. The convexity provides a effective way to bound the WP geometry.

The Fenchel-Nielsen coordinates include geodesic-length functions, as well as lengths of auxiliary segments, [3, 20, 29, 40]. A pants decomposition $\mathcal{P} = \{\alpha_1, \ldots, \alpha_{3g-3+n}\}$ decomposes the topological surface F into $2g - 2 + n$ components (pants), each homeomorphic to a sphere with a combination of three discs or points removed. A marked Riemann surface (R, f) is likewise decomposed into pants by the geodesics representing $\mathcal{P}$. Each component pants, relative to its hyperbolic metric, has a combination of three geodesic boundaries and cusps. For each component pants the shortest geodesic segments connecting boundaries determine *designated points* on each boundary. For each geodesic in the pants decomposition of R a parameter τ is defined as the displacement along the geodesic between designated points, one for each side of the geodesic. For Riemann surfaces close to an initial reference Riemann surface, the displacement τ is simply the distance between the designated points; in general the displacement is the analytic continuation (the lifting) of the distance measurement. For α in $\mathcal{P}$ define the *Fenchel-Nielsen angle* by $\theta_\alpha = 2\pi\tau_\alpha/\ell_\alpha$. The Fenchel-Nielsen coordinates for Teichmüller space for the decomposition $\mathcal{P}$ are $(\ell_{\alpha_1}, \theta_{\alpha_1}, \ldots, \ell_{\alpha_{3g-3+n}}, \theta_{\alpha_{3g-3+n}})$. The coordinates provide a real analytic equivalence of $\mathcal{T}$ to $(\mathbb{R}_+ \times \mathbb{R})^{3g-3+n}$,[3, 20, 40].

A bordification of Teichmüller space is introduced by extending the range of the Fenchel-Nielsen parameters. The interpretation of *length vanishing* is the key ingredient. For ℓ_α equal to zero, the angle θ_α is not defined and in place of the geodesic for α there appears a pair of cusps; f is now a homeomorphism of $F - \alpha$ to the (marked) hyperbolic surface R (curves parallel to α map to loops encircling the cusps; see the discussion of nodes in the following Section). The parameter space for the pair $(\ell_\alpha, \theta_\alpha)$ is the identification space $\mathbb{R}_{\geq 0} \times \mathbb{R}/\{(0, y) \sim (0, y')\}$. For the pants decomposition $\mathcal{P}$ a frontier set $\mathcal{F}_{\mathcal{P}}$ is added to the Teichmüller space by extending the Fenchel-Nielsen parameter ranges: for each $\alpha \in \mathcal{P}$, extend the range of ℓ_α to include the value 0, with θ_α not defined for $\ell_\alpha = 0$. The points of $\mathcal{F}_{\mathcal{P}}$ parameterize (*degenerate*) Riemann surfaces with each $\ell_\alpha = 0, \alpha \in \mathcal{P}$, specifying a pair of cusps. In particular for a simplex $\sigma \subset \mathcal{P}$, the σ-null stratum is $\mathcal{S}(\sigma) = \{R \mid \ell_\alpha(R) = 0 \text{ iff } \alpha \in \sigma\}$. The frontier set $\mathcal{F}_{\mathcal{P}}$ is the union of the σ-null strata for the subsimplices of $\mathcal{P}$. Neighborhood bases for points of $\mathcal{F}_{\mathcal{P}} \subset \mathcal{T} \cup \mathcal{F}_{\mathcal{P}}$ are specified by the condition that for each simplex $\sigma \subset \mathcal{P}$ the projection $((\ell_\beta, \theta_\beta), \ell_\alpha) : \mathcal{T} \cup \mathcal{S}(\sigma) \to \prod_{\beta \notin \sigma}(\mathbb{R}_+ \times \mathbb{R}) \times \prod_{\alpha \in \sigma}(\mathbb{R}_{\geq 0})$ is continuous. For a simplex σ contained in pants decompositions $\mathcal{P}$ and $\mathcal{P}'$ the specified neighborhood systems for $\mathcal{T} \cup \mathcal{S}(\sigma)$ are equivalent. The *augmented Teichmüller space* $\overline{\mathcal{T}} = \mathcal{T} \cup_{\sigma \in C(F)} \mathcal{S}(\sigma)$ is the resulting stratified topological space, [2, 5]. $\overline{\mathcal{T}}$ is not locally compact since no point of the frontier has a relatively compact neighborhood; the neighborhood bases are unrestricted in the θ_α parameters for α a σ-null. The action of Mod on $\mathcal{T}$ extends to an action by homeomorphisms on $\overline{\mathcal{T}}$ (the action on $\overline{\mathcal{T}}$ is not properly discontinuous) and the quotient $\overline{\mathcal{T}}/Mod$ is (topologically) the compactified moduli space of stable curves (see the consideration of $\overline{\mathcal{M}}$ in the next Section), [2, see Math. Rev. 56 #679]. Masur noted that the WP metric extends to $\overline{\mathcal{T}}$ and is complete on $\overline{\mathcal{M}}$, [30, Thrm. 2, Cor. 2]. $\overline{\mathcal{T}}$ is WP complete since the quotient $\overline{\mathcal{M}}$ is compact and each point of $\overline{\mathcal{T}}$ has a neighborhood with complete closure.

3 Expansion of the WP metric about the compactification divisor

Our purpose is to provide a description of *local coordinates* for the local deformation space of a Riemann surface with nodes. We will present a modification of the standard coordinates [5, 30] and use the formulation to present an improved form of Masur's expansion of the WP metric. The expansion reveals that for the moduli space of stable curves $\overline{\mathcal{M}}$, along the compactification divisor $\mathcal{D}$, the WP metric behaves to third-order in distance as a product formed with the WP metric of $\mathcal{D}$.

The description begins with the *plumbing variety* $\mathcal{V} = \{(z, w, t) \mid zw = t,\ |z|, |w|, |t| < 1\}$. The defining function $zw - t$ has differential $z\,dw + w\,dz - dt$. Consequences are that $\mathcal{V}$ is a smooth variety, (z, w) are global coordinates, while (z, t) and (w, t) are not. Consider the projection $\Pi : \mathcal{V} \to D$ onto the t-unit disc. Π is a submersion, except at $(z, w) = (0, 0)$; we can consider $\Pi : \mathcal{V} \to D$ as a (degenerate) family of open Riemann surfaces. The t-fibre, $t \neq 0$, is the hyperbola germ $zw = t$ or equivalently the annulus $\{|t| < |z| < 1,\ w = t/z\} = \{|t| < |w| < 1,\ z = t/w\}$. The 0-fibre is the intersection of the unit ball with the union of the coordinate axes in $\mathbb{C}^2$; on removing the origin the union becomes $\{0 < |z| < 1\} \cup \{0 < |w| < 1\}$. Each fibre of $\mathcal{V}_0 = \mathcal{V} - \{0\} \to D$ has a complete hyperbolic metric:

$$\begin{aligned}
&\text{for } t \neq 0, \text{ on } \{|t| < |z| < 1\} \text{ then}\\
&\qquad dh_t^2 = \Big(\frac{\pi}{\log|t|}\csc\frac{\pi\log|z|}{\log|t|}\Big|\frac{dz}{z}\Big|\Big)^2;\\
&\text{for } t = 0, \text{ on } \{0 < |z| < 1\} \cup \{0 < |w| < 1\} \text{ then}\\
&\qquad dh_0^2 = \Big(\frac{|d\zeta|}{|\zeta|\log|\zeta|}\Big)^2 \text{ for } \zeta = z,\ w.
\end{aligned} \tag{1}$$

The family of hyperbolic metrics (dh_t^2) is a continuous metric, degenerate only at the origin, for the vertical line bundle of $\mathcal{V}$. In particular we have the elementary expansion

$$\begin{aligned} dh_t^2 &= \Big(\frac{|d\zeta|}{|\zeta|\log|\zeta|}\Big)^2\Big(\Theta\csc\Theta\Big)^2 \quad \text{for} \quad \Theta = \frac{\pi\log|z|}{\log|t|} \\ &= dh_0^2\,\big(1+\frac{1}{3}\Theta^2+\frac{1}{15}\Theta^4+\dots\big). \end{aligned} \tag{2}$$

The parameter t is a boundary point of the annulus $\{|t| < |z| < 1\}$. The boundary points t, 1 will be included in the data for gluings. To describe the variation of annuli with boundary points, we now specify a quasiconformal map ζ from the pointed t-annulus to the pointed t'-annulus $\zeta(z) = zr^{\beta(r,t')}$, $z = re^{i\theta}$, with $\frac{\partial\beta}{\partial r}$ compactly supported in the annulus. The boundary conditions are $\zeta(1) = 1$, and by specification $t|t|^{\beta(|t|,t')} = t'$. On differentiating in t' and evaluating at $(|t|, t)$ we find the boundary condition $t\log|t|\,\dot\beta(|t|,t) = 1$. More generally the infinitesimal variation of the map is the vector field $\dot\zeta(z) = z\log r\,\dot\beta(r,t)$ for $\dot\zeta$, $\dot\beta$ the first t-derivatives. The map ζ varies from the identity and has Beltrami differential

$$\bar\partial\dot\zeta = \frac{z}{2\bar z}\,\frac{\partial}{\partial\log r}(\dot\beta(r,0)\log r)\,\frac{\overline{dz}}{dz}. \tag{3}$$

For sake of later application we evaluate the pairing with a quadratic differential $z^\alpha\big(\frac{dz}{z}\big)^2$,

$$\begin{aligned} \int_{\{|t|<|z|<1\}} \bar\partial\dot\zeta\, z^\alpha\big(\frac{1}{z}\big)^2 dE &= \int_{\{|t|<|z|<1\}} \frac{z^\alpha}{2z\bar z}\,\frac{\partial}{\partial\log r}(\dot\beta\log r)\,dE \\ &\text{where for } \alpha = 0, \text{ then} \\ &= \pi\dot\beta\log r\Big|_{|t|}^{1} = \frac{-\pi}{t}, \\ &\text{and otherwise, then} \\ &= 0, \end{aligned} \tag{4}$$

for dE the Euclidean area element and where we have applied the boundary condition for $\dot\beta$; the evaluation involves fixing a normalization for the Serre duality pairing and agrees with [30, Prop. 7.1].

We review the description of *Riemann surfaces with nodes*, [5, 30, 44]. A Riemann surface with nodes R is a connected complex space, such that every point has a neighborhood isomorphic to either the unit disc in $\mathbb{C}$, or the germ at the origin in $\mathbb{C}^2$ of the union of the coordinate axes. R is *stable* provided each component of $R-\{nodes\}$ has negative Euler characteristic, i.e. has a hyperbolic metric. A regular q-differential on R is the assignment of a meromorphic q-differential Θ_j for each component R_j of $R-\{nodes\}$ such that: i) each Θ_* has poles only at the punctures of R_* with orders at most q, and ii) if punctures p, p' are paired to form a node then $Res_p\Theta_* = (-1)^q\, Res_{p'}\Theta_*$, [5].

We review the deformation theory of Riemann surfaces with punctures and then with nodes. For a Riemann surface R with hyperbolic metric and punctures there is a natural *cusp coordinate* (with unique germ modulo rotation) at each puncture: at the puncture p, the coordinate z with $z(p)=0$ and the hyperbolic metric of R given as $\left(\frac{|dz|}{|z|\log|z|}\right)^2$, the germ of the hyperbolic metric for the unit-disc. If the surface is uniformized by the upper half-plane with p represented by a width-one cusp at infinity then $z=e^{2\pi i\zeta}$ for ζ the uniformization variable. Now a deformation neighborhood of the marked surface R is given by specifying smooth Beltrami differentials $\nu_1,\dots,\nu_m$ spanning the Dolbeault group $H^{0,1}_{\bar\partial}(\bar R, \mathcal{E}((\kappa p_1\cdots p_n)^{-1}))$ for κ the canonical bundle of $\bar R$ and $p_1,\dots,p_n$ the point line bundles for the punctures, [25]. For $s\in\mathbb{C}^m$ set $\nu(s)=\sum_k s_k\nu_k$; for s small there is a Riemann surface $R^{\nu(s)}$ and a diffeomorphism $\zeta: R\to R^{\nu(s)}$ satisfying $\bar\partial\zeta=\nu(s)\partial\zeta$. The family of surfaces $\{R^{\nu(s)}\}$ represents a neighborhood of the marked Riemann surface in its Teichmüller space. We showed in [45, Lemma 1.1] that the Beltrami differentials can be modified a small amount *so that* in terms of each *cusp coordinate* the diffeomorphisms $\zeta^{\hat\nu(s)}$ are simply rotations; $\zeta^{\hat\nu(s)}$is a hyperbolic isometry in a neighborhood of the cusps; $\zeta^{\hat\nu(s)}$ cannot be complex analytic in s, but is real analytic. We further note that for s small the s-derivatives of $\nu(s)$ and $\hat\nu(s)$ are close. We say that $\zeta^{\hat\nu(s)}$ *preserves cusp coordinates.* The parameterization provides a key ingredient for obtaining simplified estimates of the degeneration of hyperbolic metrics and an improved expansion for the WP metric.

We review the plumbing construction for R a Riemann surface with a pair of punctures p, p'. The data is (U,V,F,G,t) where: U and V are disjoint disc coordinate neighborhoods of p and p'; $F:U\to\mathbb{C}$, $F(p)=0$ and $G:V\to\mathbb{C}$, $G(p')=0$, are coordinate mappings and t is a sufficiently small complex number. Pick a constant $0<c<1$ such that $F(U)$ and $G(V)$ contain the disc $\{|\zeta|<c\}$. For $c'<c$ let $R^*_{c'}$ be the open surface obtained by removing from R the discs $\{|F|\le c'\}\subset U$ and $\{|G|\le c'\}\subset V$. Now we prescribe the plumbing family $\{R_t\}$ over the t-disc. Let $D_c=\{|t|<c^4\}$, $M=R^*_{c^2}\times D_c$ and $\mathcal{V}_c=\{(z,w,t)\mid zw=t,\,|z|,|w|<c \text{ and } |t|<c^4\}$. M

and $\mathcal{V}_c$ are complex manifolds with holomorphic projections to D_c. Consider the holomorphic maps from M to $\mathcal{V}_c$: $\hat{F} : (q,t) \to (F(q), t/F(q), t)$ and $\hat{G} : (q', t) \to (t/G(q'), G(q'), t)$. The maps are consistent with the projections to D_c. The identification space $M \cup \mathcal{V}_c/\{\hat{F}, \hat{G}$ equivalence$\}$ is a degenerating family $\{R_t\}$ with a projection to the disc D_c. By construction the 0-fibre has a node with local model $\mathcal{V}_c$.

We are ready to describe a *local manifold cover* of the compactified moduli space $\overline{\mathcal{M}}$. For R having nodes, $R_0 = R - \{nodes\}$ is a union of Riemann surfaces with punctures. The quasiconformal deformation space of R_0, $Def(R_0)$, is the product of the Teichmüller spaces of the components of R_0. As already noted from [45, Lemma 1.1] for $m = \dim Def(R_0)$ there is a real analytic family of Beltrami differentials $\hat{\nu}(s)$, s in a neighborhood of the origin in $\mathbb{C}^m$, such that $s \to R_s = R^{\hat{\nu}(s)}$ is a coordinate parameterization of a neighborhood of R_0 in $Def(R)$ and the prescribed mappings $\zeta^{\hat{\nu}(s)} : R_0 \to R^{\hat{\nu}(s)}$ preserve the cusp coordinates at each puncture. Further for R with n nodes we now prescribe the plumbing data $(U_k, V_k, z_k, w_k, t_k)$, $k = 1, \dots, n$, for $R^{\hat{\nu}(s)}$, where z_k on U_k and w_k on V_k are cusp coordinates relative to the $R^{\hat{\nu}(s)}$-hyperbolic metric (the plumbing data varies with s). The parameter t_k parameterizes *opening* the k th node. For all t_k suitably small, perform the n prescribed plumbings to obtain the family $R_{s,t} = R^{\hat{\nu}(s)}_{t_1,\dots,t_n}$. The tuple $(s,t) = (s_1, \dots, s_m, t_1, \dots, t_n)$ provides real analytic local coordinates, the *hyperbolic metric plumbing coordinates*, for the local manifold cover of $\overline{\mathcal{M}}$ at R, [30, 43] and [44, Secs. 2.3, 2.4]. The coordinates have a special property: for s fixed the parameterization is holomorphic in t. The property is a basic feature of the plumbing construction. The family $R_{s,t}$ parameterizes the small deformations of the marked noded surface R.

The roles of the Fenchel-Nielsen coordinates and the hyperbolic metric plumbing coordinates can be interchanged. In particular for the nodes of R given by the σ-null stratum $\{\alpha_1, \dots, \alpha_n\}$ the above local manifold cover has topological coordinates $((\ell_\beta, \theta_\beta)_{\beta \notin \sigma}, (\ell_\alpha e^{i\theta_\alpha})_{\alpha \in \sigma})$. The observation can be established by expressing the Fenchel-Nielsen coordinates solely in terms of geodesic-lengths, and then applying techniques for theta-series to analyze the differentials of geodesic-lengths. Upon interchanging the roles of the coordinates, we obtain a local description of the bordification in terms of the (s,t) tuple, [2, 5, 44]. At the point R of the σ-null stratum in $\overline{\mathcal{T}}$ the local parameters are $(s, |t|, \arg t)$ with the *arg* valued in $\mathbb{R}$.

We review the geometry of the local manifold covers. For a complex manifold M the complexification $\boldsymbol{T}^{\mathbb{C}}M$ of the $\mathbb{R}$-tangent bundle is decomposed into the subspaces of *holomorphic* and *antiholomorphic* tangent vectors. A Hermitian metric g is prescribed on the holomorphic subspace. For a general complex parameterization $s = u + iv$ the coordinate $\mathbb{R}$-tangents are expressed as $\frac{\partial}{\partial u} = \frac{\partial}{\partial s} + \frac{\partial}{\partial \bar{s}}$ and $\frac{\partial}{\partial v} = i\frac{\partial}{\partial s} - i\frac{\partial}{\partial \bar{s}}$. For the $R_{s,t}$ parameterization the s-parameters are not holomorphic while for s-parameters fixed the t-parameters are holomorphic; $\{\frac{\partial}{\partial s_j} + \frac{\partial}{\partial \bar{s}_j}, i\frac{\partial}{\partial s_j} - i\frac{\partial}{\partial \bar{s}_j}, \frac{\partial}{\partial t_k}, i\frac{\partial}{\partial t_k}\}$ is a basis over $\mathbb{R}$ for the tangent space of the local manifold cover. For a smooth Riemann surface the dual of the space of holomorphic tangents is the space of quadratic differentials. The following is now a modification of Masur's result [30, Prop. 7.1].

Proposition 1 *The hyperbolic metric plumbing coordinates (s,t) are real analytic and for s fixed the parameterization is holomorphic in t. Provided the modification $\hat{\nu}$ is small, for a neighborhood of the origin there are families in (s,t) of regular 2-differentials φ_j, ψ_j, j=1,...,m and η_k, k=1,...,n such that:*

1. *For $R_{s,t}$ with $t_k \neq 0$, all k, $\{\varphi_j, \psi_j, \eta_k, i\eta_k\}$ forms the dual basis to $\{\frac{\partial\hat{\nu}(s)}{\partial s_j} + \frac{\partial\hat{\nu}(s)}{\partial \bar{s}_j}, i\frac{\partial\hat{\nu}(s)}{\partial s_j} - i\frac{\partial\hat{\nu}(s)}{\partial \bar{s}_j}, \frac{\partial}{\partial t_k}, i\frac{\partial}{\partial t_k}\}$ over $\mathbb{R}$.*
2. *For $R_{s,t}$ with $t_k = 0$, all k, the η_k, k=1,...,n, are trivial and the $\{\varphi_j, \psi_j\}$ span the dual of the holomorphic subspace $\boldsymbol{T}Def(R_0)$.*

Proof. The situation compares to that considered by Masur. The new element: the variation of the plumbing data is prescribed by a Schiffer variation for a gluing-function real analytically depending on the parameter s, [43, pg. 410]. As already noted for s fixed, plumbing produces a holomorphic family. Following Masur the families of regular 2-differentials $\{\varphi_j, \psi_j, \eta_k\}$ are obtained by starting with a local frame $\mathcal{F}$ of regular 2-differentials and prescribing the pairings with $\{\frac{\partial\hat{\nu}}{\partial s_j}, \frac{\partial\hat{\nu}}{\partial \bar{s}_j}, \frac{\partial}{\partial t_k}\}$, [30, Sec. 5 and Prop. 7.1]. At an initial point the basis is simply given by a linear transformation of the frame $\mathcal{F}$. The prescribed basis will then exist in a neighborhood provided the pairings are continuous. We first consider the pairings with $\frac{\partial}{\partial t_k}$. From (3) we have the Beltrami differential for the pairing with $\frac{\partial}{\partial t_k}$, $k = 1, \ldots, n$. In particular for a plumbing collar of $R_{s,t}$ let z (or w) be the coordinate of the plumbing. A quadratic differential φ on $R_{s,t}$ can be factored on the collar into a product of $\left(\frac{dz}{z}\right)^2$ and a function holomorphic in z. We write $\mathcal{C}_k(\varphi)$ for the constant coefficient of the Laurent expansion of the function factor. From (4) the pairing with $\frac{\partial}{\partial t_k}$ is the linear functional $-\frac{\pi}{t_k}\mathcal{C}_k$. From Masur's considerations [30, Sec. 5, esp. 5.4, 5.5] the pairing of $\frac{\partial}{\partial t_k}$ with the local frame $\mathcal{F}$ is continuous, and there are regular 2-differentials $\{\varphi_j, \psi_j, \eta_k^*\}$ with: $\mathcal{C}_\ell(\varphi_j) = \mathcal{C}_\ell(\psi_j) = 0$, $j = 1, \dots, m$; $\mathcal{C}_\ell(\eta_k^*) = \delta_{k\ell}$, $k, \ell = 1, \dots, n$. The 2-differentials $\eta_k = \frac{-t_k}{\pi}\eta_k^*$, $k = 1, \dots, n$ have the desired pairings with $\frac{\partial}{\partial t_k}$. The final matter is to note that the pairings of $\{\varphi_j, \psi_j, \eta_k^*\}$ with $\{\frac{\partial \hat{\nu}}{\partial s_\ell}, \frac{\partial \hat{\nu}}{\partial \bar{s}_\ell}\}$ are indeed continuous in (s, t). By construction the differential $\hat{\nu}(s)$ is supported in the complement of the plumbing collars, [45, Lemma 1.1]. On the support of $\hat{\nu}(s)$ the 2-differentials are real analytic in (s, t). The pairings are continuous and even real analytic. The proof is complete.

We now note two general matters: the role of the coefficient functional $\mathcal{C}$, and the approximation of the hyperbolic metric. As above, for z a plumbing collar coordinate for $R_{s,t}$, a quadratic differential ψ can be factored on the collar as the product of $\left(\frac{dz}{z}\right)^2$ and a holomorphic function. $\mathcal{C}(\psi)$ denotes the constant coefficient of the Laurent expansion of the function. The surface $R_{s,t}$ is constructed by plumbing $\left(R_s\right)^*_{c^2}$ with the R_s-hyperbolic cusp coordinates. $R_{s,t}$ is the disjoint union of $\left(R_s\right)^*_c$, R_s with the cuspidal discs $|z_*|, |w_*| < c$ removed, and the annulus $\{|t|/c < |z| < c\}$. An approximate hyperbolic metric $d\omega^2$ is given by choosing the R_s-hyperbolic metric on $\left(R_s\right)^*_c$ and dh_t^2 on the annulus (see (1)). The metric $d\omega^2$ is the *model grafting* treated in detail in [44, Sec. 3.4.MG]; as noted in [44, pgs. 445, 446] for $dh^2_{s,t}$ the $R_{s,t}$-hyperbolic metric we have that $\left|d\omega^2/dh^2_{s,t} - 1\right|$ is $O\left(\sum_k (\log|t_k|)^{-2}\right)$. The approximation $d\omega^2$ will now be substituted for the construction of [30, Sec. 6] to obtain an improved form of the original expansion. The improved approximation of the hyperbolic metric is the new contribution. Yamada [47] presented a third-order expansion based on the technical work of Wolf [37] and Wolf-Wolpert [38].

Theorem 2 *For a noded Riemann surface R the hyperbolic metric plumbing coordinates for $R_{s,t}$ provide real analytic coordinates for a local manifold cover neighborhood for $\overline{\mathcal{M}}$. The parameterization is holomorphic in t for s fixed. On the local manifold cover the WP metric is formally Hermitian satisfying:*

1. *For $t_k = 0$, $k = 1, \dots, n$, the restriction of the metric is a smooth Kähler metric, isometric to the WP product metric for a product of Teichmüller spaces.*

2. *For the tangents $\{\frac{\partial}{\partial s_j}, \frac{\partial}{\partial \bar{s}_j}, \frac{\partial}{\partial t_k}\}$ and the quantity $\rho = \sum_{k=1}^{n} (\log|t_k|)^{-2}$ then:* $g_{WP}(\frac{\partial}{\partial t_k}, \frac{\partial}{\partial t_k})(s, t) = \dfrac{\pi^3}{|t_k|^2(-\log^3|t_k|)}\,(1 + O(\rho))$;

$g_{WP}(\frac{\partial}{\partial t_k}, \frac{\partial}{\partial t_\ell})$ is $O((|t_k t_\ell| \log^3 |t_k| \log^3 |t_\ell|)^{-1})$ for $k \neq \ell$;
and for $\mathfrak{u} = \frac{\partial}{\partial s_j}, \frac{\partial}{\partial \bar{s}_j}$; $\mathfrak{v} = \frac{\partial}{\partial s_\ell}, \frac{\partial}{\partial \bar{s}_\ell}$:
$g_{WP}(\frac{\partial}{\partial t_k}, \mathfrak{u})$ is $O((|t_k|(-\log^3 |t_k|))^{-1})$ and
$g_{WP}(\mathfrak{u}, \mathfrak{v})(s,t) = g_{WP}(\mathfrak{u}, \mathfrak{v})(s,0)\,(1 + O(\rho))$.

Proof. We begin with the expansion of the dual metric for the basis provided in Proposition 1. The behavior of the φ_j, ψ_j, η_k and their contribution to the Petersson pairing $\int \alpha\overline{\beta}(d\omega^2)^{-1}$ is straightforward. On $\left(R_s\right)^*_{c^2}$ the quadratic differentials and the approximating metric are real analytic in (s,t). The contributions to the pairing are real analytic and each differential η_k, $k = 1, \ldots, n$, contributes a factor of t_k. On the plumbing collars $\{|t|/c < |z| < c\} = \{|t|/c < |w| < c\}$ each quadratic differentials is given as the product of $\left(\frac{dz}{z}\right)^2 = \left(\frac{dw}{w}\right)^2$ and a function factor. We begin with an elementary calculation

$$\int_{\{|t|/c<|z|<c\}} \left|z^\alpha \left(\frac{dz}{z}\right)^2\right|^2 \left(dh_t^2\right)^{-1} =$$
$$\frac{2}{\pi}\int_{|t|/c}^{c} \left(\log|t| \sin\frac{\pi \log r}{\log|t|}\right)^2 r^{2\alpha}\, d\log r$$
$$\text{where for } \alpha = 0,\ \mu = \frac{\log r}{\log|t|} \text{ and } \epsilon = \frac{\log c}{\log|t|},\ \text{then}$$
$$= \frac{2}{\pi}(-\log^3|t|)\int_{\epsilon}^{1-\epsilon} \sin^2 \pi\mu\, d\mu \;=\; \frac{1}{\pi}(-\log^3|t|) + O(1),$$
$$\text{and for } \alpha = 1,\ \text{since } |\sin\mu| \leq |\mu|,\ \text{then}$$
$$= O(1).$$

We are ready to consider the contribution to the Petersson pairing from the collars. Consider the contribution for the ℓ^{th} collar. By construction η_ℓ is the unique quadratic differential from the dual basis with a nonzero $\mathcal{C}_\ell$ evaluation. In particular $\mathcal{C}_\ell(\eta^*_\ell) = 1$ and the contribution to the self pairing for η^*_ℓ is $\frac{1}{\pi}(-\log^3|t_\ell|) + O(1)$. In general we note that a quadratic differential on a plumbing collar can be factored as $\left(\frac{dz}{z}\right)^2(f_z + \boldsymbol{c} + f_w)$ for f_z holomorphic in $|z| < c$, $f_z(0) = 0$; $\boldsymbol{c}$ the $\mathcal{C}$-evaluation value and f_w holomorphic in $|w| < c$, $f_w(0) = 0$. Furthermore f_z, resp. f_w, is given as the Cauchy integral of f over $|z| = c$, resp. $|w| = c$. Further from the Schwarz Lemma $|f_z| \leq c'|z| \max_{|z|=c}|f|$ with a corresponding bound for $|f_w|$. The bounds are combined with the majorant bound $|\sin\mu| \leq |\mu|$ to show that: for φ_j, ψ_j, η^*_k on $|z| = c+\epsilon_0$ and $|w| = c+\epsilon_0$ depending analytically on (s,t)

their contribution to the Petersson pairing over the collar is also analytic in (s,t).

Combining our considerations and noting the approximation of $d\omega^2$ to the hyperbolic metric for $R_{s,t}$ we find that

$$\begin{aligned}
\langle \eta_k^*,\, \eta_k^* \rangle_{WP} \;&=\; \frac{1}{\pi}(-\log^3|t_k|)\,(1\;+\;O(\sum_{\ell=1}^{n}(\log|t_\ell|)^{-2})),\\
\langle \eta_k^*,\, \eta_\ell^* \rangle_{WP} \;&=\; O(1) \text{ for } k\neq\ell,\\
\text{and for } \mathfrak{a}=\varphi_j,\psi_j;\ \mathfrak{b}=\varphi_\ell,\psi_\ell : &\\
\langle \mathfrak{a},\, \eta_k^* \rangle_{WP} \;&=\; O(1) \text{ and}\\
\langle \mathfrak{a},\mathfrak{b} \rangle_{WP}(s,t) \;&=\; \langle \mathfrak{a},\mathfrak{b} \rangle_{WP}(s,0)\,(1\;+\;O(\sum_{k=1}^{n}(\log|t_k|)^{-2})).
\end{aligned}$$

The desired expansion now follows from the following Proposition and the relations $\eta_k=-\frac{t_k}{\pi}\eta_k^*$. The proof is complete.

For $\mathcal{A}$ a symmetric $m+n\times m+n$ matrix

$$\begin{pmatrix}
\lambda_1 & \dots & a_{\ell j} & \dots\\
\vdots & \lambda_k & \vdots & \vdots\\
a_{j\ell} & \dots & \ddots & \dots\\
\dots & \dots & \dots & B
\end{pmatrix}
\qquad
\begin{array}{l}
\text{with } \lambda_k,\ 1\le k\le n;\\
a_{j\ell},\ 1\le j\le m+n,\ j\neq\ell,\ 1\le\ell\le n \text{ and}\\
B\;=\;(b_{j\ell}) \text{ a symmetric } m\times m \text{ matrix,}
\end{array}$$

we consider the situation that $\lambda_1,\dots,\lambda_n$ are large compared to the $a_{j\ell}$ and $b_{j\ell}$.

Proposition 3 *For* $\det B\neq 0$, *and* $\rho=\sum\limits_{k=1}^{n}\lambda_k^{-1}$ *then:*

$$\det\mathcal{A}=\det B\prod_{k=1}^{n}\lambda_k(1+O(\rho))$$

and $\mathcal{A}^{-1}=(\alpha_{j\ell})$ *where: for* $1\le k\le n,\ \alpha_{kk}=\lambda_k^{-1}(1+O(\rho))$*; for* $1\le j<\ell\le n,\ \alpha_{j\ell}$ *is* $O((\lambda_j\lambda_\ell)^{-1})$*; for* $1\le j\le n<\ell\le m+n,\ \alpha_{j\ell}$ *is* $O(\lambda_j^{-1})$*, and for* $1\le j,\,\ell\le m,\ \alpha_{j+n\,\ell+n}=b^{j\ell}(1+O(\rho))$*. The constants for the O-terms are bounded in terms of* $m+n$, $\det B^{-1}$ *and* $\max\{|a_{j\ell}|,|b_{j\ell}|\}$.

Proof. We consider the general formula for the determinant as a sum over the permutation group and by the cofactor expansion. First observe that there is a dichotomy for $m+n$-fold products in the calculation of $\det \mathcal{A}$; a product either also occurs in the expansion of $\det B \prod_k \lambda_k$, or has at most $n-1$ factors $\lambda_k, 1 \leq k \leq n$. Products with less than n factors are bounded in terms of the cited product and $O(\rho)$. The determinant expansion is a consequence. We continue and apply the analog of the dichotomy when examining the cofactors of $\mathcal{A}$. For the cofactor for λ_ℓ we find the expansion $\det B \prod_{k\neq\ell} \lambda_k(1+O(\rho))$. Similarly for the cofactor of $a_{j\ell}$ we find the λ-contribution to be $\prod_{k\neq j,\ell} \lambda_k(1+O(\rho))$ for $j \neq \ell \leq n$ and to be $\prod_{k\neq j} \lambda_k(1+O(\rho))$ for $j \leq n < \ell$. Finally for the cofactor of $b_{j\ell}$ we find the expansion $b^{j\ell} \det B \prod_k \lambda_k(1+O(\rho))$ in terms of the inverse $B^{-1} = (b^{j\ell})$. The proof is complete.

By way of application we present a normal form for the quadratic form dg^2_{WP}; the result is an immediate consequence of the above Theorem.

Corollary 4 *For the prescribed hyperbolic metric plumbing coordinates:*

$$dg^2_{WP}(s,t) \;=\; \big(dg^2_{WP}(s,0) \;+\; \pi^3 \sum_{k=1}^{n} (4dr_k^2 + r_k^6 d\theta_k^2)\big)\,(1 + O(\|r\|^3))$$

for $r_k = (-\log|t_k|)^{-1/2},\ \theta_k = \arg t_k$ *and* $r = (r_1, \ldots, r_n)$.

The result provides a local expansion of the WP metric about the compactification divisor $\mathcal{D} = \{t_k = 0\}$. To the third order of approximation the WP metric is formally a product. As we will note below, a second-order approximation is already special. As already noted, the bordification $\overline{\mathcal{T}}$ has a local description in terms of the parameters $(s, |t|, \arg t)$ or equivalently in terms of $(s, (-\log|t|)^{-1/2}, \arg t)$. The above result provides the associated WP expansion.

An almost-product Riemannian metric with remainder bounded by the displacement from a submanifold is very special. We note the situation as motivation for the results of Section 5; the following considerations do not apply since $4dr^2 + r^6 d\theta^2$ is not a Riemannian metric. Consider a product $\mathbb{R}^m \times \mathbb{R}^n$ with Euclidean coordinates x for $\mathbb{R}^m$ and y for $\mathbb{R}^n$. Consider that in a neighborhood of the origin a metric has the expansion

$$dg^2 \;=\; dg_x^2 \;+\; dg_y^2 \;+\; O_{C^1}(\|y\|^2)$$

with dg_x^2, resp. dg_y^2, a C^1-metric for $\mathbb{R}^m$, resp. for $\mathbb{R}^n$, and the remainder a C^1-symmetric tensor as indicated. The expansion provides that the *second* *fundamental form* of the x-axes, $\mathbb{R}^m \times \{0\}$, vanishes identically [33, pgs. 62, 100]. In this case the x-axes is a *totally geodesic* submanifold: a geodesic initially tangent to the x-axes is contained in the x-axes [33, pg. 104]. The expansion also provides that for the x-axes the *normal connection* and the *normal curvature* vanish identically [33, pgs. 114, 115].

4 Length-minimizing curves on Teichmüller space

We begin by developing basic facts about the behavior of WP geodesics on Teichmüller space. Although Teichmüller space is topologically a cell, the behavior of geodesics is not a consequence of general results [24], since the WP metric is not complete. For instance the Hopf-Rinow theorem cannot be directly applied to obtain length-minimizing curves [10, 24, 33], and it is necessary to show that distance is measured along geodesics. We proceed though by applying our paradigm: *a filling geodesic-length sum behaves qualitatively as the distance from a point for a complete metric.* In the following we combine the paradigm and modifications of the standard arguments to find the basic behavior of geodesics.

Theorem 5 *The WP exponential map from a base point is a diffeomorphism from its open domain onto the Teichmüller space.*

Corollary 6 *Teichmüller space is a unique geodesic space. Each WP geodesic segment is the unique length-minimizing rectifiable curve connecting its endpoints.*

Proof of Corollary. Let γ be the WP geodesic connecting a pair of points p and q in the Teichmüller space. For a filling geodesic-length function $\mathcal{L}$, choose $c > 0$, such that $\gamma \subset S_c = \{\mathcal{L} < c\}$, [42]. Consider G the set of all rectifiable curves connecting p and q, contained in $\overline{S_c}$, and each with length at most $d(p,q) + 1$. Provided G is nonempty and the elements of G are parameterized proportional to arc-length on the interval $[0,1]$, then G constitutes an equicontinuous family of maps. In particular for $\beta \in G$ and $t, t' \in [0,1]$ by the proportional parameterization it follows that

$$|t - t'| = \frac{L(\beta[t,t'])}{L(\beta)} \geq \frac{d(\beta(t), \beta(t'))}{d(p,q) + 1}.$$

From the Arzelà-Ascoli Lemma [6, pg. 36] there exists a rectifiable length-minimizing (amongst elements of G) curve β_0 connecting p and q contained in $\overline{S_c}$.

We consider the behavior of a rectifiable length-minimizing (amongst elements of G) curve β_0 passing through an arbitrary point $r \in \overline{S_c}$ (r could lie on $\partial\overline{S_c}$). Since $\overline{S_c}$ is compact, there is a positive ϵ such that WP geodesics are uniquely length-minimizing in an ϵ-neighborhood of each point of $\overline{S_c}$. Since $\overline{S_c}$ is WP convex it follows for r', r'' on the chosen curve, close to r, with r' *before* r and r'' *after* r, that the segments $\widehat{r'r}$ and $\widehat{rr''}$ are necessarily WP geodesics. It further follows that $\widehat{r'r''}$ is a WP geodesic, since the segment is locally length-minimizing at r. By convexity of the geodesic-length function $\mathcal{L}$, its value at r' or r'' is greater than its value at r. Since $r', r'' \in \overline{S_c}$ it follows that $r \in S_c$. It now follows that a rectifiable length-minimizing (amongst elements of G) curve β_0 is a WP geodesic entirely contained in S_c.

In general given ϵ, $0 < \epsilon < 1$, there exists a curve β' connecting p and q in the Teichmüller space such that $L(\beta') < d(p,q) + \epsilon$. For c' large, $\beta' \subset \overline{S_{c'}}$ and thus the corresponding family of maps G is non empty. The length of β' bounds the length of a $S_{c'}$-length-minimizing curve β_0 connecting p and q: in particular $L(\beta') \geq L(\beta_0)$. From the above paragraph and the Theorem, the unique geodesic connecting p and q is $\beta_0 = \gamma$. The inequalities now provide that $L(\gamma) < d(p,q) + \epsilon$. The proof is complete.

Proof of Theorem. First we note that the domain of the exponential map is an open set. Given a geodesic γ connecting a pair of points, select $c > 0$ such that $\gamma \subset S_c$. Since S_c is open the points in neighborhoods of the γ-endpoints are also connected by WP geodesics. In particular the domain of the exponential map is open.

We next note that the exponential map is a local diffeomorphism [24]. Further note that a germ of the inverse is determined by its value at a single point. We now consider the continuation of a given germ $\boldsymbol{\iota}$, with the exponential map based at p and the germ given at $q \in \mathcal{T}$. We consider the continuation of $\boldsymbol{\iota}$ along α, a curve with initial point q. We argue that the continuation set is closed.

Choose a filling geodesic-length function $\mathcal{L}$ and value c such that $p, \alpha \subset \overline{S_c}$. First we observe that each WP geodesic connecting p and a point of α is contained in $\overline{S_c}$. This follows since the values of $\mathcal{L}$ at the endpoints are bounded by c and $\mathcal{L}$ is WP convex. Since $\overline{S_c}$ is compact there is an overall length bound for the WP geodesics contained in $\overline{S_c}$. As noted in Bridson-Haefliger a length-bounded family of geodesics is given by an equicontinuous family of maps, [6, pg. 36]. By the Arzelà-Ascoli Lemma it follows that a sequence of WP geodesics contained in $\overline{S_c}$ has a subsequence converging to a geodesic contained in $\overline{S_c}$.

Consider now that the germ $\boldsymbol{\iota}$ can be continued to a sequence of points $\{q_n\}$ along α. In particular WP geodesics $\widehat{pq_n}$ are determined. A subsequence (same notation) $\widehat{pq_n}$ converges to $\widehat{pq'}$. The WP geodesic $\widehat{pq'}$ determines a germ of the inverse of the exponential map; the germ gives *exponential inverses* for the WP geodesics $\widehat{pq_n}$. The germ is the continuation of $\boldsymbol{\iota}$; the continuation set is closed. The continuation set is necessarily open; $\boldsymbol{\iota}$ can be continued along every curve. On considering homotopies it is established that the continuation to the endpoint of α is path independent. Finally since the Teichmüller space is simply connected the continuations determine a global inverse for the exponential map. The proof is complete.

We are also interested in understanding the WP *join* of two sets, and in particular the distance between points on a pair of geodesics. For the WP inner product consider the Levi-Civita connection ∇ satisfying for vector fields X, Y and W the relations,[10, 24, 33],

$$X\big\langle Y,W\big\rangle = \big\langle \nabla_X Y, W\big\rangle + \big\langle Y, \nabla_X W\big\rangle$$
$$\nabla_X Y - \nabla_Y X = [X,Y].$$

Further consider the curvature tensor

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

A *variation of geodesics* is a smooth map $\beta(t,s)$ from $[t_0,t_1]\times(-\epsilon,\epsilon)$ to $\mathcal{T}$ such that for each s', $\beta(t,s')$ is a WP geodesic. For the vector fields $T = d\beta(\frac{\partial}{\partial t})$ and $V = d\beta(\frac{\partial}{\partial s})$, the first variation of the geodesic V satisfies the Jacobi equation

$$\nabla_T\nabla_T V = R(T,V)T;$$

solutions are Jacobi fields, [10]. The Jacobi equation is a linear second-order system of ordinary differential equations for vector fields along $\beta(t,s')$. The space of solutions has dimension $2\dim\mathcal{T}$; a solution is uniquely prescribed by its initial value and its initial derivative. Furthermore since $\mathcal{T}$ has negative curvature there are no conjugate points along a geodesic and the linear map $(V|_{t=t_0}, \nabla_T V|_{t=t_0})$ to $(V|_{t=t_0}, V|_{t=t_1})$ is an isomorphism [10, pg. 19]. Solutions are uniquely prescribed by their values at the endpoints. This property is needed for the understanding of the join of sets. In particular Jacobi fields provide a mechanism for analyzing the exponential map.

The WP exponential map $(p,v) \xrightarrow{\boldsymbol{e}} exp_p v$ has domain $\boldsymbol{\mathcal{D}}$ the open set $\{(p, exp_p^{-1}(\mathcal{T}))\} \subset \mathbf{T}\mathcal{T}$ of the tangent bundle. We are ready to consider the behavior of geodesics.

Proposition 7 *The WP exponential map* $(p,v) \xrightarrow{\boldsymbol{e}} (p, exp_p v)$ *is a diffeomorphism from* $\boldsymbol{\mathcal{D}} \subset \mathbf{T}\mathcal{T}$ *to* $\mathcal{T} \times \mathcal{T}$*. For a pair of disjoint WP geodesics parameterized proportional to arc-length, the WP distance between corresponding points is a strictly convex function.*

Proof. The map $\boldsymbol{e}$ is smooth with differential $d\boldsymbol{e} = (id, d\,exp)$. As already noted since the WP metric has negative curvature $d\,exp$ has maximal rank and thus $\boldsymbol{e}$ is a local diffeomorphism of $\mathcal{D}$ to $\mathcal{T} \times \mathcal{T}$. A consequence of Corollary 6 is that $\boldsymbol{e}$ is a global diffeomorphism.

We are ready to consider the distance between corresponding points of a pair of disjoint WP geodesics. From the above result a one-parameter variation of geodesics is determined $\beta(t,s)$, $(t,s) \in [t_0,t_1] \times [s_0,s_1]$. For a value $s' \in [s_0,s_1]$ we write T for the tangent field of $\beta(t,s')$ and V for its variation field; we assume $\|T\| = 1$. The second variation in s at s' of the length of $\beta(t,s)$ is given by the classical formula [10, (1.14)]

$$\big\langle \nabla_V V, T \big\rangle\big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \big\langle \nabla_T V, \nabla_T V \big\rangle - \big\langle \nabla_T V, T \big\rangle \big\langle \nabla_T V, T \big\rangle - \big\langle R(V,T)T, V \big\rangle dt. \quad (5)$$

Observations are in order. First by hypothesis the curves $\beta(t_0,s)$ and $\beta(t_1,s)$ are geodesics with constant speed parameterization; the acceleration $\nabla_V V$ vanishes at t_0 and t_1. Second the first two terms of the integrand combine to give the length-squared of the projection $\nabla_T V$ onto the normal space of T. And the third term of the integrand is strictly positive given strictly negative curvature [10]. In summary the distance is a strictly convex function. The proof is complete.

We are ready to show that $\overline{\mathcal{T}}$ is a geodesic space. For points p and q of the completion let $\{p_n\}$ and $\{q_n\}$ be sequences from $\mathcal{T}$ converging to p, resp. to q. Note for the distance we have $d(p,q) = \lim_n d(p_n,q_n)$. Consider the sequence of curves $\gamma_n = \widehat{p_n q_n}$ of $\mathcal{T}$ parameterized proportional to arc-length by the unit-interval. Since the sequences $\{p_n\}$ and $\{q_n\}$ are Cauchy it follows from Proposition 7 that for each $t \in [0,1]$ the sequence $\{\gamma_n(t)\}$ is also Cauchy (*without* passing to a subsequence). The sequence $\{\gamma_n\}$ prescribes a function γ with domain the unit-interval and values in $\overline{\mathcal{T}}$. Furthermore since the γ_n are distance proportional-parameterized, for $t, t' \in [0,1]$ then

$$\frac{d(\gamma_n(t), \gamma_n(t'))}{d(p_n, q_n)} = |t - t'|.$$

It follows that $d(\gamma(t), \gamma(t')) = |t - t'|d(p, q)$; γ is a continuous function, in particular a geodesic. We summarize the considerations with the following.

Proposition 8 *The completion $\overline{\mathcal{T}}$ is a geodesic space.*

5 Length-minimizing curves on the completion

For the complex of curves $C(F)$ a k-simplex is a set of $k+1$ free homotopy classes of nontrivial, nonperipheral, mutually disjoint simple closed curves for the reference surface F. A simplex σ *precedes* a simplex σ' provided $\sigma \subseteq \sigma'$; preceding is a partial ordering. With the convention that the -1-*simplex* is the null set, there is a natural function Λ from the completion $\overline{\mathcal{T}}$ to the complex $C(F) \cup \{\emptyset\}$ determined by the classes of the nodes. For a marked noded Riemann surface (R, f) with $f : F \to R$, the labeling $\Lambda((R, f))$ is the simplex of free homotopy classes on F mapped to the nodes on R. The level sets of Λ are the strata of $\overline{\mathcal{T}}$. We write $\mathcal{S}(\sigma)$ for the stratum determined by the simplex σ. The stratum for a k-simplex has complex dimension $3g(F) - 3 - k$.

We now consider first properties of length-minimizing curves on $\overline{\mathcal{T}}$. We are able to make the analysis without first establishing that a length-minimizing curve is a limit of WP geodesics. In this section we build on the following result and present an alternative approach to the basic observation of S. Yamada [47] that except possibly for its endpoints, a length-minimizing curve is contained in a single stratum of $\overline{\mathcal{T}}$.

Proposition 9 *For a length-minimizing curve γ on $\overline{\mathcal{T}}$ the composition $\Lambda \circ \gamma$ has a left and right limit at each point. The composition is continuous at a point where the left and right limits agree.*

Proof. First observe that only a finite number of simplices precede a given simplex. There is a continuous analog for strata: in a suitable neighborhood of a point of $\overline{\mathcal{T}}$ there are only a finite number of strata, and each precedes or coincides with the stratum of the point. If a left or right limit fails to exist for $\Lambda \circ \gamma$ at t_0, then there is a monotonic convergent sequence of parameter values $\{t_n\}$, $t_n \to t_0$ with $\Lambda \circ \gamma$ having value σ on $\{t_{2n}\}$ and a different value τ on $\{t_{2n+1}\}$. We may choose that σ precedes τ and further that σ, resp. τ, is a maximal, resp. minimal, such value. Maximal connected segments of γ contained in the stratum of σ are determined by the positivity of the geodesic-length functions of the classes in $\tau - \sigma$. In particular each maximal segment is parameterized by an open parameter interval; by Corollary 6 each (closed subsegment of each) maximal segment is length-minimizing.

Consider two points $\gamma(t_{2n})$ and $\gamma(t_{2n+2})$ on different maximal segments. By Corollary 6 there exists a WP geodesic β contained in the stratum of σ connecting $\gamma(t_{2n})$ and $\gamma(t_{2n+2})$. We now compare the segments $\gamma|_{[t_{2n},\,t_{2n+2}]}$ and β. Assume each segment is parameterized by arc-length; the segments necessarily have the same length. On the stratum σ the curves β and the maximal segment of γ at t_{2n} are solutions of an ordinary differential equation. If the initial tangents of β and $\gamma|_{[t_{2n},\,t_{2n+2}]}$ coincide, then by the uniqueness of solutions and the maximality, the segments must coincide for the length of β. The coinciding contradicts $\Lambda\circ\gamma$ having different values at t_{2n} and t_{2n+1}. The alternative is that the initial (unit) tangents of β and $\gamma|_{[t_{2n},\,t_{2n+2}]}$ differ. In this case γ can be modified by first substituting the segment β for the parameter interval $[t_{2n}, t_{2n+2}]$ and then *smoothing the corner* (inside the stratum) at $\gamma(t_{2n})$, to obtain a new curve $\widetilde{\gamma}$ of strictly smaller length, again a contradiction. A sequence $\{t_n\}$ as described cannot exist. In summary the composition $\Lambda\circ\gamma$ is locally constant to the left and right of each point of its domain.

Finally if the left and right limits have a common value at t_0 then either $\Lambda\circ\gamma(t_0)$ also has the common value, or the common value precedes $\Lambda\circ\gamma(t_0)$. In the second instance we can again construct a modification $\widetilde{\gamma}$ of strictly smaller length. The proof is complete.

We are interested in a class of singular *metrics* that model the WP metric in a neighborhood of a point on the compactification divisor $\mathcal{D}\subset\overline{\mathcal{M}}$. Consider now the product $(\mathbb{R}^2)^{m+n}$ with Euclidean coordinates (x,y) for x the $2m$-tuple with Euclidean metric dx^2 and y the $2n$-tuple with Euclidean metric dy^2. We refer to $\mathbb{R}^{2m}\times\{0\}$, resp. to $\{0\}\times\mathbb{R}^{2n}$, as the x-axes, resp. the y-axes. Here the x-axes represent coordinates on a stratum of dimension $2m$ and codimension $2n$, while the y-axes represent the parameters which open nodes. We write (r_j,θ_j) for the polar coordinates for the 2-plane (y_{2j-1},y_{2j}) and (r,θ) for the product polar coordinates for the $2n$-tuple of y-coordinates. We consider the singular metric

$$\sum_{j=1}^{n} 4\,dr_j^2 + r_j^6\,d\theta_j^2$$

for the y-axes which we simply abbreviate as $dr^2 + r^6 d\theta^2$.

Definition 10 *A continuous symmetric* 2*-tensor* ds^2 *is a product cuspidal metric for a neighborhood of the origin in* $(\mathbb{R}^2)^{m+n}$ *provided:*

1. ds^2 is a smooth Riemannian metric on $\bigcap_{j=1}^{n}\{r_j > 0\}$;

2. the restriction of ds^2 to the x-axes is a smooth Riemannian metric ds_x^2;

3. $ds^2 = (d\mu^2 + dr^2 + r^6 d\theta^2)(1 + O(\|r\|^2))$ for $d\mu^2$ the pullback of ds_x^2 to $(\mathbb{R}^2)^{m+n}$ by the projection onto the x-axes ,and $\|r\|$ denoting the Euclidean norm of the radius vector for the y-axes.

We are ready to continue our consideration of a length-minimizing curve γ and a point of discontinuity t_* interior to the domain of the label composition $\Lambda \circ \gamma$. The first circumstance to consider is that $\Lambda \circ \gamma$ is continuous from one side, say the right. In particular for $\Lambda \circ \gamma$ discontinuous from the left the simplex $\sigma = \Lambda \circ \gamma(t_*^-)$ strictly precedes the simplex $\sigma' = \Lambda \circ \gamma(t_*)$. Since γ is length-minimizing, the curve is a WP geodesic in the stratum $\mathcal{S}(\sigma')$ for an initial interval to the right of t_*. Observe by analogy that in the circumstance that ds^2, $d\mu^2$ and $dr^2 + r^2 d\theta^2$ were smooth, then the x-axes would be totally geodesic (the *second fundamental form* would be trivial; see the discussion after Corollary 4) and the suggested refracting behavior of γ would not be possible. We will now show by a scaling argument that the behavior is also not possible for a product cuspidal metric.

We observe that the individual strata of $\overline{\mathcal{T}}$ branch cover strata in $\overline{\mathcal{M}}$ and that certain curves in $\overline{\mathcal{M}}$ have unique lifts determined by their initial point. $\overline{\mathcal{M}}$ is a V-manifold; consider first the local-manifold cover $\widehat{\mathcal{U}}$ for a neighborhood of the point given by $\gamma(t_*)$ (a neighborhood of $\gamma(t_*)$ in $\overline{\mathcal{M}}$ is described as $\widehat{\mathcal{U}}/Aut(\gamma(t_*)))$. From Section 2 the preimage of $\widehat{\mathcal{U}}$ in $\overline{\mathcal{T}}$ is a disjoint union of sets, including a neighborhood $\mathcal{U}$ of $\gamma(t_*)$. The local stratum $\sigma \cap \mathcal{U} \subset \overline{\mathcal{T}}$ is a covering of its image $\widehat{\sigma} \subset \widehat{\mathcal{U}}$, with covering group the lattice of Dehn twists for the set of loops $\sigma' - \sigma$. The local stratum $\sigma' \cap \mathcal{U} \subset \overline{\mathcal{T}}$ coincides with its local projection $\widehat{\sigma'}$ to $\widehat{\mathcal{U}}$. In the following paragraphs we will use the simple observation that: a rectifiable curve in $\widehat{\mathcal{U}}$ with first segment in $\widehat{\sigma}$ and second segment in $\widehat{\sigma'}$ has WP isometric lifts to $\mathcal{U}$, each uniquely determined by prescribing an initial point. We can choose coordinates so $\widehat{\mathcal{U}} \cap \widehat{\sigma}$ is given by a neighborhood of the origin in $(\mathbb{R}^2)^{m+n}$ and $\widehat{\mathcal{U}} \cap \widehat{\sigma'}$ is given by a neighborhood of the origin in the x-axes $\mathbb{R}^{2m} \times \{0\}$.

We study the WP length of an oriented curve γ having first segment off the x-axes and second segment a geodesic in the x-axes. Let $\mathbf{o}$ be the first contact point of the curve with the x-axes. Consider the Euclidean ball of radius δ about $\mathbf{o}$. From Corollary 4 we can choose δ small for the metric to have the coordinate description of a product cuspidal metric in the ball. Along γ let a be the first intersection point of γ with the δ-sphere at $\mathbf{o}$ to the left of $\mathbf{o}$. Along γ let b be the first intersection point of γ with the δ-sphere at $\mathbf{o}$ to the right of $\mathbf{o}$. We investigate the lengths of curves from a to $\mathbf{o}$ to b as δ varies. We use the coordinates (x, y) for the following constructions. Let a_x, resp. a_y, be the Euclidean projection of a to the x, resp. the y, axes. Let β be the unit-speed $d\mu^2$ geodesic in the x-axes from a_x to b. For the same arc-length parameter let $\widehat{\beta}$ be the curve from a to b whose Euclidean projection to the y-axes is a constant speed radial line. On $\widehat{\beta}$ the tensor $dr^2 + r^6 d\theta^2$ restricts to dr^2 and in particular

$$\int_{\widehat{\beta}} (ds^2)^{1/2} = \int_{\widehat{\beta}} (1 + O(\|y\|^2))(d\mu^2 + dr^2)^{1/2}.$$

Since the length of $\widehat{\beta}$ is bounded in terms of δ and the Euclidean height of $\widehat{\beta}$ is bounded by $\|a_y\|$ it follows that the length of $\widehat{\beta}$ is given as

$$\int_{\widehat{\beta}} (d\mu^2 + dr^2)^{1/2} + O(\delta\|a_y\|^2)$$

for the Euclidean norm of a_y. The integral immediately evaluates to $(|||\beta|||^2 + \|a_y\|^2)^{1/2}$ for $|||\ |||$ denoting the $d\mu^2$ length. In summary the length of $\widehat{\beta}$ is bounded above by $(|||\beta|||^2 + \|a_y\|^2)^{1/2} + O(\delta\|a_y\|^2)$.

We next consider a lower bound for the length of the segment of γ from a to b. The first expansion is provided similar to the above consideration. Select a subsequence of values δ' tending to zero such that for each δ' the maximum of the y-height $\|y\|$ on the γ segment from a to $\mathbf{o}$ actually occurs at the initial point a. For the subsequence the length of γ is

$$\int_{\gamma} (d\mu^2 + dr^2 + r^6 d\theta^2)^{1/2} + O(\delta'\|a_y\|^2).$$

The metric $d\xi^2 = (d\mu^2 + dr^2 + r^6 d\theta^2)$ is a product and the local behavior of its geodesics is understood. Let $\widehat{\gamma}$ be a comparison curve (a $d\xi^2$ piecewise geodesic) with segments from a to $\mathbf{o}$ and from $\mathbf{o}$ to b. The projection of the first segment of $\widehat{\gamma}$ to the x-axes is the $d\mu^2$ geodesic from a_x to $\mathbf{o}$. The projection of the first segment of $\widehat{\gamma}$ to the y-axes is the constant speed radial line from a_y to $\mathbf{o}$. The second segment of $\widehat{\gamma}$ is the geodesic from $\mathbf{o}$ to b. The integral of γ is minorized by the integral of $\widehat{\gamma}$ and consequently the length of γ is minorized by

$$(|||a_x|||^2 + \|a_y\|^2)^{1/2} + |||b|||^2 + O(\delta'\|a_y\|^2)$$

where we have written $|||x_0|||$ for the $d\mu^2$-distance from $\mathbf{o}$ to the point x_0 of the x-axes.

We are prepared to analyze the length of γ in a small neighborhood of the point $\mathbf{o}$.

Proposition 11 *A curve having first segment off the x-axes and second segment a geodesic in the x-axes is not ds^2 length-minimizing between its endpoints. There is a shorter curve of the same description.*

Proof. We first consider the rescaling limit of a neighborhood of $\mathbf{o}$ with the substitution $\delta u = x, \delta v = r$ and $d\eta^2 = \delta^{-2}ds^2$. The curves $\widehat{\beta}$ and $\widehat{\gamma}$ considered above have radial lines as their projections to the y-axes; it suffices for length considerations to consider the projection of the y-axes to its radial component r. The rescaling limit of $(d\mu^2 + dr^2)(1 + O(\|y\|^2))$ is the Euclidean metric and for a subsequence the points a, b limit to points of the unit sphere (same notation). The curve $\widehat{\beta}$ limits to the chordal line connecting a to b; the curve $\widehat{\gamma}$ limits to the segmented curve of line-segments connecting a to $\mathbf{o}$ and $\mathbf{o}$ to b. If a is not antipodal to b then (on the subsequence) $\widehat{\beta}$ is strictly shorter than $\widehat{\gamma}$. On a neighborhood of $\mathbf{o}$, γ is now modified by substituting a segment of $\widehat{\beta}$ to obtain a strictly shorter curve, a desired conclusion.

It remains to consider that the rescaling limit as δ tends to zero of a is the antipode to b. In this circumstance we have that $|||a_x|||$ is comparable to δ, $|||\beta|||$ is comparable to 2δ and $\|a_y\|$ by hypothesis is $o(\delta)$. Pick $\epsilon < 1$ such that $\epsilon^2|||\beta||| > |||a_x|||$ for all small δ. Now from the preliminary considerations for small δ the length of $\widehat{\beta}$ is bounded above by

$$(|||\beta|||^2 + \|a_y\|^2)^{1/2} + O(\delta\|a_y\|^2) \leq |||\beta||| + \frac{1}{2\epsilon|||\beta|||}\|a_y\|^2 + O(\delta\|a_y\|^2)$$

and for a suitable subsequence the length of γ is bounded below by

$$(|||a_x|||^2 + \|a_y\|^2)^{1/2} + |||b||| + O(\delta\|a_y\|^2) \geq \\ |||a_x||| + |||b||| + \frac{\epsilon}{2|||a_x|||}\|a_y\|^2 + O(\delta'\|a_y\|^2)$$

for $\|a_y\|(|||a_x|||)^{-1}$ sufficiently small, which is ensured for δ sufficiently small. As specified above, δ' are the special values for which the maximum of the y-height $\|y\|$ on the γ sement from a to $\mathbf{o}$ occurs at the initial point a. Since β is a geodesic we have that $|||\beta||| \leq |||a_x||| + |||b|||$. Observe that the coefficient of the $\|a_y\|^2$-term for $\widehat{\beta}$ is strictly less than that for γ. Since

$\|a_y\|^2$ is positive for δ positive, it now follows that $\widehat{\beta}$ is strictly shorter than γ and in particular that γ is not length-minimizing in a neighborhood of **o**. The proof is complete.

The second circumstance to consider is that for a length-minimizing curve γ there is an interior domain discontinuity point t_* with the label composition $\Lambda \circ \gamma$ not continuous from either side. The curve γ connects points in different strata by passing through a higher codimension stratum. In particular from Proposition 11 it follows that the values of $\Lambda \circ \gamma$ for t_*^-, t_* and t_*^+ are all distinct; furthermore the values for t_*^- and t_*^+ strictly precede the value at t_*. We have the coordinate description of the local strata $\widehat{\mathcal{U}} \cap (\mathcal{S} \circ \Lambda \circ \gamma(t_*^-) \cup \mathcal{S} \circ \Lambda \circ \gamma(t_*) \cup \mathcal{S} \circ \Lambda \circ \gamma(t_*^+))$ given in a neighborhood of the origin in $(\mathbb{R}^2)^{m+n}$. For suitable $n_- + n_+ = n$ the neighborhood is given as a neighborhood of the origin in $(\mathbb{R}^2)^{m+n_-+n_+}$ with coordinates (x, y_-, y_+). In a neighborhood of the origin the three strata are given by germs of the coordinate axes: $\mathcal{S} \circ \Lambda \circ \gamma(t_*)$ by the x-axes; $\mathcal{S} \circ \Lambda \circ \gamma(t_*^-)$ by the y_--axes; and $\mathcal{S} \circ \Lambda \circ \gamma(t_*^+)$ by the y_+-axes. Again a rectifiable curve with the prescribed behavior for $\Lambda \circ \gamma$ has WP isometric lifts to $\mathcal{U}$, each uniquely determined now by prescribing an initial and terminal point.

Proposition 12 *A curve having endpoints distinct from the origin and in distinct coordinate proper subspaces of the y-axes and further having the origin as an intermediate point is not ds^2 length-minimizing between its endpoints. There is a shorter curve avoiding the origin.*

Proof. The considerations are simplified since in effect the subspaces corresponding to $\mathbb{R}^{2n_-}$ and $\mathbb{R}^{2n_+}$ are orthogonal. Choose $\epsilon > 0$ such that $(1+\epsilon) < (1-\epsilon)\sqrt{2}$; from Definition 10 the restriction of ds^2 to the y-axes is estimated above, resp. below, by the $(1+\epsilon)$, resp. $(1-\epsilon)$, multiple of $(dr^2 + r^6 d\theta^2)$. For a^- in $\mathbb{R}^{2n_-} - \{0\}$ and a^+ in $\mathbb{R}^{2n_+} - \{0\}$ the $(dr^2 + r^6 d\theta^2)$ geodesic in $\mathbb{R}^{2n}$ connecting a^- to a^+ and the piecewise geodesic connecting a^- to the origin and then to a^+ are Euclidean line segments. The line connecting a^- to a^+ has ds^2 length at most $(1+\epsilon)(\|a^-\|^2 + \|a^+\|^2)^{1/2}$. The line segments connecting a^- to the origin to a^+ have length at least $(1-\epsilon)(\|a^-\| + \|a^+\|)$.

For an oriented curve γ with the prescribed strata behavior consider a Euclidean radius δ sphere at the origin and let a^-, resp. a^+, be the first intersection point along γ to the left, resp. right, of the origin. Since the radial component of the metric is comparable to the Euclidean metric, the maximum value of $\|r\|$ along the segments of γ is comparable to δ. Apply the above estimate for δ small to obtain the desired conclusion. The proof is complete.

We are ready to present our counterpart of S. Yamada's Theorem 2, [47].

Theorem 13 *$\overline{\mathcal{T}}$ is a unique geodesic space. The length-minimizing curve connecting points p, $q \in \overline{\mathcal{T}}$ is contained in the closure of the stratum with label $\Lambda(p) \cap \Lambda(q)$. The open segment $\gamma - \{p, q\}$ is a solution of the WP geodesic equation on the stratum with label $\Lambda(p) \cap \Lambda(q)$. For a point p the stratum with label $\Lambda(p)$ is the union of the length-minimizing open segments containing p. The closure of each stratum is a convex set, complete in the induced metric.*

Proof. $\overline{\mathcal{T}}$ is a geodesic space from Proposition 8. For a length-minimizing curve γ we consider the label behavior of $\Lambda \circ \gamma$. From Proposition 9 $\Lambda \circ \gamma$ only has a finite number of discontinuities. From Propositions 11 and 12, as well as the lifting property of the indicated curves on the local-manifold covers of $\overline{\mathcal{M}}$, it follows that $\Lambda \circ \gamma$ is at most discontinuous at an endpoint. Since each stratum is a relatively open subset of its closure in $\overline{\mathcal{T}}$, it further follows that the value of Λ on the open segment of γ precedes its value at each endpoint (the open segment value is a lower bound for the partial ordering). It also follows that $\Lambda(p) \subset \Lambda(q)$ is a necessary condition for p to be an interior point of a length-minimizing curve with endpoint q.

A free homotopy class α of a simple closed curve is represented by a vertex in $\Lambda(p) \cap \Lambda(q)$. For the geodesic-length function ℓ_α, the composition $\ell_\alpha \circ \gamma$ is a continuous function. The composition vanishes at its domain endpoints and is convex on its domain interior. It follows that the composition is identically zero and consequently that the open segment of γ is contained in the stratum with label $\Lambda(p) \cap \Lambda(q)$. A stratum is a product of Teichmüller spaces. The maximal open segment of γ is a solution of the product WP geodesic equation on the stratum. Consider WP geodesics $\gamma\, , \gamma'$ parameterized proportional to arc-length by the unit-interval with common endpoints. The distance between corresponding points is a continuous function, vanishing at 0 and 1, and convex on $(0, 1)$ from Proposition 7. The distance is identically zero and the geodesics coincide. $\overline{\mathcal{T}}$ is a unique geodesic space.

As note above, $\Lambda(p) \subset \Lambda(q)$ is a necessary condition for p to be an interior point of a length-minimizing curve with endpoint q. Since a stratum is a product of Teichmüller spaces for which length-minimizing curves are solutions of the geodesic equation and since solutions can be extended, it follows that the condition is sufficient for extension. The final conclusion follows since $\overline{\mathcal{T}}$ is a geodesic space. The closure of a stratum is convex from the above description of geodesics. The closure of a stratum is complete in the induced metric from the completeness of $\overline{\mathcal{T}}$. The proof is complete.

We are ready to present the basic result. $CAT(0)$ is a generalized condition for a non-positively curved, uniquely geodesic space, [6, Chap. II.1]. With the above result there is little further need to distinguish between length-minimizing curves and solutions of the geodesic differential equation. We now also refer to length-minimizing curves parameterized proportional to arc-length as *geodesics*.

Theorem 14 *$\overline{\mathcal{T}}$ is a CAT(0) space.*

Proof. A length-minimizing curve on $\overline{\mathcal{T}}$ is approximated by WP geodesics on $\mathcal{T}$ by choosing sequences of points converging to the endpoints, and considering the joins parameterized on the original interval. From Proposition 7 the joins converge to the designated geodesics, and for a pair of geodesics the relative distance functions converge. In particular a limit of geodesic triangles satisfying the $CAT(0)$ inequality will also satisfy the inequality, [6, Chap. II.1]. Since the WP metric on $\mathcal{T}$ has negative curvature, geodesic triangles satisfy the $CAT(0)$ inequality [6, Chap. II.1, Remark 1A.8]. The proof is complete.

The local geometry of geodesics on $\overline{\mathcal{M}}$ differs from that of $\overline{\mathcal{T}}$. A product cuspidal metric is not uniquely geodesic.

Proposition 15 *$\overline{\mathcal{M}}$ is not locally uniquely geodesic at the compactification divisor and in particular is not locally a CAT(0) space.*

Proof. We show that the local manifold cover for a neighborhood of a Riemann surface having a single node is not uniquely geodesic [6, Chap. II.1, Prop. 1.4]. Introduce hyperbolic metric plumbing coordinates. For a reference base point $(s',t')=(s'_1,\dots,s'_m,t')$ off the s-axes, we consider the curves based at (s',t'), disjoint from the s-axes, and linking the s-axes. The base point (s',t') lifts to a point of the Teichmüller space $\mathcal{T}$. From Corollary 6 for each possible value of the linking number there is a corresponding length-minimizing curve in $\mathcal{T}$ (minimizing for curves disjoint from the s-axes). We first bound the lengths of such linking curves. A general comparison curve is prescribed by the sequence: a radial line segment in the t-coordinate, followed by an integer number of rotations about a t-coordinate circle and finally a radial line segment returning to the base point. From Corollary 4 the comparison curves can be prescribed with length uniformly bounded by a multiple of $|t'|$. It follows for $|s'|,|t'|$ small that the length-minimizing linking curves are all contained in a small neighborhood of the origin.

We consider length bounds involving the linking number. For a curve with linking number n and the minimal absolute value of the coordinate t

on the curve $|t_0|$, then by Corollary 4 the length of the curve is at least a uniform multiple of $|n\, t_0^3|$. It follows that $|t_0|$ is bounded by $|t'/n|^{1/3}$; it further follows by considering only the t-radial component of the length that the linking curve has length at least a uniform multiple of $|t'| - |t'/n|^{1/3}$; the desired bound. A comparison curve with linking number one is the t-coordinate circle of radius $|t'|$; its length is bounded by a multiple of $|t'|^3$. We draw a simple conclusion: there is a length-minimizing linking curve γ of minimal length (presumably with linking number ± 1).

We bisect γ. Let p denote (s', t') and q denote the γ-midpoint. The length of γ is $O(|t'|^3)$; from Corollary 4 the length of a curve connecting p to the s-axes is at least a multiple of $|t'|$. With the length bounds and the fact that γ is a solution of the geodesic differential equation it now follows that the segments of γ connecting p to q are length-minimizing for the neighborhood of the origin. The neighborhood is not uniquely geodesic. The proof is complete.

6 Applications

We are interested in understanding the flat subspaces of $\overline{\mathcal{T}}$. Our purpose is to understand the flat geodesic simplices and in particular the flat geodesic triangles. Consider a geodesic triangle with distinct vertices o, p and q. Parameterize proportional to arc-length the sides $\widehat{op}$ and $\widehat{oq}$ by geodesics $\gamma(t)$ and $\gamma'(t)$, $t \in [0,1]$ with $\gamma(0) = \gamma'(0) = o$. The distance function $d(\gamma(t), \gamma'(t))$ is an important measure of the triangle. We also require a numerical invariant for noded Riemann surfaces. Let $\nu(R)$ be the number of components of $R - \{nodes\}$ that are not thrice-punctured spheres. The maximal dimension of a flat subspace of the stratum corresponding to R is given by $\nu(R)$. We use the description of flat subspaces to give a different proof of a Brock-Farb result: the WP metric is in general not Gromov-hyperbolic, [9]. Recall that a metric space (M, d) is *Gromov-hyperbolic* provided there exists a positive number δ such that for each geodesic triangle the δ-neighborhood of a pair of sides contains the third side, [16].

Proposition 16 *On the augmented Teichmüller space $\overline{\mathcal{T}}$ of genus g, n punctured surfaces*

1. *For geodesics γ, γ' as above and an interior parameter value, consider that the values of the distance function $d(\gamma(t), \gamma'(t))$ and its supporting linear function coincide. The interiors of the geodesics γ, γ' then lie*

on a submanifold of $\overline{\mathcal{T}} - \mathcal{T}$ given as the Cartesian product of geodesics from component Teichmüller spaces.

2. *For a stratum corresponding to a noded Riemann surface R, the maximal dimension of a locally Euclidean isometric submanifold is $\nu(R)$. The maximal value of ν is $g - 1 + \lfloor \frac{g+n}{2} \rfloor$, which is achieved for an arrangement with g once-punctured tori and $\lfloor \frac{g+n}{2} \rfloor - 1$ four-punctured spheres.*

3. *For $3g - 3 + n \geq 3$ the Teichmüller space with the WP metric is not Gromov-hyperbolic.*

Proof. First, a convex function is necessarily linear if it shares a common interior value with the supporting linear function. From Bridson-Haefliger [6, Chap. I.1, Defn. 1.10 and Chap. II.3, Prop. 3.1] provided $d(\gamma(t), \gamma'(t))$ is linear then the comparison angles formed by the point triples $(\gamma(t), o, \gamma'(t))$ all coincide. The *flat triangle lemma* of A. D. Alexandrov can now be applied [6, Chap. II.2, Prop. 2.9]. The convex hull of o, p and q in $\overline{\mathcal{T}}$ is consequently isometric to the convex hull of a Euclidean triangle with the corresponding side lengths. An isometry is prescribed. There is an associated variation of geodesics $\beta(t,s)$, parameterized proportional to arc-length, such that $\beta(t,0) = \gamma(t)$, $\beta(t,1) = \gamma'(t)$, $\beta(0,s) = o$ and $\beta(1,s)$ lies on $\widehat{pq}$ with $d(p, \beta(1,s)) = sd(p,q)$. By Theorem 13 it follows for interior parameter values that $\beta(t,s)$ lies in a single stratum; by Proposition 7 it follows for interior parameter values that $\beta(t,s)$ is smooth. The stratum is a product of Teichmüller spaces. We may apply the techniques of Riemannian geometry, [10]. Since the triangle is flat the contribution to (5) from the term $\langle R(V,T)T, V \rangle$ is zero. For a product of negatively-curved metrics $\langle R(V,T)T, V \rangle$ vanishes only if the variation fields V and T everywhere have collinear projections to the tangent spaces of the factors [33, Chap. 3, Lemmas 39, 58]. Since the projections are collinear the triangle also projects to a geodesic segment in each component Teichmüller space. The desired first conclusion.

Second, as already indicated for a Riemannian product of negatively-curved metrics the maximal dimension of a flat subspace is the number of factors. The dimension of a maximal flat is given by $\nu(R)$ since a punctured Riemann surface has a positive dimensional Teichmüller space provided the surface is not the thrice-punctured sphere. Once-punctured tori have Euler characteristic -1 and $\dim \mathcal{T} > 0$; the general surface with $\dim \mathcal{T} > 0$ has Euler characteristic strictly less than -1. The maximal statement follows. The desired second conclusion.

Third, for the Euclidean plane, a positive number δ, and a non degenerate triangle, a large-scaling provides a triangle δ with a δ-neighborhoold of a pair of sides omitting an open segment on the third side. A stratum corresponding to a noded surface R with $\nu(R) \geq 2$ contains triangles isometric to Δ. For such a triangle a triple of points in $\mathcal{T}$, one close to each vertex, prescribes a triangle with measurements close to those of Δ. Independent of the length of an edge, the geodesic triangle in $\mathcal{T}$ is uniformly close to the triangle Δ with distance estimated only by the distance separating corresponding vertices. For a suitable triple, a δ-neighborhood of two joining sides omits an open segment on the third joining side. A stratum with $\nu(R) \geq 2$ exists provided $\dim \mathcal{T} \geq 3$. The proof is complete.

The maximal simplices in $C(F)$ serve an important role for the geometry of $\overline{\mathcal{T}}$. Since thrice-punctured spheres are conformally rigid, a $3g-4$-simplex σ in $C(F)$ corresponds by Λ^{-1} to a unique marked maximally noded Riemann surface R_σ in $\overline{\mathcal{T}}$. The Mod stabilizer of a maximally noded Riemann surface R_σ is an extension of a finite group by a rank $3g-3$ Abelian group, the mapping classes of products of Dehn twists about the elements of σ. The maximally noded Riemann surfaces serve the role of the *maximal rank cusps* for the moduli space. Brock studied the finite length WP geodesics from a point of $\mathcal{T}$ to the marked noded Riemann surfaces, [7]. The geodesics from a point can be extended to include their endpoints in $\overline{\mathcal{T}}$. As a consequence of the $CAT(0)$ geometry the initial unit tangents for the family of geodesics from a point to a stratum provide for a Lipschitz map from the stratum to the unit tangent sphere. Accordingly the image of $\overline{\mathcal{T}}-\mathcal{T}$ in each unit tangent sphere has measure zero and thus the infinite length geodesic rays have tangents dense in each tangent sphere. Brock's method for approximating infinite length rays by finite length rays now provides the following.

Theorem 17 ([7]) *The geodesic rays from a point of $\mathcal{T}$ to the maximally noded Riemann surfaces have initial tangents dense in the tangent space.*

The following is a new consequence of the result.

Corollary 18 *The geodesics connecting maximally noded Riemann surfaces have tangents dense in the tangent bundle of $\mathcal{T}$.*

Proof. We consider unit-speed geodesics. Given a unit-tangent v at a point p of $\mathcal{T}$ and a positive number ϵ, we proceed to determine an approximating geodesic. By the above Theorem let γ_- be a unit-speed geodesic connecting a point q, representing a maximally noded Riemann surface to p, with the final tangent w of γ_- within ϵ of v. For a small positive number δ similarly let γ_+ be a unit-speed geodesic connecting p to a point r, representing a maximally noded Riemann surface, with the initial tangent of γ_+ within δ of w. The geodesics γ_- and γ_+ form a vertex at p with angle in the interval $[\pi - \delta, \pi]$. The three points p, q and r determine a geodesic triangle $\Delta(p, q, r)$ in $\overline{\mathcal{T}}$ for which there is a comparison triangle $\Delta(\bar{p}, \bar{q}, \bar{r})$ in the Euclidean plane. By [6, Chap. II.4, Lem. 4.11] the vertex angles for $\Delta(p, q, r)$ are bounded by the corresponding vertex angles for the Euclidean triangle $\Delta(\bar{p}, \bar{q}, \bar{r})$. In particular the vertex angle at $\bar{p}$ is also in the interval $[\pi - \delta, \pi]$. It follows for $\Delta(\bar{p}, \bar{q}, \bar{r})$ that the angle at $\bar{q}$ is at most δ. For $\delta < \pi/2$ it follows that $d(\bar{q}, \bar{r}) > d(\bar{q}, \bar{p})$ and there is a point $\bar{s}$ with $d(\bar{q}, \bar{s}) = d(\bar{q}, \bar{p})$, $\bar{s}$ on the geodesic segment $\widehat{\bar{q}\bar{r}}$. By trigonometry $d(\bar{s}, \bar{p}) = 2d(\bar{p}, \bar{q}) \sin \delta/2$ and thus the comparison point s on the geodesic segment $\widehat{qr}$ satisfies $d(s, p) \leq 2d(p, q) \sin \delta/2$. Similarly the midpoints s_{mid} of $\widehat{qs}$ and p_{mid} of $\widehat{qp}$ satisfy $2d(s_{mid}, p_{mid}) \leq d(s, p)$. To summarize the considerations, the geodesic $\widehat{qr}$ contains a point s that is within distance $2d(p, q) \sin \delta/2$ of p, and the midpoints s_{mid} of $\widehat{qs}$ and p_{mid} of $\widehat{qp}$ are within distance $d(p, q) \sin \delta/2$.

From Proposition 7 the geodesic segments $\widehat{s_{mid}s}$ and $\widehat{p_{mid}p}$ are sufficiently close in the C^1-topology for δ sufficiently small. We now choose r' and γ'_+ to provide that the tangent at s' is within ϵ of the final tangent of γ_-, which in turn was chosen to be within ϵ of v. The proof is complete.

The following is an immediate consequence of the above result.

Corollary 19 *$\overline{\mathcal{T}}$ is the closed convex hull of the subset of marked maximally noded Riemann surfaces.*

We combine the above and follow the outline of the Masur-Wolf approach to give an immediate proof of the Masur-Wolf theorem, [31, Theorem A]. The classification of simplicial automorphisms of the curve complex $C(F)$ by N. Ivanov [22], M. Korkmaz [27], and F. Luo [28] is an essential consideration.

Theorem 20 *For $3g - 3n > 1$ and $(g, n) \neq (1, 2)$, every WP isometry of $\mathcal{T}$ is induced by an element of the extended mapping class group.*

Proof. An isometry of $\mathcal{T}$ extends to an isometry of the completion $\overline{\mathcal{T}}$. By Theorem 13 an isometry of $\overline{\mathcal{T}}$ necessarily preserves the strata structure and the incidence relations. It follows that an isometry induces a simplicial automorphism of $C(F)$. From the results of Ivanov [22], Korkmaz [27] and Luo [28], every simplicial automorphism coincides with the induced automorphism of an extended mapping class. In particular for a WP isometry of

$\overline{\mathcal{T}}$, there is an extended mapping class such that the two mappings coincide on the subset of maximally noded Riemann surfaces. The conclusion now follows from Corollary 19. The proof is complete.

A complete, convex subset $\mathcal{C}$ of a $CAT(0)$ space is the base for an *orthogonal projection*, [6, Chap. II.2, Prop. 2.4]. A fibre of the projection is the unique geodesic realizing the distance between its points and the base. The projection is a retraction that does not increase distance. The distance $d_{\mathcal{C}}$ to $\mathcal{C}$ is a convex function satisfying $|d_{\mathcal{C}}(p) - d_{\mathcal{C}}(q)| \leq d(p,q)$, [6, Chap. II.2, Prop. 2.5]. Examples of complete, convex sets $\mathcal{C}$ are: points, complete geodesics, and fixed-point sets of isometry groups. Furthermore in the case of $\overline{\mathcal{T}}$ with Theorem 13 the closure of each individual stratum is the base of a projection (local projections are prescribed on $\overline{\mathcal{M}}$). On $\overline{\mathcal{T}}$ a tubular neighborhood of an stratum is fibered by the projection-geodesics. For the local understanding of the distance to the stratum we now consider a refinement of the prescription for a product cuspidal metric. In particular by Corollary 4 the WP metric has an expansion with an order-three approximation about the x-axes

$$ds^2 = (d\mu^2 + dr^2 + r^6 d\theta^2)(1 + O(\|r\|^3)).$$

Corollary 21 *For a stratum σ defined by vanishing of the geodesic-length sum $\ell = \ell_1 + \cdots + \ell_n$, the distance to the stratum is given locally as $d(p,\sigma) = (2\pi\,\ell)^{1/2} + O(\ell^2)$.*

Proof. We begin by considering distances on $\overline{\mathcal{M}}$. For a prescribed stratum and point choose a local-manifold cover $\widehat{\mathcal{U}}$ with the point corresponding to the origin and (image) stratum σ corresponding to the x-axes for the normal form of ds^2. The distance to the stratum $d(p,\sigma)$ is estimated from above by considering the radial line from a point p to the origin. The bound is $\|p_y\| + O(\|p_y\|^4)$ in terms of the y-projection of p. We next consider a lower bound for $d(p,\sigma)$. A curve in $\widehat{\mathcal{U}}$ connecting p to σ can be isometrically lifted to $\overline{\mathcal{T}}$; it suffices to examine curves that are length-minimizing on $\overline{\mathcal{T}}$. From [6, Chap. II.2, Prop. 2.4] for p close to the origin there is a geodesic $\gamma \subset \widehat{\mathcal{U}}$ connecting p to $q \in \sigma$ and γ provides the length-minimizing curve connecting to σ for each point of γ. For $p' \in \gamma$, as noted, $d(p',\sigma)$ is bounded above by the Euclidean norm of p'_y. Since ds^2 is likewise bounded below in terms of dr^2 it follows that $d(p',\sigma)$ is actually comparable to $\|p'_y\|$ and consequently that the maximum Euclidean y-height of the γ-segment $\widehat{p'q}$ is likewise bounded. It now follows overall that

$$d(p',\sigma) = \int_{\widehat{p'q}} (d\mu^2 + dr^2 + r^6 d\theta^2)^{1/2} + O(\|p'\|^4).$$

The explicit integral is minorized by choosing the radial line in the y-coordinate; the resulting lower bound is $\|p'_y\|$. Thus the distance to the stratum on $\overline{\mathcal{M}}$ and on $\overline{\mathcal{T}}$ is $d(p,\sigma) = \|p_y\| + O(\|p_y\|^4)$. In [44, Example 4.3] a relationship is provided for the geodesic-length functions and the present hyperbolic metric plumbing coordinates; it is shown that

$$\ell_j = \frac{2\pi^2}{-\log|t_j|} + O\big(\frac{1}{(\log^2|t_j|)}\sum_{k=1}^{n}\frac{1}{(\log^2|t_k|)}\big).$$

We may rearrange terms and substitute the relation $\varrho_j^2 = \frac{4\pi^3}{-\log|t_j|}$ to obtain the expansion

$$\varrho_j^2 = 2\pi\ell_j(1 + O\big(\ell_j\sum_k \ell_k^2)).$$

The distance expansion now follows from Corollary 4. The proof is complete.

The WP gradients of the geodesic-length functions also have general expansions, [45, II: Sec. 2.2, Lemmas 2.3 and 2.4]. We have for ϵ positive and geodesic-length functions $\ell_\alpha \le \ell_\beta \le \epsilon$ that

$$\begin{aligned}\big\langle \operatorname{grad}\ell_\alpha, \operatorname{grad}\ell_\alpha\big\rangle &= \frac{2}{\pi}\ell_\alpha \ + \ O(\ell_\alpha^3)\\ \big\langle \operatorname{grad}\ell_\alpha, \operatorname{grad}\ell_\beta\big\rangle &= O(\ell_\alpha\ell_\beta^2)\end{aligned}$$

with constants independent of the surface, or in particular for $\lambda_* = (2\pi\ell_*)^{1/2}$ that

$$\begin{aligned}\big\langle \operatorname{grad}\lambda_\alpha, \operatorname{grad}\lambda_\alpha\big\rangle &= 1 \ + \ O(\lambda_\alpha^4)\\ \big\langle \operatorname{grad}\lambda_\alpha, \operatorname{grad}\lambda_\beta\big\rangle &= O(\lambda_\alpha\lambda_\beta^3).\end{aligned}$$

We consider applications of the gradient bounds. We now consider simplices σ and τ with free homotopy classes α in σ and β in τ with (all) representatives intersecting. The simplice σ and τ do not precede a common simplex. There is also a positive lower bound for the corresponding geodesic-length sum $\ell_\alpha + \ell_\beta$. In particular from the *collar result* there is a positive constant $\ell_0 < 2$ such that about a geodesic α with $\ell_\alpha \le \ell_0$, there is an embedded collar of width $2\log 2/\ell_\alpha$: in which case ℓ_β is at least the width, [34]. Consider a WP length-minimizing curve γ connecting the strata $\mathcal{S}(\sigma)$ and $\mathcal{S}(\tau)$. On the curve γ the geodesic-length functions ℓ_α and ℓ_β each vanish (ℓ_α on $\mathcal{S}(\sigma)$; ℓ_β on $\mathcal{S}(\tau)$) and are each unbounded. The following is now the consequence of the universal bounds for the gradients.

Corollary 22 *There is a positive constant δ such that null strata $\mathcal{S}(\sigma)$ and $\mathcal{S}(\tau)$ either have intersecting closures or $d(\mathcal{S}(\sigma),\mathcal{S}(\tau)) \geq \delta$.*

We also recognize from the relations for a stratum σ, defined by the vanishing of the geodesic-length sum $\ell = \ell_1 + \cdots + \ell_n$, that the vector fields $\{\operatorname{grad}\lambda_1,\dots,\operatorname{grad}\lambda_n\}$ play the role of *normal Fermi fields* [17, Sec. 2.3]. Recall in particular for a tubular neighborhood of a submanifold in a Riemannian manifold, Fermi coordinates are the relative analog of *normal* coordinates about a point. First consider a neighborhood $\mathcal{N}$ of the 0-section of the normal subbundle of the tangent bundle of the submanifold. With the restriction of the exponential map $\mathcal{N}$ is identified with a tubular neighborhood of the submanifold. An orthonormal frame for the normal bundle provides Fermi coordinates for the fibres of $\mathcal{N}$, and also for a tubular neighborhood of the submanifold upon composition with the inverse of the exponential map. The unit-speed geodesics normal to the submanifold are given in terms of the Fermi coordinates as the unit-speed linear rays from the 0-section. The tangent fields of the unit-speed geodesics normal to the submanifold are constant sums of the Fermi coordinate tangent fields (the *normal Fermi fields*).

We are ready to present the analogy. For constant sums of the vector fields $\operatorname{grad}\lambda_*$, the WP distance between endpoints of integral curves nearly equals the integral curve length. Consider for a positive vector $c = (c_1,\dots,c_n)$ the integral curves of $\mathfrak{v} = -\sum_j c_j\operatorname{grad}\lambda_j$. The time-one integral curve of $\mathfrak{v}$ with initial point $2\pi(\ell_1,\dots,\ell_n) = (c_1^2,\dots,c_n^2)$ has terminal point at distance $O(\|c\|^4)$ to σ. Further from Corollary 21 the point $2\pi(\ell_1,\dots,\ell_n) = (c_1^2,\dots,c_n^2)$ is at WP distance $\|c\| + O(\|c\|^4)$ from σ. From the gradient relations above we have that $\|\mathfrak{v}\|_{WP} = \|c\| + O(\|c\|^4)$ and that the time-one integral curve has the same WP length. For $\|c\|$ small, the time-one integral curves of $\mathfrak{v}$ have endpoints at distance nearly equal to the curve length. The integral curves approximate WP geodesics. The integral curves of $\mathfrak{n} = \sum_j \ell^{-1/2}\lambda_j\operatorname{grad}\lambda_j$ also have length nearly equal to the distance between endpoints. The time $(2\pi\ell)^{1/2}$ integral curve of $\mathfrak{n}$ connects $(\ell_1,\dots,\ell_n)$ and σ; for ℓ small $\mathfrak{n}$ is approximately the WP unit normal field to σ. Also by Corollary 21 $\mathfrak{n}$ approximates $\operatorname{grad} d_{WP}(\cdot,\sigma)$.

7 The structure of geodesic limits

We investigate sequences of geodesics. $\overline{\mathcal{T}}$ is a complete metric space with a compact quotient $\overline{\mathcal{M}}$. We anticipate that the compactness is manifested in the structure of the space of geodesics for $\overline{\mathcal{T}}$. We find that geometric limits of geodesics are described by *polygonal paths* and products of Dehn twists. Specifically for a sequence of bounded length geodesics there is a subsequence of Mod-translates that converges geometrically (sequences of products of Dehn twists are applied to subsegments) to a polygonal path, a piecewise geodesic curve connecting strata. We consider an application of the result and show that each fixed-point free element of the mapping class group Mod has a geodesic axis in $\overline{\mathcal{T}}$; the axis is unique and lies in $\mathcal{T}$ when the element is irreducible. Furthermore irreducible elements have either coinciding or divergent axes. The present results provide a different approach for the considerations of G. Daskalopoulos and R. Wentworth [13].

Sequences of geodesics can have special behavior for product cuspidal metrics. We present an example. Consider the half-plane $\mathbb{R}_{\geq 0} \times \mathbb{R}$ with coordinates (r, θ) and the identification space $\mathbb{R}_{\geq 0} \times \mathbb{R}/\{(0,\theta) \sim (0,\theta')\}$ with *metric* $dr^2 + r^6 d\theta^2$. Denote the special point $\{(0,\theta) \sim (0,\theta')\}$ by O. For the isometry $T : (r,\theta) \to (r, \theta + 1)$ consider the unit-speed geodesics γ_n connecting (r_0, θ_0) and $T^n(r_1, \theta_1)$. For n large the length of γ_n is nearly $r_0 + r_1$; we can provide that the γ_n are essentially parameterized on the interval $[0, r_0 + r_1]$. By elementary considerations of differential equations, on the parameter interval $[0, r_0]$ the sequence $\{\gamma_n\}$ converges to the $\theta = \theta_0$ line segment $\widehat{(r_0, \theta_0)O}$. On the parameter interval $[r_0, r_0 + r_1]$ the sequence $\{T^{-n}\gamma_n\}$ converges to the $\theta = \theta_1$ line segment $\widehat{O(r_1, \theta_1)}$. In effect the geodesic sequence $\{\gamma_n\}$ is described by the polygonal path $\widehat{(r_0, \theta_0)O} \cup \widehat{O(r_1, \theta_1)}$ and the sequence of transformations $\{T^n\}$. Furthermore the curve $\widehat{(r_0, \theta_0)O} \cup T^n\widehat{O(r_1, \theta_1)}$ is continuous and has distance in the sense of parameterized unit-speed curves to γ_n that tends to zero as n tends to infinity.

We consider the local description of the map from $\overline{\mathcal{T}}$ to $\overline{\mathcal{M}}$. Associated to a k-simplex σ is the rank $k+1$ Abelian group Mod_σ of mapping classes of products of Dehn twists about the elements of σ. Mod_σ stabilizes the σ-null stratum $\mathcal{S}(\sigma)$. For a point $p \in \mathcal{S}(\sigma)$ the stabilizer $Mod(p) \subset Mod$ is a group extension of a finite group $G(p)$ by Mod_σ. Furthermore for a point p we can prescribe a suitable basis $\{\mathcal{U}\}$ of $Mod(p)$ invariant neighborhoods. Neighborhoods of the projection of p to $\overline{\mathcal{M}}$ are given as $\mathcal{U}/Mod(p)$. Furthermore each quotient $\mathcal{U}/Mod_\sigma$ is a local manifold cover. We can further prescribe that the quotients $\mathcal{U}/Mod_\sigma$ are relatively compact in a fixed quotient $\mathcal{U}'/Mod_\sigma$; the quotients $\partial\mathcal{U}/Mod_\sigma$ are accordingly compact.

Since $\overline{\mathcal{M}}$ is compact, given a sequence of points of $\overline{\mathcal{T}}$ there exists a subsequence and associated elements of Mod such that the sequence of image points converges. Accordingly we consider sequences of unit-speed parameterized geodesics with initial points converging.

Proposition 23 *Consider a sequence of unit-speed geodesics $\{\gamma'_n\}$ with initial points converging to p_0, lengths converging to a positive value L' and parameter intervals converging to $[t', t'']$ with $L' = t'' - t'$. There exists an associated partition $t' = t_0 < t_1 < \cdots < t_k = t''$ of the interval; simplices $\sigma_0 = \Lambda(p_0), \sigma_1, \ldots, \sigma_k$; and points $p_1 \in \mathcal{S}(\sigma_1), \ldots, p_k \in \mathcal{S}(\sigma_k)$ on the null strata.*

The data satisfies $L(\widehat{p_j p_{j+1}}) = t_{j+1} - t_j$ for $j = 0, \ldots, k-1$ and for the stratum with label $\tau_j = \Lambda(p_j) \cap \Lambda(p_{j+1})$ then: τ_0 strictly precedes σ_1 if $k > 1$; τ_{k-1} strictly precedes σ_{k-1} if $k > 1$; τ_j strictly precedes σ_j and σ_{j+1} for $j = 1, \ldots, k-2$. The concatenation of geodesic segments $\widehat{p_0p_1} \cup \widehat{p_1p_2} \cup \cdots \cup \widehat{p_{k-1}p_k}$ is the unique length-minimizing curve connecting p_0 to p_k and intersecting in order the closures of the strata $\mathcal{S}(\sigma_1), \ldots, \mathcal{S}(\sigma_{k-1})$.

There is a subsequence $\{\gamma_n\}$ of the geodesics and sequences of products of Dehn twists $T_{(j,n)} \in Mod_{\sigma_j - \tau_j}$, $j = 0, \ldots, k-1$, such that on the parameter interval $[t_j, t_{j+1}]$ the geodesic segments $T_{(j,n)} \circ \cdots \circ T_{(0,n)}\gamma_n$ converge to $\widehat{p_j p_{j+1}}$ in the sense of parameterized unit-speed curves. Furthermore the distance between the parameterized unit-speed curves γ_n and $\widehat{p_0 p_{(k,n)}}$, $p_{(k,n)} = (T_{(k-1,n)} \circ \cdots \circ T_{(0,n)})^{-1} p_k$, tends to zero for n tending to infinity. The sequence of transformations $\{T_{(0,n)}\}$ is either trivial or unbounded. The sequences of transformations $\{T_{(j,n)}\}$, $j = 1, \ldots, k-1$ are unbounded.

Proof. The main argument is to provide the two steps for determining the individual geodesic segments $\widehat{p_j p_{j+1}}$. The overall argument is then a finite induction. For the first step choose a neighborhood $\mathcal{U}$ of p_0 with $\partial\mathcal{U}/Mod_{\sigma_0}$ compact. For each geodesic γ'_n let q_n be the first point of intersection with $\partial\mathcal{U}$. Either a subsequence of the points q_n converges to a point q' or we select elements $T_{(0,n)} \in Mod_{\sigma_0}$ such that the images $T_{(0,n)}q_n$ lie in a relatively compact fundamental domain for the action of Mod_{σ_0} on $\partial\mathcal{U}$. For the situation of selecting elements $T_{(0,n)}$ there is a subsequence $\{T_{(0,n)}q_n\}$ convergent to a point q' and the sequence $\{T_{(0,n)}\}$ is unbounded. Now the group Mod_{σ_0} fixes p_0 and a sequence of points converges to q'. The group Mod_{σ_0} is a direct product with a factor $Mod_{\sigma_0 - \tau_0}$ for $\tau_0 = \Lambda(q')$; for $\{T_{(0,n)}q_n\}$ converging to q' it is a basic feature of the $\overline{\mathcal{T}}$ topology that the $T_{(0,n)}$ can be replaced with their $Mod_{\sigma_0-\tau_0}$ factors and the resulting sequence also converges. Finally since geodesics in a $CAT(0)$ space depend continuously on endpoints [6, Chap. II.1, Prop. 1.4], the appropriate geodesic segments from p_0 to q_n or to $T_{(0,n)}q_n$ converge to $\widehat{pq'}$, as claimed.

The second step is to show for a subsequence of the geodesics that maximal initial segments converge to segments of the prolongation of $\widehat{pq'}$ in the stratum τ_0. We now write $T_{(0,n)}\gamma_n'$ whether the transformations are trivial or not. We preliminarily note that the subsequence necessarily converges on a closed interval. In particular for a subsequence converging uniformly to a segment $\widehat{p_0q''}-\{q''\}$ and ϵ small, consider the point on $\widehat{p_0q''}$ distance ϵ before q''; a corresponding sequence of points, one on each $T_{(0,n)}\gamma_n'$, is determined. On each $T_{(0,n)}\gamma_n'$ consider the point ϵ further along than the referenced point; the resulting points are at distance 3ϵ from q'' for n large. The interval of convergence is indeed closed. There are now three possibilities for the interval: i) a subsequence $\{T_{(0,n)}\gamma_n'\}$ converges on the entire parameter interval $[t',t'']$ to a segment of the forward prolongation of $\widehat{p_0q'}$ (and the overall convergence argument is complete), ii) a subsequence $\{T_{(0,n)}\gamma_n'\}$ converges on $[t',t_1]$, $t_1<t''$, to $\widehat{p_0p_1}-\{p_1\}$ and $\Lambda(p_1)$ properly succeeds τ_0, or iii) a subsequence $\{T_{(0,n)}\gamma_n'\}$ converges on $[t',t_*]$, $t_*<t''$, to $\widehat{p_0q''}$, with $\widehat{p_0q''}$ having a nontrivial forward prolongation in the stratum τ. We examine case iii). We examine the behavior of the subsequence $\{T_{(0,n)}\gamma_n'\}$ in a neighborhood of q''. We can again apply the argument from the beginning of the proof to determine elements $S_n\in Mod_{\tau_0}$ such that a subsequence of $\{S_n\circ T_{(0,n)}\gamma_n'\}$ converges in a neighborhood of q''. The limit is length-minimizing. From Proposition 11 the subsequence $\{S_n\circ T_{(0,n)}\gamma_n'\}$ converges to the prolongation of $\widehat{p_0q''}$; from the above observation concerning the $\overline{\mathcal{T}}$ topology and the Mod_{τ_0} action the subsequence $\{T_{(0,n)}\gamma_n'\}$ also converges to the prolongation. We now summarize the convergence considerations: for a maximal parameter interval of convergence of a subsequence $\{T_{(0,n)}\gamma_n'\}$ either: i) the parameter interval is $[t',t'']$, or ii) the interval is $[t',t_1]$, $t_1<t''$, and the limit is $\widehat{p_0p_1}$ with $\Lambda(p_1)$ strictly succeeding $\Lambda(p_0)\cap\Lambda(p_1)$.

We now proceed and apply the considerations of the above two paragraphs to the subsequence $\{T_{(0,n)}\gamma_n'\}$ considered on the interval $[t_1,t'']$. The initial points converge to p_1. Elements $T_{(1,n)}\in Mod_{\sigma_1-\tau_1}$ are determined such that a further subsequence $\{T_{(1,n)}\circ T_{(0,n)}\gamma_n'\}$ converges as in case i) or case ii). A stratum τ_1 is prescribed. The simplex σ_1 properly succeeds τ_1 for otherwise (from the observation concerning the $\overline{\mathcal{T}}$ topology and the Mod_* action) elements $T_{(1,n)}$ are not required and the subsequence $\{T_{(0,n)}\gamma_n'\}$ on

$[t', t_1 + \epsilon)$ converges to a curve that is not length-minimizing by Proposition 11. We now note that Mod preserves the strata structure of $\overline{\mathcal{T}}$. Since the entire considerations including the initial-point convergence can be applied to the sequence $\{\gamma'_n\}$ *starting* from an arbitrary value t''' and proceeding in the negative t-direction, we observe that consequently a finite partition $t' = t_0 < t_1 < \cdots < t_k = t''$ is determined. Points $p_0, \ldots, p_k$; strata $\sigma_0, \ldots, \sigma_k, \tau_0, \ldots, \tau_{k-1}$; and sequences $\{\gamma_n\}$, $\{T_{(0,n)}\}, \ldots, \{T_{(k-1,n)}\}$ are determined. The desired properties are provided in the above construction with only two remaining matters: the sequences $\{T_{(j,n)}\}$, $j = 1, \ldots, k-1$, are unbounded and the length-minimizing property of the concatenation of the points $p_0, \ldots, p_k$.

We consider the length property first. A candidate length-minimizing curve is given as a concatenation $\mathcal{C} = \widehat{p_0q_1} \cup \widehat{q_1q_2} \cup \cdots \cup \widehat{q_{k-1}p_k}$ with $p_0 \in \mathcal{S}(\sigma_0)$ and $q_j \in \mathcal{S}(\sigma_j)$ for $j = 1, \ldots, k-1$. Since the group $Mod_{\sigma_j - \tau_j}$ stabilizes $\overline{\mathcal{S}(\sigma_j)}$ it follows that the concatenation $\mathcal{C}_T = T^{-1}_{(0,n)}\widehat{p_0q_1} \cup T^{-1}_{(0,n)} \circ T^{-1}_{(1,n)}\widehat{q_1q_2} \cup \cdots \cup T^{-1}_{(0,n)} \circ \cdots \circ T^{-1}_{(k-1,n)}\widehat{q_{k-1}p_k}$ is a continuous curve connecting p_0 to $p_{(k,n)} = (T_{(k-1,n)} \circ \cdots \circ T_{(0,n)})^{-1}p_k$. From the overall construction $d(\gamma_n(t''), p_{(k,n)})$ tends to zero; from [6, Chap. II.1, Prop. 1.4] the distance between γ_n and $\widehat{p_0p_{(k,n)}}$ is consequently small for n large, as claimed. The three curves $\gamma_n, \mathcal{C}_T$ and $\widehat{p_0p_{(k,n)}}$ each approximately connect p_0 and $p_{(k,n)}$. Since $L(\mathcal{C}) = L(\mathcal{C}_T)$, it follows that $L(\mathcal{C}) \geq \lim_n L(\gamma_n) = L'$. Thus each suitable concatenation has length at least L' with the minimum achieved for the arrangement of points $\{p_0, p_1, \ldots, p_k\}$. It remains for $k > 1$ to establish uniqueness. Consider geodesics $\alpha_j(s)$, parameterized on the unit-interval, connecting p_j to q_j, $j = 1, \ldots, k-1$. The concatenation $\widehat{p_0\alpha_1(s)} \cup \widehat{\alpha_1(s)\alpha_2(s)} \cup \cdots \cup \widehat{\alpha_{k-1}(s)p_k}$ satisfies the strata hypothesis and by Theorem 14 has length a convex function of the parameter s. Now for a $CAT(0)$ space either the distance from a point to a geodesic is a strictly convex function, or the point lies on the prolongation of the geodesic, [6, Chap. II.1, Defn. 1.1]. The points $\{p_0, p_1, q_1\}$ do not lie on a common geodesic since $\tau_0 = \Lambda(p_0) \cap \Lambda(p_1)$ strictly precedes σ_1 and $\sigma_1 \subset \Lambda(p_1) \cap \Lambda(q_1)$. In consequence either $L(\widehat{p_0\alpha_1(s)})$ is a strictly convex function or $\alpha_1(s)$ is constant. It follows that the length of a non constant family is a strictly convex function. The minimal-length concatenation is unique.

The final matter is the unboundedness of the sequences $\{T_{(j,n)}\}$. The elements $T_{(j,n)}$ are selected to provide that the initial curve segments lie in relatively compact sets. We can prescribe that each sequence $\{T_{(j,n)}\}$ is either trivial or unbounded. In a neighborhood of the parameter value

t_j the concatenation $\mathcal{C}_j = \widehat{p_{j-1}p_j} \cup T^{-1}_{(j,n)}\widehat{p_jp_{j+1}}$ is the approximation to $T_{(j-1,n)} \circ \cdots \circ T_{(0,n)}\gamma_n$. Since γ_n is length-minimizing, it follows for n large that the concatenation $\mathcal{C}_j$ is arbitrarily close to length-minimizing. If $\{T_{(j,n)}\}$ were trivial then $\widehat{p_{j-1}p_j} \cup \widehat{p_jp_{j+1}}$ would be length-minimizing in contradiction of Theorem 13, since $p_j \in \overline{\mathcal{S}(\sigma_j)}$ and σ_j strictly succeeds the stratum of one of the connecting geodesic segments. The proof is complete.

We now introduce terminology for the data for a convergent sequence of geodesics. Consider as above a convergent sequence $\{\gamma_n\}$ with data $\{p_j\}$ and $\{T_{(j,n)}\}$.

Definition 24 *For a convergent sequence of geodesics $\{\gamma_n\}$ the vertices are the points p_j, $j = 0, \dots, k$; the vertex concatenation is $\widehat{p_0p_1} \cup \widehat{p_1p_2} \cup \cdots \cup \widehat{p_{k-1}p_k}$ and an approximating concatenation is $T^{-1}_{(0,n)}\widehat{p_0p_1} \cup T^{-1}_{(0,n)} \circ T^{-1}_{(1,n)}\widehat{p_1p_2} \cup \cdots \cup T^{-1}_{(0,n)} \circ \cdots \circ T^{-1}_{(k-1,n)}\widehat{p_{k-1}p_k}$.*

We are ready to consider the matter of existence of axes for elements of Mod. We present a different approach towards certain results of G. Daskalopoulos and R. Wentworth [13]. They show that each irreducible (pseudo Anosov) mapping class has a unique invariant axis and that non commuting irreducible mapping classes have divergent axes. To provide a context we first recall the Thurston-Nielsen classification of mapping classes [1, Exposés 9, 11]. A mapping class is *irreducible* provided no power fixes the free homotopy class of a simple closed curve. A mapping class is precisely one of: periodic, irreducible or reducible, [1]. Reducible classes are first analyzed in terms of mappings of proper subsurfaces. For a reducible mapping class $[h]$ an invariant is $\sigma_{[h]}$ the maximal simplex fixed by a power. A general invariant of a transformation S is its translation length: $\inf_p d(p, Sp)$.

Theorem 25 *A mapping class S acting on $\overline{\mathcal{T}}$ either has fixed-points or positive translation length realized on a closed, convex set $\mathcal{A}_S$, isometric to a metric space product $\mathbb{R} \times Y$. In the latter case the isometry S acts on $\mathbb{R} \times Y$ as the product of a translation of $\mathbb{R}$ and id_Y. For S irreducible the translation length is positive and $\mathcal{A}_S$ is a geodesic in $\mathcal{T}$. For S reducible the null stratum $\mathcal{S}(\sigma_{[h]})$ is a product of Teichmüller spaces $\prod \mathcal{T}' \times \prod \mathcal{T}''$ with a power S^m fixing the factors, acting by a product of: irreducible elements S' on $\mathcal{T}'$ with axis $\gamma_{S'}$ and the identity on each $\mathcal{T}''$; $\mathcal{A}_S \subset \prod \gamma_{S'} \times \prod \overline{\mathcal{T}}''$.*

Proof. The first matter is to establish that the translation length is realized. We consider a sequence of geodesics $\{\gamma'_n\}$ parameterized on $[a, b]$ with S connecting endpoints, $S(\gamma'_n(a)) = \gamma'_n(b)$, and $\lim_n L(\gamma'_n)$ the translation length.

Apply elements of Mod and according to Proposition 23 select a convergent subsequence $\{\gamma_n\}$ with vertices $\{p_0,\dots,p_k\}$; each γ_n has endpoints connected by a conjugate of S. For the special situation of translation length zero then $a=b$ and the vertex concatenation is the singleton $\{p_0\}$. The main matter is to determine the distance between the Mod orbit of p_0 and that of p_k. For an ϵ approximating concatenation to γ_n and a conjugate Q we have $Q(\gamma_n(a))=\gamma_n(b)$ with $\gamma_n(a)$ within ϵ of p_0 and $\gamma_n(b)$ within ϵ of the prescribed endpoint $p_{(k,n)}$; it follows that $d(Q(p_0),p_{(k,n)})<2\epsilon$. It follows that the distance between the Mod orbit of p_0 and that of p_k is zero. The distance is also given by considering Mod translates of p_k in a neighborhood of p_0. From the preliminary discussion there is a positive lower bound for the distance between the points of the p_k orbit. It now follows that for a suitable ϵ and n' above, the distance inequality implies that $Q(p_0)=p_{(k,n')}$. For the value n' the approximating concatenation connects p_0 and $p_{(k,n')}$ and has length $\lim_n L(\gamma_n)$, the minimal translation length. It follows that there is only one geodesic segment in the concatenation and that Q realizes its translation length for the segment $\widehat{p_0Q(p_0)}$. S realizes its translation length for an image of the segment.

By general considerations for positive translation length S realizes its translation length on axes in $\overline{\mathcal{T}}$, [6, Chap. II.6, Defn 6.3, Thrm. 6.8]. An axis is isometric to $\mathbb{R}$ and may not be unique; we consider specifics. S stabilizes each axis and thus stabilizes the stratum of an axis. Since in fact an irreducible mapping class only stabilizes the single stratum $\mathcal{T}$, it follows that an irreducible class is fixed-point free with axes in $\mathcal{T}$. Since axes are parallel and the distance between geodesics in a Teichmüller space is a strictly convex function, it follows that an irreducible axis is unique in $\overline{\mathcal{T}}$. Now for a reducible mapping class S a power P fixes a simplex if and only if the power fixes the corresponding null stratum. For the maximal simplex σ_S the geodesic-length sum $\mathcal{L}_S=\sum_{\alpha\in\sigma_S}\ell_\alpha$ restricts on each geodesic of $\overline{\mathcal{T}}$ to either the zero-function or a strictly convex function. Since $\mathcal{L}_S$ is P-invariant on an axis of P the restriction is the zero-function. Thus the axes of P are contained in the null stratum $\mathcal{S}(\sigma_S)$. On considering a power of P we can further arrange that P^m stabilizes the components $F-\cup_{\alpha\in\sigma_S}\{\alpha\}$ of the reference surface, and by [1, Exposé 11, Thrm 4.2] that on each component of $F-\cup_{\alpha\in\sigma_S}\{\alpha\}$ the restriction of P^m is either the trivial or an irreducible mapping class. For positive translation length at least one factor is an irreducible mapping class since by [6, Chap. II.6, Thrm. 6.8(2)] the translation length of the power is also positive. It follows that P^m realizes its translation length on a product $\mathcal{A}_{P^m}=\prod\gamma'\times\prod\overline{\mathcal{T}}''$ contained in the closure of $\mathcal{S}(\sigma_S)$. By [6, Chap. II.6, Thrm. 6.8(4)] $\mathcal{A}_{P^m}$ is isometric to a product $\mathbb{R}\times Y$ with S stabilizing the product and acting thereon by the product of a translation and a periodic element. $\mathcal{A}_{P^m}$ is itself a complete $CAT(0)$ space. It follows that Y is a complete $CAT(0)$ space and by [6, Chap. II.2, Cor. 2.8] that a periodic element has a non-empty closed convex fixed-point set. The proof is complete.

We recall notions regarding the behavior of geodesic rays. Unit-speed rays $\gamma(t)$, $\gamma'(t)$ are *asymptotic* provided the limit $\lim_{t\to\infty} d(\gamma(t),\gamma'(t))$ is zero and are *divergent* provided the limit is infinity. For values t, τ the points $\gamma(0)$, $\gamma'(0)$, $\gamma'(\tau)$ and $\gamma(t)$ determine a quadrilateral; on comparing side lengths we have that $|\tau - t| \leq d(\gamma(0),\gamma'(0)) + d(\gamma(t),\gamma'(\tau))$. Since also $d(\gamma(t),\gamma'(t)) \leq d(\gamma(t),\gamma'(\tau)) + |\tau - t|$ it follows on substituting for $|\tau - t|$ that $d(\gamma(t),\gamma'(t)) \leq 2d(\gamma(t),\gamma'(\tau)) + d(\gamma(0),\gamma'(0))$. In particular for divergent rays the distance from a point on one ray to the other ray tends to infinity with the point.

Corollary 26 *A geodesic ray in $\mathcal{T}$ and the axes of an irreducible mapping class are either asymptotic or divergent. Two irreducible mapping classes have axes that coincide or are divergent.*

Proof. We first consider a ray γ and an irreducible element S with axis γ_S. By Proposition 7 for unit-speed parameterizations the distance between corresponding points of γ and γ_S is a convex function. In particular the distance has a limit $L_\infty = \lim_{t\to\infty} d(\gamma(t),\gamma'(t))$. We will use that the WP geometry along γ_S is periodic to show that: either L_∞ is zero or there is a positive lower bound for the convexity of the distance and thus L_∞ is infinite.

We revisit formula (5) for the one-parameter variation $\beta(t,s)$ (t is now the parameter for the geodesics *connecting* γ_S to γ.) The integrand of (5) is bounded below by the contribution of the curvature term $-\langle R(V,T)T,V\rangle$, which in turn is non-negative. The curve $\beta(t,s')$, $t_0 \leq t \leq t_1$, is a geodesic with initial point on γ_S and length at least L_∞. T is the tangent field of $\beta(t,s')$ and V is a Jacobi field along $\beta(t,s')$ with initial vector having unit length. Now choose t', $t_0 < t' \leq L_\infty$ such that the neighborhood of radius $t' - t_0$ about a point of γ_S is relatively compact in $\mathcal{T}$. For the parameter range $t_0 \leq t \leq t'$ the geodesic segments $\beta(t,s')$, the tangent fields T, and the Jacobi fields V are all modulo the action of S supported on a compact set: the closure of the neighborhood of a fundamental segment of γ_S. Since the WP curvature is strictly negative on the compact set, we obtain a positive lower bound for the evaluations, the desired convexity bound for the distance function. It follows that L_∞ is either zero or infinite.

Consider next irreducible elements S, resp. Q, with translation lengths L_S, resp. L_Q, and axis γ_S, resp. γ_Q; assume the axes are asymptotic in the forward direction. Choose a reference point p' on γ_S. Given ϵ positive, choose positive integers n, m such that $|nL_S - mL_Q| < \epsilon$. Further choose a positive integer k_0 such that for $k \geq k_0$ the point $p = S^k p'$ on γ_S and the corresponding point q on γ_Q are at distance at most ϵ. We have then that $d(S^n p, Q^m q) < 2\epsilon$; we thus have that $d(Q^{-m}S^n p, p) < 3\epsilon$. Since there is a positive lower bound for the distance between distinct points of the Mod orbit of p', it follows for ϵ small that the transformation $Q^{-m}S^n$ fixes the sequence of points $S^k p'$, $k \geq k_0$. It follows that Q^m stabilizes γ_S. Since the axes are asymptotic, for p far along γ_S the displacement $d(Q^m p, p)$ is close to $d(Q^m q, q)$; Q^m realizes its translation length on γ_S; by the Theorem the axes coincide. The proof is complete.

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