High-Order Central WENO Schemes for 1D Hamilton-Jacobi Equations

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1 Introduction

We consider high-order central approximations for solutions of one-dimensional Hamilton-Jacobi (HJ) equations of the form

$$\frac{\partial}{\partial t}\phi(x,t) + H\left(\phi_x, x\right) = 0, \qquad x \in \mathbb{R}, \tag{1}$$

subject to the initial data $\phi(x, t\!=\!0) = \phi_0(\mathbf{x})$. Solutions for (1) with smooth initial data typically remain continuous but develop discontinuous derivatives in finite time. Such solutions are not unique; the physically relevant solution is known as the *viscosity solution* (see [1, 3, 4, 5, 8, 15] and the references therein).

Various numerical methods were proposed in order to approximate the solutions of (1). Examples for such methods are the high-order Godunov-type schemes that were introduced in [20, 21], and were based on an Essentially Non-Oscillatory (ENO) reconstruction step [7] that was evolved in time with a first-order monotone flux. The least dissipative monotone flux, the Godunov flux, requires solving Riemann problems at cell interfaces. A fifth-order Weighted ENO (WENO) scheme, based on [10, 18], was introduced by Jiang and Peng [9].

Recently, Lin and Tadmor introduced in [16, 17] central schemes for approximating solutions of the HJ equation. These schemes are based on the Nessyahu-Tadmor scheme for approximating solutions of hyperbolic conservation laws [19]. Unlike upwind schemes, central schemes do not require Riemann solvers, which makes them attractive for solving systems of equations and for multi-dimensional problems. A second-order semi-discrete version of these schemes was introduced by Kurganov and Tadmor in [12]. While less dissipative, the semi-discrete scheme requires the estimation of the local speed of propagation, which is computationally intensive in particular in multi-dimensional problems. In a later work [11], the numerical viscosity was further reduced by computing more precise information about local speed of propagation. To address the problem of schemes that are too computationally intensive, we introduced in [2] efficient first- and second-order central schemes for approximating the solutions of multi-dimensional versions of (1).

Unlike the previous attempts, our schemes in [2] scale well with increasing dimension.

In this paper we derive fully-discrete Central WENO (CWENO) schemes for approximating solutions of (1), which combine our previous works [2, 13, 14]. We introduce third- and fifth-order accurate schemes, which are the first central schemes for the HJ equations of order higher than two. The core ingredient in the derivation of our schemes is a high-order CWENO reconstructions in space.

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2 CWENO Schemes for HJ Equations

We are interested in approximating solutions of (1) subject to the initial data $\phi(x, t = 0) = \phi_0(x)$. For simplicity we assume a uniform grid grid in space and time with mesh spacings, $h := \Delta x$ and Δt. We denote the grid points by $x_i = i\Delta x$, $t^n = n\Delta t$, and the fixed mesh ratio by $\lambda = \Delta t/\Delta x$. Let φ_i^n denote the approximate value of $\phi\left(x_i, t^n\right)$, and $\left(\varphi_x\right)_i^n$ denote the approximate value of the derivative $\phi_x\left(x_i, t^n\right)$. We define $\Delta^+\varphi_i^n := \varphi_{i+1}^n - \varphi_i^n$, $\Delta^-\varphi_i^n := \varphi_i^n - \varphi_{i-1}^n$ and $\Delta^0\varphi_i^n := \varphi_{i+1}^n - \varphi_{i-1}^n$.

We assume that the approximate solution at time t^n, φ_i^n is given. In order to approximate the solution at the next time step t^{n+1}, φ_i^{n+1}, we start by reconstructing a continuous piecewise-polynomial from the data, φ_i^n, and sample it at the half-integer points, $\{x_{i+1/2}\}$, in order to obtain the point-values of the interpolant at these points $\varphi_{i+1/2}^n$ as well as the derivative, $\varphi'_{i+1/2}$. We then evolve $\varphi_{i+\frac{1}{2}}^n$ from time t^n to time t^{n+1} according to (1),

$$\varphi\left(x_{i+\frac{1}{2}}, t^{n+1}\right) = \varphi\left(x_{i+\frac{1}{2}}, t^n\right) - \int_{t^n}^{t^{n+1}} H\left(\varphi_x\left(x_{i+\frac{1}{2}}, t\right)\right) dt. \tag{2}$$

This evolution is done at the half-integer grid points where the reconstruction is smooth (as long as the CFL condition $\lambda\left|H'\left(\varphi_x\right)\right| \leq 1/2$ is satisfied). Finally, in order to return to the original grid, we project $\varphi_{i+1/2}^{n+1}$ back onto the integer grid points $\{x_i\}$ to end up with φ_i^{n+1}.

Since the evolution step (2) is done at points where the solution is smooth, we can approximate the time integral at the RHS of (2) using a sufficiently accurate quadrature rule. For example, for a third- and fourth-order method, this integral can be replaced by a Simpson's quadrature,

$$\begin{aligned}\int_{t^n}^{t^{n+1}} H\left(\varphi_x\left(x_{i+\frac{1}{2}}, t\right)\right) dt \approx \frac{\Delta t}{6}\Big[&H\left(\varphi_x\left(x_{i+\frac{1}{2}}, t^n\right)\right) \\ &+4H\left(\varphi_x\left(x_{i+\frac{1}{2}}, t^{n+\frac{1}{2}}\right)\right) + H\left(\varphi_x\left(x_{i+\frac{1}{2}}, t^{n+1}\right)\right)\Big].\end{aligned} \tag{3}$$

The intermediate values of the derivative in time, $\varphi_x\left(x_{i+1/2}, t^{n+1/2}\right)$, and $\varphi_x\left(x_{i+1/2}, t^{n+1}\right)$, which are required in the quadrature (3), can be predicted using a Taylor expansion or with a Runge-Kutta (RK) method. For details we refer the reader to [13, 19] and the references therein.

The remaining ingredient is the piecewise-polynomial reconstruction in space. A careful study of the above procedure reveals that there are actually three different quantities that should be recovered in every time step. First, given φ_i at time t^n we need to reconstruct the point-values at the half-integer grid points, $\varphi_{i+1/2}$, at the same time t^n. This is the first term on the RHS of (2). The second term on the RHS of (2) requires evaluating the Hamiltonian H at the derivative $\varphi'_{i+1/2}$. Hence, the second quantity we should recover is $\varphi'_{i+1/2}$ from φ_i. Finally, the predictor step that provides the values at the quadrature nodes in (3), require us to estimate $\varphi'_{i+1/2}$ from $\varphi_{i+1/2}$ at every step of the RK method. In the next two sections we will focus on the reconstruction of these three quantities, first for a third-order method and then for a fifth-order method.

The projection from $\varphi^{n+1}_{i+1/2}$ onto the original grid points to get φ_i^{n+1} is accomplished using the same reconstruction used to approximate $\varphi^n_{i+1/2}$ from φ_i^n.

2.1 A Third-Order Scheme

Following the above procedure, a third-order scheme can be generated by combining a third-order accurate ODE solver in time with a sufficiently high-order reconstruction in space. Here we present fourth-order CWENO reconstructions of the point values of $\varphi_{i+1/2}$ and its derivative $\varphi'_{i+1/2}$.

The reconstruction of $\varphi_{i+1/2}$ from φ_i.
In order to obtain a fourth-order reconstruction of $\varphi_{i+1/2}$ we will write a convex combination of two quadratic polynomials, $\varphi_-^{[2]}$ constructed on a stencil which is left-biased with respect to $x_{i+1/2}$, and the right-biased $\varphi_+^{[2]}$,

$$\varphi_-^{[2]}(x) = \varphi_i + \frac{1}{h}\left(\Delta^+\varphi_i\right)(x-x_i) + \frac{1}{2h^2}\left(\Delta^+\Delta^-\varphi_i\right)(x-x_i)(x-x_{i+1}) + O\left(h^3\right),$$
$$\varphi_+^{[2]}(x) = \varphi_i + \frac{1}{h}\left(\Delta^+\varphi_i\right)(x-x_i) + \frac{1}{2h^2}\left(\Delta^+\Delta^+\varphi_i\right)(x-x_i)(x-x_{i+1}) + O\left(h^3\right).$$

An evaluation of these approximations at $\{x_{i+\frac{1}{2}}\}$ reads

$$\varphi_-^{[2]}\left(x_{i+\frac{1}{2}}\right) = \frac{1}{8}(-\varphi_{i-1}+6\varphi_i+3\varphi_{i+1}), \quad \varphi_+^{[2]}\left(x_{i+\frac{1}{2}}\right) = \frac{1}{8}(3\varphi_i+6\varphi_{i+1}-\varphi_{i+2}).$$

A straightforward computation shows that

$$\frac{1}{2}\varphi_-^{[2]}(x_{i+\frac{1}{2}}) + \frac{1}{2}\varphi_+^{[2]}(x_{i+\frac{1}{2}}) = \varphi_{i+\frac{1}{2}} + O\left(h^4\right).$$

The fourth-order WENO estimate of $\varphi_{i+1/2}$ is therefore given by the convex combination

$$\varphi_w^{[4]}\left(x_{i+\frac{1}{2}}\right) = w^-_{i+\frac{1}{2}}\varphi_-^{[2]}\left(x_{i+\frac{1}{2}}\right) + w^+_{i+\frac{1}{2}}\varphi_+^{[2]}\left(x_{i+\frac{1}{2}}\right),$$

where the weights satisfy $w^-_{i+1/2} + w^+_{i+1/2} = 1$, $w^\pm_{i+1/2} \geq 0$, $\forall i$. In smooth regions we would like to satisfy $w_i^- \approx w_i^+ \approx \frac{1}{2}$ to attain an $O\left(h^4\right)$ error, while when the stencil $\{x_{i-1}, x_i, x_{i+1}, x_{i+2}\}$ supporting $\varphi_w\left(x_{i+\frac{1}{2}}\right)$ contains a discontinuity, the weight of the more oscillatory polynomial should vanish. Following [10, 18], we meet these requirements by setting

$$w^k_{i+\frac{1}{2}} = \frac{\alpha^k_{i+\frac{1}{2}}}{\sum_l \alpha^l_{i+\frac{1}{2}}}, \qquad \alpha^k_{i+\frac{1}{2}} = \frac{c^k}{\left(\epsilon + S^k_{i+\frac{1}{2}}\right)^p} \tag{4}$$

where $k, l \in \{+, -\}$ (k and l will range over a larger space of symbols when we use more interpolants). The constants $c^\pm = 1/2$ and are independent of the grid-point. We choose ϵ as 10^{-6} to prevents the denominator in (4) from vanishing, and set $p = 2$ (see [10]). The smoothness measures $S_i^\pm$ should be large when φ is nearly singular. Following the standard practice with WENO-type schemes [10], we take $S_i^\pm$ to be the sum of the L^2-norms of the first and second derivatives on the stencil supporting $\varphi_\pm^{[2]}$. If we approximate the first derivative at $x_{i+1/2}$ by $\frac{1}{h}\Delta^+\varphi_{i+1/2}$, the second derivative by $\frac{1}{h^2}\Delta^+\Delta^-\varphi_{i+1/2}$, and define the smoothness measure

$$S_{i+\frac{1}{2}}\left[r,s\right] = h\sum_{j=r}^{s}\left(\frac{1}{h}\Delta^+\varphi_{i+j+\frac{1}{2}}\right)^2 + h\sum_{j=r+1}^{s}\left(\frac{1}{h^2}\Delta^+\Delta^-\varphi_{i+j+\frac{1}{2}}\right)^2, \tag{5}$$

then for the fourth-order interpolation of $\varphi_w\left(x_{i+\frac{1}{2}}\right)$ we have $S^-_{i+1/2} = S_{i+1/2}\left[-1,0\right]$ and $S^+_{i+1/2} = S_{i+1/2}\left[0,1\right]$.

The reconstruction of $\varphi'_{i+1/2}$ from φ_i.

To obtain a fourth-order estimate of the derivative $\varphi'(x_{i+1/2})$ from $\varphi(x_i)$, we start from the cubic interpolants

$$\begin{aligned}
\varphi_-^{[3]}(x) &= \varphi_i + \frac{1}{h}\left(\Delta^+\varphi_i\right)(x - x_i) + \frac{1}{2h^2}\left(\Delta^+\Delta^-\varphi_i\right)(x - x_i)(x - x_{i+1}) \\
&\quad + \frac{1}{6h^3}\left(\Delta^-\Delta^+\Delta^-\varphi_i\right)(x - x_i)(x - x_{i+1})(x - x_{i-1}) + O\left(h^4\right), \\
\varphi_+^{[3]}(x) &= \varphi_i + \frac{1}{h}\left(\Delta^+\varphi_i\right)(x - x_i) + \frac{1}{2h^2}\left(\Delta^+\Delta^+\varphi_i\right)(x - x_i)(x - x_{i+1}) \\
&\quad + \frac{1}{6h^3}\left(\Delta^+\Delta^+\Delta^+\varphi_i\right)(x - x_i)(x - x_{i+1})(x - x_{i+2}) + O\left(h^4\right).
\end{aligned}$$

Differentiating $\varphi_\pm^{[3]}$ at $x_{i+\frac{1}{2}}$

$$\varphi'^{[3]}_{-,i+\frac{1}{2}} = \frac{1}{24h}\left(\varphi_{i-2} - 3\varphi_{i-1} - 21\varphi_i + 23\varphi_{i+1}\right),$$
$$\varphi'^{[3]}_{+,i+\frac{1}{2}} = \frac{1}{24h}\left(-23\varphi_i + 21\varphi_{i+1} + 3\varphi_{i+2} - \varphi_{i+3}\right).$$

Again,

$$\frac{1}{2}\varphi'^{[3]}_{-,i+\frac{1}{2}} + \frac{1}{2}\varphi'^{[3]}_{+,i+\frac{1}{2}} = \varphi'_{i+\frac{1}{2}} + O\left(h^4\right),$$

and a fourth-order WENO reconstruction of $\varphi'\left(x_{i+\frac{1}{2}}\right)$ is

$$\varphi'^{[4]}_{i+1/2} = w^-_{i+\frac{1}{2}}\varphi'^{[3]}_{-,i+\frac{1}{2}} + w^+_{i+\frac{1}{2}}\varphi'^{[3]}_{+,i+\frac{1}{2}}$$

where the weights are of the form (4) with $c^{\pm} = 1/2$ and $S^-_{i+1/2} = S_{i+1/2}\left[-2,0\right]$ and $S^+_{i+1/2} = S_{i+1/2}\left[0,2\right]$.

The reconstruction of $\varphi'_{i+1/2}$ from $\varphi_{i+1/2}$.
Repeating the above procedure, this time with three quadratic interpolants

$$\tilde{\varphi}^{[2]}_{-}\left(x\right) = \varphi_{i+\frac{1}{2}} + \frac{1}{h}\left(\Delta^-\varphi_{i+\frac{1}{2}}\right)\left(x - x_{i+\frac{1}{2}}\right) + \frac{1}{2h^2}\left(\Delta^+\Delta^-\varphi_{i+\frac{1}{2}}\right)\left(x - x_{i+\frac{1}{2}}\right)\left(x - x_{i+\frac{3}{2}}\right) + O\left(h^3\right),$$
$$\tilde{\varphi}^{[2]}_{0}\left(x\right) = \varphi_{i+\frac{1}{2}} + \frac{1}{2h}\left(\Delta^0\varphi_{i+\frac{1}{2}}\right)\left(x - x_{i+\frac{1}{2}}\right) + \frac{1}{2h^2}\left(\Delta^+\Delta^-\varphi_{i+\frac{1}{2}}\right)\left(x - x_{i-\frac{1}{2}}\right)\left(x - x_{i+\frac{3}{2}}\right) + O\left(h^3\right),$$
$$\tilde{\varphi}^{[2]}_{+}\left(x\right) = \varphi_{i+\frac{1}{2}} + \frac{1}{h}\left(\Delta^+\varphi_{i+\frac{1}{2}}\right)\left(x - x_{i+\frac{1}{2}}\right) + \frac{1}{2h^2}\left(\Delta^+\Delta^+\varphi_{i+\frac{1}{2}}\right)\left(x - x_{i+\frac{1}{2}}\right)\left(x - x_{i+\frac{3}{2}}\right) + O\left(h^3\right),$$

results with

$$\frac{1}{6}\tilde{\varphi}'^{[2]}_{-,i+\frac{1}{2}} + \frac{2}{3}\tilde{\varphi}'^{[2]}_{0,i+\frac{1}{2}} + \frac{1}{6}\tilde{\varphi}'^{[2]}_{+,i+\frac{1}{2}} = \varphi'_{i+\frac{1}{2}} + O\left(h^4\right),$$

where

$$\tilde{\varphi}'^{[2]}_{-,i+\frac{1}{2}} = \frac{1}{2h}(\varphi_{i-\frac{3}{2}} - 4\varphi_{i-\frac{1}{2}} + 3\varphi_{i+\frac{1}{2}}), \quad \tilde{\varphi}'^{[2]}_{0,i+\frac{1}{2}} = \frac{1}{2h}(\varphi_{i+\frac{3}{2}} - \varphi_{i-\frac{1}{2}}),$$
$$\tilde{\varphi}'^{[2]}_{+,i+\frac{1}{2}} = \frac{1}{2h}(-3\varphi_{i+\frac{1}{2}} + 4\varphi_{i+\frac{3}{2}} - \varphi_{i+\frac{5}{2}}).$$

The fourth-order WENO estimate of $\varphi'_{i+1/2}$ is

$$\tilde{\varphi}'^{[4]}_{i+1/2} = w^-_{i+\frac{1}{2}}\tilde{\varphi}'^{[2]}_{-,i+\frac{1}{2}} + w^0_{i+\frac{1}{2}}\tilde{\varphi}'^{[2]}_{0,i+\frac{1}{2}} + w^+_{i+\frac{1}{2}}\tilde{\varphi}'^{[2]}_{+,i+\frac{1}{2}}$$

where the weights w are of the form (4) with $c^- = c^+ = 1/6, c^0 = 2/3$, and the oscillatory indicators $S^-_{i+1/2} = S_{i+1/2}\left[-2,-1\right]$, $S^-_{i+1/2} = S_{i+1/2}\left[-1,0\right]$, and $S^+_{i+1/2} = S_{i+1/2}\left[0,1\right]$.

2.2 A Fifth-Order Scheme

Once again, similarly to the third-order scheme, we need to reconstruct the point-values of φ and φ'. We start with the reconstruction of $\varphi_{i+1/2}$ and $\varphi'_{i+1/2}$ from φ_i. We write sixth-order interpolants as a convex combination of cubic interpolants, $\varphi_-^{[3]}(x)$ and $\varphi_+^{[3]}(x)$ introduced above and

$$\varphi_0^{[3]}(x) = \varphi_i + \frac{1}{h}\left(\Delta^+\varphi_i\right)(x-x_i) + \frac{1}{2h^2}\left(\Delta^+\Delta^-\varphi_i\right)(x-x_i)(x-x_{i+1}) + \frac{1}{6h^3}\left(\Delta^+\Delta^-\Delta^+\varphi_i\right)(x-x_i)(x-x_{i+1})(x-x_{i+2}) + O\left(h^4\right).$$

In this case

$$\frac{3}{16}\varphi^{[3]}_{-,i+\frac{1}{2}} + \frac{5}{8}\varphi^{[3]}_{0,i+\frac{1}{2}} + \frac{3}{16}\varphi^{[3]}_{+,i+\frac{1}{2}} = \varphi_{i+\frac{1}{2}} + O\left(h^6\right),$$

where

$$\begin{aligned}
\varphi^{[3]}_{-,i+\frac{1}{2}} &= \frac{1}{16}(\varphi_{i-2} - 5\varphi_{i-1} + 15\varphi_i + 5\varphi_{i+1}),\\
\varphi^{[3]}_{0,i+\frac{1}{2}} &= \frac{1}{16}(-\varphi_{i-1} + 9\varphi_i + 9\varphi_{i+1} - \varphi_{i+2}),\\
\varphi^{[3]}_{+,i+\frac{1}{2}} &= \frac{1}{16}(5\varphi_i + 15\varphi_{i+1} - 5\varphi_{i+2} + \varphi_{i+3}).
\end{aligned}$$

In a similar way,

$$-\frac{9}{80}\varphi'^{[3]}_{-,i+\frac{1}{2}} + \frac{49}{40}\varphi'^{[3]}_{0,i+\frac{1}{2}} - \frac{9}{80}\varphi'^{[3]}_{+,i+\frac{1}{2}} = \varphi'_{i+1/2} + O\left(h^6\right),$$

where

$$\begin{aligned}
\varphi'^{[3]}_{-,i+\frac{1}{2}} &= \frac{1}{24h}(\varphi_{i-2} - 3\varphi_{i-1} - 21\varphi_i + 23\varphi_{i+1}),\\
\varphi'^{[3]}_{0,i+\frac{1}{2}} &= \frac{1}{24h}(\varphi_{i-1} - 27\varphi_i + 27\varphi_{i+1} - \varphi_{i+2}),\\
\varphi'^{[3]}_{+,i+\frac{1}{2}} &= \frac{1}{24h}(-23\varphi_i + 21\varphi_{i+1} + 3\varphi_{i+2} - \varphi_{i+3}).
\end{aligned}$$

The sixth-order WENO estimates for $\varphi_{i+1/2}$ and $\varphi'_{i+1/2}$ are

$$\begin{aligned}
\varphi^{[6]}_{i+\frac{1}{2}} &= w^-_{i+\frac{1}{2}}\varphi^{[3]}_{-,i+\frac{1}{2}} + w^0_{i+\frac{1}{2}}\varphi^{[3]}_{0,i+\frac{1}{2}} + w^+_{i+\frac{1}{2}}\varphi^{[3]}_{+,i+\frac{1}{2}},\\
\varphi'^{[6]}_{i+\frac{1}{2}} &= w'^-_{i+\frac{1}{2}}\tilde{\varphi}'^{[3]}_{-,i+\frac{1}{2}} + w'^0_{i+\frac{1}{2}}\tilde{\varphi}'^{[3]}_{0,i+\frac{1}{2}} + w'^+_{i+\frac{1}{2}}\tilde{\varphi}'^{[3]}_{+,i+\frac{1}{2}},
\end{aligned}$$

where the weights for φ are given by (4), with $c_- = c_+ = 3/16, c_0 = 5/8$ and the oscillatory indicators are $S^-_{i+1/2} = S_{i+1/2}[-2,0]$, $S^0_{i+1/2} = S_{i+1/2}[-1,1]$ and $S^+_{i+1/2} = S_{i+1/2}[0,2]$. The negative weights for φ' require special treatment (see [22] for details). Following [22] we split the positive and negative weights in the following way: first, we set $\gamma_-^- = \gamma_-^+ = 9/40$, $\gamma_-^0 = 49/40$ and $\gamma_+^- = \gamma_+^+ = 9/80$, $\gamma_+^0 = 49/20$. Then, For $k, l \in \{-, 0, +\}$, set $\sigma_\pm = \sum_k \gamma_\pm^k$ so that similarly to (4),

$$\alpha^k_{\pm,i+\frac{1}{2}} = \frac{\gamma^k_\pm}{\sigma_\pm \left(\epsilon + S^k_{i+\frac{1}{2}}\right)^p}$$

and

$$w'^k_{i+\frac{1}{2}} = \sigma_+ \frac{\alpha^k_{+,i+\frac{1}{2}}}{\sum_l \alpha^l_{+,i+\frac{1}{2}}} - \sigma_- \frac{\alpha^k_{-,i+\frac{1}{2}}}{\sum_l \alpha^l_{-,i+\frac{1}{2}}}.$$

Because $\varphi^{[3]}_{i+1/2}$ and $\varphi'^{[3]}_{i+1/2}$ are defined on the same stencils, they use the same smoothness measures $S_{i+1/2}$.

All that is left is the reconstruction of $\varphi'_{i+1/2}$ from $\varphi_{i+1/2}$. In this case a sixth-order approximation to $\varphi'_{i+1/2}$ requires a weighted sum of four cubic interpolants. This reconstruction is similar to the previous ones. We skip the details and summarize the result:

$$\tilde{\varphi}'^{[6]}_{i+\frac{1}{2}} = w^-_{i+\frac{1}{2}} \tilde{\varphi}'^{[3]}_{-,i+\frac{1}{2}} + w^{0-}_{i+\frac{1}{2}} \tilde{\varphi}'^{[3]}_{0-,i+\frac{1}{2}} + w^{0+}_{i+\frac{1}{2}} \tilde{\varphi}'^{[3]}_{0+,i+\frac{1}{2}} + w^+_{i+\frac{1}{2}} \tilde{\varphi}'^{[3]}_{+,i+\frac{1}{2}},$$

where

$$\begin{aligned}
\tilde{\varphi}'^{[3]}_{-,i+\frac{1}{2}} &= \frac{1}{6h}(-2\varphi_{i-\frac{5}{2}} + 9\varphi_{i-\frac{3}{2}} - 18\varphi_{i-\frac{1}{2}} + 11\varphi_{i+\frac{1}{2}}),\\
\tilde{\varphi}'^{[3]}_{0-,i+\frac{1}{2}} &= \frac{1}{6h}(\varphi_{i-\frac{3}{2}} - 6\varphi_{i-\frac{1}{2}} + 3\varphi_{i+\frac{1}{2}} + 2\varphi_{i+\frac{3}{2}}),\\
\tilde{\varphi}'^{[3]}_{0+,i+\frac{1}{2}} &= \frac{1}{6h}(-2\varphi_{i-\frac{1}{2}} - 3\varphi_{i+\frac{1}{2}} + 6\varphi_{i+\frac{3}{2}} - \varphi_{i+\frac{5}{2}}),\\
\tilde{\varphi}'^{[3]}_{+,i+\frac{1}{2}} &= \frac{1}{6h}(-11\varphi_{i+\frac{1}{2}} + 18\varphi_{i+\frac{3}{2}} - 9\varphi_{i+\frac{5}{2}} + 2\varphi_{i+\frac{7}{2}}).
\end{aligned}$$

Here, $c_- = c_+ = 1/20, c_0- = c_0+ = 9/20$, $S^-_{i+1/2} = S_{i+1/2}[-3,-1]$, $S^{0-}_{i+1/2} = S_{i+1/2}[-2,0]$, $S^{0+}_{i+1/2} = S_{i+1/2}[-1,1]$ and $S^+_{i+1/2} = S_{i+1/2}[0,2]$.

3 Numerical Examples

In all our numerical simulations, the ODE solvers we use are the non-linear fourth-order Strong-Stability Preserving Runge-Kutta (SSP-RK) methods of [6].

We start by testing the accuracy of our new CWENO methods when approximating the solution of the linear advection equation, $\varphi_t + \varphi_x = 0$. The initial data is taken as $\varphi(x,0) = \sin^4(\pi x)$, the mesh ratio $\lambda = 0.9$ and the time $T = 4$. The results obtained with the fifth-order method of §2.2 are shown in Table 1.

Table 1. Error and convergence rate for linear advection with initial condition $\varphi(x,0) = \sin^4(\pi x)$

N	L_1 error	L_1 order
50	5.03×10^{-2}	–
100	8.36×10^{-5}	9.23
200	2.56×10^{-6}	5.03
400	8.24×10^{-8}	4.96
800	2.99×10^{-9}	4.78

Next, we test the CWENO methods with two nonlinear Hamiltonians: a convex Hamiltonian $\varphi_t + \frac{1}{2}(\varphi_x + 1)^2 = 0$ and a non-convex Hamiltonian $\varphi_t - \cos(\varphi_x + 1) = 0$. The interval is $[0, 2]$, the boundary conditions are periodic and the initial conditions for both Hamiltonians are taken as $\varphi(x, 0) = -\cos(\pi x)$. The exact solution to both problems is smooth until $t \approx 1/\pi^2$, after which a singularity forms. A second singularity forms in the non-convex H example at $t \approx 1.29/\pi^2$.

The results of the accuracy test with the fifth-order method are shown in Table 2, and the solution at time $T = 1.5/\pi$ is plotted in Figure 1. Following [9] the errors in Table 2 after the formation of the singularity are computed at a distance of 0.1 away from any singularities.

Table 2. L_1 Error and convergence rate estimates for convex and non-convex Hamiltonians. *top*: $T = 0.5/\pi^2$, *bottom*: $T = 1.5/\pi^2$. $\lambda = 0.3$

N	convex L_1 error	convex L_1 order	non-convex L_1 error	non-convex L_1 order
50	6.35×10^{-6}	–	4.17×10^{-5}	–
100	1.62×10^{-7}	5.30	1.49×10^{-6}	4.81
200	5.72×10^{-9}	4.82	4.19×10^{-8}	5.15
400	2.73×10^{-10}	4.39	1.34×10^{-8}	4.97
800	1.45×10^{-11}	4.23	4.20×10^{-8}	4.99

N	convex L_1 error	convex L_1 order	non-convex L_1 error	non-convex L_1 order
50	2.12×10^{-4}	–	2.56×10^{-5}	–
100	1.03×10^{-5}	4.37	7.80×10^{-7}	5.03
200	9.68×10^{-8}	6.73	1.70×10^{-8}	5.52
400	6.20×10^{-10}	7.29	5.02×10^{-10}	5.08
800	1.90×10^{-11}	5.03	1.71×10^{-11}	4.88

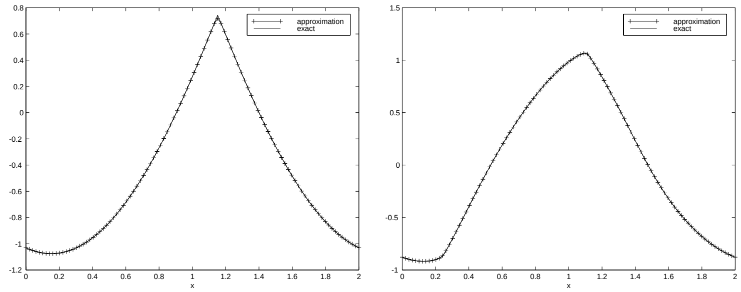

Fig. 1. *left:* Convex Hamiltonian *right:* non-convex Hamiltonian at $T = \frac{1.5}{\pi^2}$ compared with the exact solution, $N = 100$.

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