On a Chemotaxis Model with Saturated Chemotactic Flux

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Abstract

We propose a PDE chemotaxis model, which can be viewed as a regularization of the Patlak-Keller-Segel (PKS) system. Our modification is based on a fundamental physical property of the chemotactic flux—its boundedness. This means that the cell velocity is proportional to the magnitude of the chemoattractant gradient only when the latter is small, while when the chemoattractant gradient tends to infinity the cell velocity saturates. Unlike the original PKS system, the solutions of the modified model do not blow up in either finite or infinite time in any number of spatial dimensions, thus making it possible to use bounded spiky steady states to model cell aggregation. After obtaining local and global existence results, we use the local and global bifurcation theories to show the existence of one-dimensional spiky steady states; we also study the stability of bifurcating steady states. Finally, we numerically verify these analytical results, and then demonstrate that solutions of the two-dimensional model with nonlinear saturated chemotactic flux typically develop very complicated spiky structures.

1 Introduction

Chemotaxis is a phenomenon of collective movement of microorganisms in the direction of increasing chemical concentration. The simplest and classical PDE model of chemotaxis was introduced in [28] and [20, 21]. In this model, which we will refer to as the Patlak-Keller-Segel (PKS) model, the cell density $\rho(\mathbf{x}, t)$ and the chemoattractant concentration $c(\mathbf{x}, t)$ are governed by the following system of convection-diffusion-reaction equations:

$$\begin{cases} \rho_t + \chi \boldsymbol{\nabla} \cdot (\rho \, \nabla c) = \nu \Delta \rho, \\ c_t = \Delta c - \gamma_c c + \gamma_\rho \rho. \end{cases} \tag{1.1}$$

Here, $\mathbf{x} = (x_1, \ldots, x_d)^T$ is a d-dimensional spatial variable, t is time, γ_c, γ_ρ and ν are positive constants, and χ is a nonnegative chemosensitivity constant (by rescaling the spatial variable $\mathbf{x}$ one can assume, without loss of generality, that the diffusion coefficient of c is equal to 1).

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The most important phenomenon in chemotaxis is the aggregation of cells, namely, the concentration of ρ as t increases. The biological phenomenon behind this is that even when the cells are initially distributed almost evenly over the habitat Ω, later on they, being chemotactic to a chemical released by themselves, start to aggregate in a number of "centers" ([1, 4–6, 9, 32, 37]). In the literature, two ways of mathematically modeling cell aggregation have been proposed: (i) solutions of (1.1) blow up in finite time and at the blowup time, ρ is a linear combination of several δ-functions, plus a regular part—see [8, 14, 25]; (ii) time-dependent solutions converge to *bounded* but *spiky* steady states—see [24, 26, 35, 36] for such results on several modifications (regularizations) of (1.1). See also the survey papers [18, 19].

While the blowup and the formation of the δ-function are not an unreasonable *modeling* of the cell aggregation phenomenon, they create enormous, and also unnecessary, challenges to numerics and analysis (be it formal or rigorous). Thus we prefer chemotaxis models that only have bounded, global-in-time solutions that approach spiky steady states as time increases. Such models may be obtained by regularizing the PKS system. A variety of regularizations has been proposed over the past decades, see the review papers [17–19], the monograph [31] and references therein.

In this paper, we consider a regularization of the PKS model, which is based on a fundamental physical property of the chemotactic flux—its boundedness (this feature is almost always lost in weakly nonlinear, small gradients expansions, underlying the derivation of most continuum models). To derive the modified system we replace the linear chemotactic flux $\rho \nabla c$ by a nonlinear saturated one, $\rho \mathbf{Q}(\nabla c)$, which is proportional to the magnitude of the chemoattractant gradient only when the latter is small and is bounded when the chemoattractant gradient tends to infinity. The regularized model then reads:

$$\begin{cases} \rho_t + \chi \boldsymbol{\nabla} \cdot (\rho \, \mathbf{Q} \, (\nabla c)) = \nu \Delta \rho, \\ c_t = \Delta c - \gamma_c c + \gamma_\rho \rho, \end{cases} \tag{1.2}$$

where a smooth saturated chemotactic flux $\mathbf{Q}(u_1, \ldots, u_d) = \mathbf{Q}(\mathbf{u}) = (Q_1(\mathbf{u}), \ldots, Q_d(\mathbf{u}))$ satisfies the following properties:

$$\mathbf{Q}(\mathbf{0}) = \mathbf{0}, \quad |Q_i| \le C_i, \quad \frac{\partial Q_i}{\partial u_i} > 0 \;\; \forall \mathbf{u}, \;\; \forall i = 1, \ldots, d, \tag{1.3}$$

where C_i are constants. Without loss of generality, one may assume that $\max_{1 \le i \le d} C_i = 1$.

The synthesized form of the saturated flux is a Pade approximate which connects universal features present at both very small and very large gradients. There is a certain arbitrariness in the choice of the chemotactic flux function $\mathbf{Q}$. A typical example of a saturated chemotactic flux, which is used in all of our numerical experiments, is

$$\mathbf{Q}(\nabla c) = \begin{cases} \nabla c, & \text{if } |\nabla c| \le s^*, \\ \left(\dfrac{|\nabla c| - s^*}{\sqrt{1 + (|\nabla c| - s^*)^2}} + s^* \right) \dfrac{\nabla c}{|\nabla c|}, & \text{otherwise,} \end{cases} \tag{1.4}$$

where s^* is a switching parameter, which defines a small gradient values, for which the system (1.2) reduces to the original PKS system (1.1) so that the effect of saturated chemotactic flux

is felt at large gradient regimes only. Note that when $s^* = 0$, the flux (1.4) becomes a mean curvature type flux:

$$\mathbf{Q}(\nabla c) = \frac{\nabla c}{\sqrt{1 + |\nabla c|^2}}. \tag{1.5}$$

The chemotaxis system (1.2) studied here is similar to the regularization (M7) from [17], where a specific form of the bounded function $\mathbf{Q}$ was considered (see also [2, 27, 33]). In [16], a more general type of the cell density equation was considered and an a-priori L^∞-bound on its solutions was established. The result from [16] also applies to the ρ-equation in (1.2). In §3, we give an alternative proof of the L^∞ bounds on both ρ and c, which also applies to the time independent version of the system (1.2). We then proceed in §4 with the proof of the local existence result for a more general chemotaxis system (4.1)—to the best of our knowledge, this general local existence result is new. This result directly applies to the system (1.2) and together with the a-priori bounds obtained in §3 leads to the global existence of the solution of (1.2). This result is supported by our numerical experiments, in which we compare the blowing up solutions of the PKS system (1.1) with the spiky, but bounded solutions of (1.2), (1.4).

In §5, we study one-dimensional (1-D) steady-state solutions and use both the local [11] and global [30, 34] bifurcation theories to investigate existence of nontrivial steady states. We then show that when the chemotaxis coefficient χ is large, the ρ component of the steady state is spiky, while the c component is close to a bounded function given explicitly; we also use [10] to study the stability of bifurcating steady states. Our extensive numerical experiments support the analytical findings. Two-dimensional (2-D) steady states are considered in §6, where formation and stability of the spiky solutions are illustrated numerically.

All numerical results reported in this paper have been obtained using a second-order positivity preserving central-upwind scheme developed in [7]. A brief description of the 2-D version of the scheme is presented in Appendix A.

2 Several Basic Definitions and Results

In this section, we provide the reader with a briefly review of several basic definitions and results, which will be used afterward.

We begin with Young's inequality, which states that for all positive real numbers a, b, p and q, such that $1/p + 1/q = 1$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{2.1}$$

This inequality also gives rise to the so-called Young's inequality with ε (valid for any $\varepsilon > 0$),

$$ab \leq \frac{\varepsilon a^2}{2} + \frac{b^2}{2\varepsilon} \tag{2.2}$$

We will use the following Sobolev and Hölder spaces:

Definition 2.1 (Sobolev and Hölder Spaces)
   • *For $p \geq 1$, the Sobolev space $W^{2,p}(\Omega)$ consists of functions in $L^p(\Omega)$ whose weak derivatives up to order 2 exist and are in $L^p(\Omega)$;*

- *For finite $T > 1$ and $\alpha \in (0,1)$, the anisotropic Hölder space $C^{\alpha,\alpha/2}(\overline{\Omega} \times [0,T])$ consists of functions defined on $\overline{\Omega} \times [0,T]$ that are Hölder continuous in $\mathbf{x}$ and t with exponents α and $\alpha/2$, respectively;*
- *$C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,T])$ is the space of functions whose partial derivatives up to order 2 in $\mathbf{x}$, order 1 in t are continuous on $\overline{\Omega} \times [0,T]$ with the highest derivatives in $C^{\alpha,\alpha/2}(\overline{\Omega} \times [0,T])$.*

We will use notation such as $C^{\alpha,\alpha/2}(\overline{\Omega} \times (0,T])$, by which we mean the functions belonging to $C^{\alpha,\alpha/2}(\overline{\Omega} \times [\varepsilon,T])$ for every small $\varepsilon > 0$.

Lemma 2.1 (Gagliardo-Ladyženskaja-Nirenberg Inequality) [22, p. 63]
Let Ω be a bounded domain in $\mathbb{R}^d$ with a smooth boundary. Then for any $u \in H^1(\Omega)$

$$\|u - \bar{u}\|^2_{L^2(\Omega)} \le M\|\nabla u\|^{2a}_{L^2(\Omega)}\|u\|^{2(1-a)}_{L^1(\Omega)}, \tag{2.3}$$

where $\bar{u} = \frac{1}{|\Omega|}\int_\Omega u\,d\mathbf{x},\ a = d/(d+2)$, and M is a constant depending only on d and Ω.

Corollary 2.1 *Let Ω be a bounded domain in $\mathbb{R}^d$ with a smooth boundary. Then for any $u \in H^1(\Omega)$ and any $\varepsilon > 0$*

$$\|u\|^2_{L^2(\Omega)} \le \varepsilon\|\nabla u\|^2_{L^2(\Omega)} + K\left(1 + \varepsilon^{-\frac{d}{2}}\right)\|u\|^2_{L^1(\Omega)}, \tag{2.4}$$

where $K \ge 1$ depends only on d and Ω.

Proof: The result immediately follows from the Gagliardo-Ladyženskaja-Nirenberg inequality (2.3) and Young's inequality (2.1) with $p = 1/a$ and $q = 1/(1-a)$. ∎

3   A Priori Estimates

Consider the following initial-boundary value problem (IBVP):

$$\begin{cases} \rho_t + \chi\boldsymbol{\nabla}\cdot(\rho\,\mathbf{Q}\,(\nabla c)) = \nu\Delta\rho, & \mathbf{x} \in \Omega,\ t > 0, \\ c_t = \Delta c - \gamma_c c + \gamma_\rho\rho, & \mathbf{x} \in \Omega,\ t > 0, \\ \rho(\mathbf{x},0) = \rho_0(\mathbf{x}),\quad c(\mathbf{x},0) = c_0(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \dfrac{\partial\rho}{\partial\mathbf{n}} = \dfrac{\partial c}{\partial\mathbf{n}} = 0, & \mathbf{x} \in \partial\Omega,\ t > 0, \end{cases} \tag{3.1}$$

where Ω is a bounded domain in $\mathbb{R}^d$ with a smooth boundary $\partial\Omega$ and $\mathbf{n}$ is the outer normal vector field on $\partial\Omega$.

We first prove that positive solutions of the above IBVP remain bounded for all times.

Theorem 3.1 *Let $(\rho(\mathbf{x},t), c(\mathbf{x},t))$ be a positive classical solution of the IBVP (3.1) with bounded nonnegative initial data. Then, for all $\mathbf{x} \in \bar{\Omega}$ and $t \ge 0$,*

$$\rho(\mathbf{x},t) \le C\left(1 + \frac{\chi}{\nu}\right)^d \max\left\{\|\rho_0\|_{L^\infty(\Omega)}, \|\rho_0\|_{L^1(\Omega)}\right\}, \tag{3.2}$$

$$c(\mathbf{x},t) \le \|c_0\|_{L^\infty(\Omega)} + \frac{\gamma_\rho}{\gamma_c}C\left(1 + \frac{\chi}{\nu}\right)^d \max\left\{\|\rho_0\|_{L^\infty(\Omega)}, \|\rho_0\|_{L^1(\Omega)}\right\}, \tag{3.3}$$

where $C = C(d,\Omega)$ is a constant, which depends on d and Ω only.

Proof: We begin by multiplying the first equation in (3.1) by ρ^{s-1} ($s \geq 2$) and integrating over Ω. Then, integrating by parts, applying the chain rule, using the boundedness of $|\mathbf{Q}|$ from (1.3) and the inequality (2.2) with $\varepsilon = \dfrac{2\nu}{s\chi}$ yields

$$\frac{1}{s}\frac{d}{dt}\int_\Omega \rho^s \, d\mathbf{x} = -\nu \int_\Omega \nabla\rho \cdot \nabla\left(\rho^{s-1}\right) \, d\mathbf{x} + \chi \int_\Omega \rho\mathbf{Q}(\nabla c)\cdot\nabla\left(\rho^{s-1}\right)\, d\mathbf{x}$$

$$\leq -\frac{4\nu(s-1)}{s^2}\int_\Omega \left|\nabla\left(\rho^{\frac{s}{2}}\right)\right|^2 \, d\mathbf{x} + \frac{2\chi(s-1)}{s}\int_\Omega \rho^{\frac{s}{2}}\left|\nabla\left(\rho^{\frac{s}{2}}\right)\right| \, d\mathbf{x}$$

$$\leq -\frac{4\nu(s-1)}{s^2}\int_\Omega \left|\nabla\left(\rho^{\frac{s}{2}}\right)\right|^2 \, d\mathbf{x} + \frac{\chi(s-1)}{s}\int_\Omega \left(\frac{2\nu}{s\chi}\left|\nabla\left(\rho^{\frac{s}{2}}\right)\right|^2 + \frac{s\chi}{2\nu}\rho^s\right) \, d\mathbf{x}$$

$$\leq -\frac{2\nu(s-1)}{s^2}\int_\Omega \left|\nabla\left(\rho^{\frac{s}{2}}\right)\right|^2 \, d\mathbf{x} + \frac{\chi^2(s-1)}{2\nu}\int_\Omega \rho^s \, d\mathbf{x}. \tag{3.4}$$

The last term in (3.4) is estimated using the inequality (2.4) with $u = \rho^{\frac{s}{2}}$ and ε such that

$$\frac{\chi^2(s-1)}{2\nu} = \frac{2\nu(s-1)}{s^2\varepsilon} - \frac{\chi^2(s-1)}{2\nu} \quad\Longleftrightarrow\quad \varepsilon = 2\left(\frac{\nu}{s\chi}\right)^2.$$

This results in

$$\frac{\chi^2(s-1)}{2\nu}\int_\Omega \rho^s \, d\mathbf{x} \leq \frac{2\nu(s-1)}{s^2}\left\|\nabla\left(\rho^{\frac{s}{2}}\right)\right\|^2_{L^2(\Omega)} + \frac{2\nu(s-1)K(1+\varepsilon^{-\frac{d}{2}})}{s^2\varepsilon}\left\|\rho^{\frac{s}{2}}\right\|^2_{L^1(\Omega)}$$
$$-\frac{\chi^2(s-1)}{2\nu}\int_\Omega \rho^s \, d\mathbf{x}. \tag{3.5}$$

Substituting (3.5) into (3.4), we obtain

$$\frac{d}{dt}\int_\Omega \rho^s \, d\mathbf{x} \leq -\frac{\chi^2 s(s-1)}{2\nu}\int_\Omega \rho^s \, d\mathbf{x} + \frac{\chi^2 s(s-1)K\left(1+\left(\frac{s\chi}{\sqrt{2}\nu}\right)^d\right)}{\nu}\left(\int_\Omega \rho^{\frac{s}{2}} \, d\mathbf{x}\right)^2.$$

We then fix $T \in (0,\infty)$, multiply both sides of the last inequality by the integrating factor $\exp\left(\dfrac{\chi^2 s(s-1)}{2\nu}t\right)$ and integrate over the time interval $[0,t]$ for $t \in [0,T]$ to obtain the following estimate:

$$\int_\Omega \rho^s(\mathbf{x},t) \, d\mathbf{x} \leq \int_\Omega \rho_0^s(\mathbf{x}) \, d\mathbf{x} + 2K\left(1+\frac{s\chi}{\nu}\right)^d \sup_{0\leq t\leq T}\left(\int_\Omega \rho^{\frac{s}{2}}(\mathbf{x},t) \, d\mathbf{x}\right)^2. \tag{3.6}$$

Let us now define the function

$$M(s) = \max\left\{\|\rho_0\|_{L^\infty(\Omega)}, \sup_{0\leq t\leq T}\left(\int_\Omega \rho^s \, d\mathbf{x}\right)^{\frac{1}{s}}\right\}, \tag{3.7}$$

which satisfies (from (3.6)):

$$M(s) \leq \left(\widetilde{K} \left(1 + \frac{s\chi}{\nu} \right)^d \right)^{\frac{1}{s}} M(s/2), \quad \forall s \geq 2,$$

where the constant $\widetilde{K}$ depends on d and Ω only. Taking $s = 2^k$, $k = 1, 2, \ldots$, we obtain

$$M(2^k) \leq \widetilde{K}^{2^{-k}} \left(1 + \frac{2^k \chi}{\nu} \right)^{2^{-k}d} M(2^{k-1})$$

$$\leq \widetilde{K}^{2^{-k} + 2^{-(k-1)}} \left(1 + \frac{2^k \chi}{\nu} \right)^{2^{-k}d} \left(1 + \frac{2^{k-1} \chi}{\nu} \right)^{2^{-(k-1)}d} M(2^{k-2}) \leq \ldots$$

$$\leq \widetilde{K}^{2^{-k} + \cdots + 2^{-1}} \left(1 + \frac{2^k \chi}{\nu} \right)^{2^{-k}d} \cdots \left(1 + \frac{2\chi}{\nu} \right)^{2^{-1}d} M(1)$$

$$\leq \widetilde{K} \left(1 + \frac{\chi}{\nu} \right)^{(2^{-k} + \cdots + 2^{-1})d} 2^{(k2^{-k} + \cdots + 2^{-1})d} M(1) \leq C \left(1 + \frac{\chi}{\nu} \right)^d M(1),$$

where C is a constant, which depends only on d and Ω. Sending $k \to \infty$, we conclude that

$$\|\rho(\cdot, t)\|_{L^\infty(\Omega)} \leq C \left(1 + \frac{\chi}{\nu} \right)^d M(1), \quad \forall t \geq 0. \tag{3.8}$$

Finally, we note the total mass of the cells remains constant in time (this can be verified by integrating the first equation in (3.1) over Ω), and therefore

$$M(1) = \max \left\{ \|\rho_0\|_{L^\infty(\Omega)}, \|\rho_0\|_{L^1(\Omega)} \right\}, \tag{3.9}$$

and the estimate (3.2) for the cell density $\rho(\mathbf{x}, t)$ follows from (3.8), (3.9).

To obtain a bound on chemoattractant concentration $c(\mathbf{x}, t)$, we compare it with the solution of the following initial value problem (IVP):

$$\begin{cases} \dfrac{dw}{dt} = -\gamma_c w + \gamma_\rho C \left(1 + \dfrac{\chi}{\nu} \right)^d M(1), \\ w(0) = \|c_0\|_{L^\infty(\Omega)}, \end{cases}$$

which can be easily calculated. The comparison principle then yields

$$0 \leq c(\mathbf{x}, t) \leq w(t) = e^{-\gamma_c t} \|c_0\|_{L^\infty(\Omega)} + \left(1 - e^{-\gamma_c t} \right) \frac{\gamma_\rho}{\gamma_c} C \left(1 + \frac{\chi}{\nu} \right)^d M(1)$$

$$\leq \|c_0\|_{L^\infty(\Omega)} + \frac{\gamma_\rho}{\gamma_c} C \left(1 + \frac{\chi}{\nu} \right)^d \max \left\{ \|\rho_0\|_{L^\infty(\Omega)} \|\rho_0\|_{L^1(\Omega)} \right\}, \tag{3.10}$$

and the proof of Theorem 3.1 is now complete. ∎

Remark 3.1 It is simpler to show that a positive steady state solution $(\rho(\mathbf{x}), c(\mathbf{x}))$ of the system (3.1) satisfies

$$\rho(\mathbf{x}) \leq C(d, \Omega) \left(1 + \frac{\chi}{\nu} \right)^d \|\rho\|_{L^1(\Omega)},$$

$$c(\mathbf{x}) \leq \frac{\gamma_\rho}{\gamma_c} C(d, \Omega) \left(1 + \frac{\chi}{\nu} \right)^d \|\rho\|_{L^1(\Omega)},$$

for all $\mathbf{x} \in \bar{\Omega}$.

Remark 3.2 By using the L^∞-bounds established in Theorem 3.1, parabolic boundary L^p-estimates and Schauder estimates (see, e.g., [22]), one can obtain that ρ_t, c_t and all spatial partial derivatives of ρ and c up to order two are bounded on $\bar{\Omega} \times [0, \infty)$.

4   Existence Results

In this section, we consider the IBVP for a more general chemotaxis system,

$$
\begin{cases}
\rho_t + \boldsymbol{\nabla} \cdot (\boldsymbol{P}(\rho, c, \nabla c)) = \nu \Delta \rho + R(\rho, c), & \mathbf{x} \in \Omega, \ t > 0, \\
c_t = \Delta c + S(\rho, c), & \mathbf{x} \in \Omega, \ t > 0, \\
\rho(\mathbf{x}, 0) = \rho_0(\mathbf{x}), \quad c(\mathbf{x}, 0) = c_0(\mathbf{x}), & \mathbf{x} \in \Omega, \\
\dfrac{\partial \rho}{\partial \mathbf{n}} = \dfrac{\partial c}{\partial \mathbf{n}} = 0, & \mathbf{x} \in \partial\Omega, \ t > 0,
\end{cases}
\tag{4.1}
$$

for which we shall establish a local existence result. This result together with the a-priori estimates proved in §3 will lead to a global existence result for the IBVP (3.1).

4.1   Notations, Definitions, Introductory Results

We will use the semigroup theory, for which [13] and [29] are good references. Let $X_1 = C(\overline{\Omega})$ and $X_2 = L^p(\Omega)$. We define the linear operator $A_1 = -\nu\Delta + I$ in X_1 with the domain $\mathscr{D}(A_1) = \left\{ u \in W^{2,q}(\Omega), \ \forall \ q > d \ \middle| \ \Delta u \in C(\overline{\Omega}), \ \frac{\partial u}{\partial \mathbf{n}}\big|_{\partial\Omega} = 0 \right\}$ and the linear operator $A_2 = -\Delta + I$ in X_2 with the domain $\mathscr{D}(A_2) = \left\{ u \in W^{2,p}(\Omega) \ \middle| \ \frac{\partial u}{\partial \mathbf{n}}\big|_{\partial\Omega} = 0 \right\}$. Then, if the boundary $\partial\Omega$ is $C^{2+\omega}$ smooth for some $\omega > 0$, $-A_i$ is a generator of an analytic semigroup in X_i, denoted by $e^{-A_i t}$, $i = 1, 2$. For any $u_0 \in X_i$, $e^{-A_i t} u_0$ is the solution of the linear parabolic equation $u_t + A_i u = 0$ on $\Omega \times (0, \infty)$ with the homogeneous Neuman boundary condition and the initial condition $u(\mathbf{x}, 0) = u_0(\mathbf{x})$.

Since the smallest eigenvalue of both A_1 and A_2 is 1 and the operators are the generators of the analytic semigroups, we can define their fractional powers A_i^a for $a \geq 0$, with domain $\mathscr{D}(A_i^a)$ and target space X_i. The basic facts are:

(i) The larger a is, the smaller the domain $\mathscr{D}(A_i^a)$ is, with $\mathscr{D}(A_i^0) = X_i$;

(ii) $A_i^a A_i^b = A_i^{a+b}$ on $\mathscr{D}(A_i^{a+b})$ for any $b \geq 0$;

(iii) For any $u_0 \in X_i$ and $t > 0$, $e^{-A_i t} u_0$ is in $\mathscr{D}(A_i^a)$ and $A_i^a e^{-A_i t} = e^{-A_i t} A_i^a$ on $\mathscr{D}(A_i^a)$;

(iv) $\mathscr{D}(A_i^a)$ equipped with the norm

$$
\|u\|_{\mathscr{D}(A_i^a)} = \|A_i^a u\|_{X_i}
$$

is a Banach space.

We shall repeatedly use the following facts:

$$
\begin{aligned}
&\mathscr{D}(A_2^a) \text{ is continuously embedded into } C^{1+\alpha}(\overline{\Omega}) \ \ \forall \ 0 \leq \alpha < 2a - 1 - \tfrac{d}{p}, \ a > \tfrac{1}{2}(1 + \tfrac{d}{p}), \\
&\mathscr{D}(A_1^b) \text{ is continuously embedded into } C^\alpha(\overline{\Omega}) \ \ \forall \ 0 \leq \alpha < 2b, \ b \in (0, 1),
\end{aligned}
\tag{4.2}
$$

and that for $i = 1, 2$

$$\|A_i^a e^{-A_i t}\|_{X_i \to X_i} \leq \frac{C_a^*}{t^a} e^{-\sigma t}, \quad \forall\, t > 0, \ a \geq 0, \tag{4.3}$$

$$\|(e^{-A_i t} - I) u\|_{X_i} \leq \frac{C_a}{a} t^a\, e^{-\sigma t} \|u\|_{\mathscr{D}(A_i^a)}, \quad \forall\, t > 0, \ 0 \leq a \leq 1, \ u \in \mathscr{D}(A_i^a), \tag{4.4}$$

where C_a, C_a^* and $\sigma \in (0, 1)$ are constants.

We now formally convert the IBVP (4.1) into the following system of integral equations:

$$\begin{cases} \rho(\cdot, t) = e^{-A_1 t} \rho_0 - \displaystyle\int_0^t \boldsymbol{\nabla} \cdot e^{-A_1(t-s)} \left(\boldsymbol{P}\left(\rho(\cdot, s), c(\cdot, s), \nabla c(\cdot, s)\right)\right)\, ds \\[2mm] \qquad\qquad + \displaystyle\int_0^t e^{-A_1(t-s)} \left(\rho(\cdot, s) + R(\rho(\cdot, s), c(\cdot, s))\right)\, ds, \\[4mm] c(\cdot, t) = e^{-A_2 t} c_0 + \displaystyle\int_0^t e^{-A_2(t-s)} \left(c(\cdot, s) + S(\rho(\cdot, s), c(\cdot, s))\right)\, ds. \end{cases} \tag{4.5}$$

Note that for technical reasons, in the first integral of the ρ-equation, we write the product of the operator $e^{-A_1(t-s)}$ and the divergence operator in the reversed order. These two operators commute only if everything else in that integral is smooth enough; thus at this moment, (4.1) and (4.5) are only formally equivalent. Nonetheless, in this paper we call a solution of (4.5) a *weak solution* of (4.1).

4.2   Local Existence

We shall first prove that (4.5) has a unique local solution (Lemma 4.1), then show that this local solution is smooth and thus it is a classical solution of IBVP (4.1) (Lemmas 4.2 and 4.3).

Lemma 4.1 *Let $\partial\Omega \in C^{2+\omega}, \omega > 0$, $\boldsymbol{P} \in C^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$ and R and S be locally Lipschitz continuous in (ρ, c). For each fixed $\rho_0 \in C(\overline{\Omega})$, $c_0 \in \mathscr{D}(A_2^a)$ with $\frac{1}{2}(1 + \frac{d}{p}) < a < 1$, there exists a small $\tau_0 > 0$ such that the system (4.5) has a unique solution $(\rho(x, t), c(x, t)) \in C([0, \tau_0], C(\overline{\Omega})) \times C([0, \tau_0], \mathscr{D}(A_2^a))$.*

Proof: For any $\tau_0 \in (0, 1)$, we define the following space:

$$\mathscr{T} = \left\{ (\rho(\cdot, t), c(\cdot, t)) \in C([0, \tau_0], C(\overline{\Omega})) \times C([0, \tau_0], \mathscr{D}(A_2^a)) \right\},$$

and for each fixed $(\rho_0, c_0) \in C(\overline{\Omega}) \times \mathscr{D}(A_2^a)$, we define its subset $\mathscr{S}$ by

$$\mathscr{S} = \left\{ (\rho(\cdot, t), c(\cdot, t)) \in \mathscr{T} \ \middle|\ \|\rho(\cdot, t) - \rho_0\|_{C(\overline{\Omega})} + \|c(\cdot, t) - c_0\|_{\mathscr{D}(A_2^a)} \leq 1, \ 0 \leq t \leq \tau_0 \right\}. \tag{4.6}$$

We also define the operator T_1 by

$$T_1 \begin{pmatrix} \rho \\ c \end{pmatrix} = \begin{pmatrix} \varphi(\cdot, t) \\ \psi(\cdot, t) \end{pmatrix}$$

with the right-hand side (RHS) equal to the RHS of (4.5).

We shall first verify that there exists a small $\tau_0 > 0$ such that $T_1 : \mathscr{S} \to \mathscr{S}$. To this end, we establish several bounds.

For any $(\rho(\cdot, t), c(\cdot, t)) \in \mathscr{S}$, (4.2) and (4.6) imply that there exists a constant K_0 depending only on $\|\rho_0\|_{C(\overline{\Omega})}$ and $\|c_0\|_{\mathscr{D}(A_2^a)}$, such that for any $t \in [0, \tau_0]$

$$\|\rho(\cdot, t)\|_{C(\overline{\Omega})} \leq K_0, \ \|c(\cdot, t)\|_{C^1(\overline{\Omega})} \leq K_0, \ \|\boldsymbol{P}(\rho(\cdot, t), c(\cdot, t), \nabla c(\cdot, t))\|_{C(\overline{\Omega})} \leq K_0,$$

$$\|\rho(\cdot, t) + R(\rho(\cdot, t), c(\cdot, t))\|_{C(\overline{\Omega})} \leq K_0, \|c(\cdot, t) + S(\rho(\cdot, t), c(\cdot, t))\|_{C(\overline{\Omega})} \leq K_0. \tag{4.7}$$

Thus, for any $t \in [0, \tau_0]$ and a fixed $b \in (\frac{1}{2}, 1)$, we use the bounds (4.3) and (4.7) to obtain

$$
\begin{aligned}
\|\varphi - \rho_0\|_{C(\overline{\Omega})} &\leq \|e^{-A_1 t}\rho_0 - \rho_0\|_{C(\overline{\Omega})} + \int_0^t \|A_1^b e^{-A_1(t-s)} \boldsymbol{P}\left(\rho(\cdot, s), c(\cdot, s), \nabla c(\cdot, s)\right)\|_{C(\overline{\Omega})}\, ds \\
&\quad + \int_0^t \|e^{-A_1(t-s)}\left(\rho(\cdot, s) + R(\rho(\cdot, s), c(\cdot, s))\right)\|_{C(\overline{\Omega})}\, ds \\
&\leq \|e^{-A_1 t}\rho_0 - \rho_0\|_{C(\overline{\Omega})} + C\int_0^t \left(\frac{e^{-\sigma(t-s)}}{(t-s)^b}K_0 + e^{-\sigma(t-s)}K_0\right) ds \\
&\leq \|e^{-A_1 t}\rho_0 - \rho_0\|_{C(\overline{\Omega})} + C_1 t^{1-b},
\end{aligned}
\tag{4.8}
$$

and

$$
\begin{aligned}
\|\psi - c_0\|_{\mathscr{D}(A_2^a)} &\leq \|(e^{-A_2 t} - I)c_0\|_{\mathscr{D}(A_2^a)} + \int_0^t \|A_2^a e^{-A_2(t-s)}\left(c(\cdot, s) + S(\rho(\cdot, s), c(\cdot, s))\right)\|_{L^p(\Omega)}\, ds \\
&\leq \|(e^{-A_2 t} - I)A_2^a c_0\|_{L^p(\Omega)} + C\int_0^t (t-s)^{-a}e^{-\sigma(t-s)}\, ds \\
&\leq \|(e^{-A_2 t} - I)A_2^a c_0\|_{L^p(\Omega)} + C_2 t^{1-a},
\end{aligned}
\tag{4.9}
$$

where C_1 and C_2 are constants independent of t and τ_0.

Using the fact that $\lim_{t \to 0^+} e^{-A_1 t}\rho_0 = \rho_0$ in $C(\overline{\Omega})$ and $\lim_{t \to 0^+} e^{-A_2 t}A_2^a c_0 = A_2^a c_0$ in $L^p(\Omega)$, it follows from (4.8) and (4.9) that there exists a small τ_0 (depending only on ρ_0 and c_0) such that $(\varphi, \psi) \in \mathscr{T}$ and

$$\|\varphi - \rho_0\|_{C(\overline{\Omega})} + \|\psi - c_0\|_{\mathscr{D}(A_2^a)} \leq 1, \quad \forall\, 0 \leq t \leq \tau_0,$$

and therefore $(\varphi, \psi) \in \mathscr{S} \implies T_1 : \mathscr{S} \to \mathscr{S}$.

In the following we shall prove that for a sufficiently small τ_0, the operator T_1 is a contracting mapping from $\mathscr{S}$ to $\mathscr{S}$. To this end, we first note that for any $(\rho_1, c_1), (\rho_2, c_2) \in \mathscr{S}$, we have

$$\|\boldsymbol{P}(\rho_1, c_1, \nabla c_1) - \boldsymbol{P}(\rho_2, c_2, \nabla c_2)\|_{C(\overline{\Omega})} \leq K^* \left(\|\rho_1 - \rho_2\|_{C(\overline{\Omega})} + \|c_1 - c_2\|_{\mathscr{D}(A_2^a)}\right),$$

$$\|\rho_1 - \rho_2 + R(\rho_1, c_1) - R(\rho_2, c_2)\|_{C(\overline{\Omega})} \leq K^* \left(\|\rho_1 - \rho_2\|_{C(\overline{\Omega})} + \|c_1 - c_2\|_{\mathscr{D}(A_2^a)}\right),$$

$$\|c_1 - c_2 + S(\rho_1, c_1) - S(\rho_2, c_2)\|_{C(\overline{\Omega})} \leq K^* \left(\|\rho_1 - \rho_2\|_{C(\overline{\Omega})} + \|c_1 - c_2\|_{\mathscr{D}(A_2^a)}\right),$$

where K^* is a constant depending only on $\|\rho_0\|_{C(\overline{\Omega})}$ and $\|c_0\|_{\mathscr{D}(A_2^a)}$.

Similarly to (4.8) and (4.9), we obtain that for any $t \in [0, \tau_0]$ with a sufficiently small $\tau_0 > 0$, the following estimate holds:

$$\|\varphi_1(\cdot, t) - \varphi_2(\cdot, t)\|_{C(\overline{\Omega})} + \|\psi_1(\cdot, t) - \psi_2(\cdot, t)\|_{\mathscr{D}(A_2^a)}$$

$$\leq C \left[\int_0^t \frac{e^{-\sigma(t-s)}}{(t-s)^b} ds + \int_0^t e^{-\sigma(t-s)} ds + \int_0^t \frac{e^{-\sigma(t-s)}}{(t-s)^a} ds \right] \left(\|\rho_1 - \rho_2\|_{C(\overline{\Omega})} + \|c_1 - c_2\|_{\mathscr{D}(A_2^a)} \right)$$

$$\leq C^* \left[\tau_0^{1-b} + \tau_0 + \tau_0^{1-a} \right] \left(\|\rho_1 - \rho_2\|_{C(\overline{\Omega})} + \|c_1 - c_2\|_{\mathscr{D}(A_2^a)} \right)$$

$$\leq \frac{1}{2} \left(\|\rho_1 - \rho_2\|_{C(\overline{\Omega})} + \|c_1 - c_2\|_{\mathscr{D}(A_2^a)} \right).$$

We then use the contracting mapping theorem to conclude that there exists a unique local solution of the system (4.5) in $C([0, \tau_0], C(\overline{\Omega})) \times C([0, \tau_0], \mathscr{D}(A_2^a))$. ∎

Lemma 4.2 *Let $(\rho(\cdot, t), c(\cdot, t))$ be the unique local solution of (4.5) obtained in Lemma 4.1. Then:*

(i) *For each fixed $\beta \in [0, \frac{1}{2})$, there exists a small $\delta_0 > 0$ such that $\rho \in C^{\delta_0}((0, \tau_0], \mathscr{D}(A_1^\beta))$; moreover there exists a constant C such that for all small $h > 0$ and $t \in (0, \tau_0 - h]$, the estimate*

$$\|\rho(\cdot, t+h) - \rho(\cdot, t)\|_{\mathscr{D}(A_1^\beta)} \leq C h^{\delta_0} \left(t^{-(\delta_0 + \beta)} + 1 \right) \tag{4.10}$$

is true provided $\beta + \delta_0 < \frac{1}{2}$;

(ii) *For each fixed $\gamma \in (0, a)$, there exists a small $\delta_0 > 0$ such that $c \in C^{\delta_0}([0, \tau_0], \mathscr{D}(A_2^\gamma))$.*

Proof: For each sufficiently small $h > 0$ and any $t \in (0, \tau_0 - h]$, we use (4.2)–(4.5) and (4.7) to obtain

$$\|\rho(\cdot, t+h) - \rho(\cdot, t)\|_{\mathscr{D}(A_1^\beta)} \leq \|(e^{-A_1 h} - I) A_1^\beta e^{-A_1 t} \rho_0\|_{C(\overline{\Omega})}$$

$$+ \int_0^t \|(e^{-A_1 h} - I) A_1^\beta \left(\boldsymbol{\nabla} \cdot e^{-A_1(t-s)}(-\boldsymbol{P}(\rho, c, \nabla c)) + e^{-A_1(t-s)}(\rho + R(\rho, c)) \right) \|_{C(\overline{\Omega})} ds$$

$$+ \int_t^{t+h} \| - A_1^\beta \boldsymbol{\nabla} \cdot e^{-A_1(t+h-s)} \boldsymbol{P}(\rho, c, \nabla c) + A_1^\beta e^{-A_1(t+h-s)}(\rho + R(\rho, c))\|_{C(\overline{\Omega})} ds$$

$$\leq \frac{C}{\delta} \frac{h^\delta}{t^{\delta+\beta}} \|\rho_0\|_{C(\overline{\Omega})} + \frac{C}{\delta} h^\delta \int_0^t \left(\frac{1}{(t-s)^{\beta+\delta+\frac{1}{2}+\varepsilon}} + \frac{1}{(t-s)^{\beta+\delta}} \right) ds$$

$$+ C \int_t^{t+h} \left(\frac{1}{(t+h-s)^{\beta+\frac{1}{2}+\varepsilon}} + \frac{1}{(t+h-s)^\beta} \right) ds$$

$$\leq C_\delta h^\delta \left(t^{-(\delta+\beta)} + t^{1/2-\delta-\beta-\varepsilon} + t^{1-\delta-\beta} \right) + C_\delta \left(h^{\frac{1}{2}-\beta-\varepsilon} + h^{1-\beta} \right) \leq C h^\delta \left(t^{-(\delta+\beta)} + 1 \right),$$

which is true for sufficiently small $\delta > 0$ and $\varepsilon > 0$. This completes the proof of *(i)*.

For fixed $\gamma \in (0, a)$ and $t \in [0, \tau_0 - h]$, we use once again (4.2)–(4.5) and (4.7) to obtain the following estimate:

$$
\begin{aligned}
\|c(\cdot, t+h) - c(\cdot, t)\|_{\mathscr{D}(A_2^\gamma)} &\leq \|(e^{-A_2 h} - I) A_2^\gamma e^{-A_2 t} c_0\|_{\mathrm{L}^p(\Omega)} \\
&\quad + \int_0^t \|(e^{-A_2 h} - I) A_2^\gamma e^{-A_2(t-s)} \left(c(\cdot, s) + S(\rho(\cdot, s), c(\cdot, s))\right)\|_{\mathrm{L}^p(\Omega)} ds \\
&\quad + \int_t^{t+h} \|A_2^\gamma e^{-A_2(t+h-s)} \left(c(\cdot, s) + S(\rho(\cdot, s), c(\cdot, s))\right)\|_{\mathrm{L}^p(\Omega)} ds \\
&\leq C h^{a-\gamma} \|c_0\|_{\mathscr{D}(A_2^a)} + C h^{a-\gamma} \int_0^t \frac{ds}{(t-s)^a} + \int_t^{t+h} \frac{ds}{(t+h-s)^\gamma} \\
&\leq C_1 h^{a-\gamma} \left(1 + t^{1-a}\right) + C_1 h^{1-\gamma} \leq C h^{\delta_1},
\end{aligned}
$$

which completes the proof of *(ii)* and thus of Lemma 4.2. $\blacksquare$

Lemma 4.3 *Let* $(\rho(\cdot, t), c(\cdot, t))$ *be the unique solution of (4.5). Suppose* $\rho_0 \in \mathscr{D}(A_1^b)$ *for some* $b \in (\frac{1}{2}, 1)$ *(or equivalently,* $\rho_0 \in \mathrm{C}^{1+\alpha}(\overline{\Omega})$ *for some* $\alpha > 0$ *and* $\frac{\partial \rho_0}{\partial n}|_{\partial \Omega} = 0$*). Then, there exists a small* $\delta > 0$ *such that*

$$
\begin{aligned}
\rho(\mathbf{x}, t) &\in C([0, \tau_0], \mathscr{D}(A_1^b)) \cap C^\delta([0, \tau_0], C^{1+\delta}(\overline{\Omega})) \cap C^{2+\delta, 1+\delta/2}(\overline{\Omega} \times (0, \tau_0]), \\
c(\mathbf{x}, t) &\in C([0, \tau_0], \mathscr{D}(A_2^a)) \cap C^\delta\left([0, \tau_0], C^{1+\delta}(\overline{\Omega})\right) \cap C^{2+\delta, 1+\delta/2}(\overline{\Omega} \times (0, \tau_0]).
\end{aligned}
\tag{4.11}
$$

Thus, (ρ, c) *is a classical solution of (4.1) on* $\overline{\Omega} \times [0, \tau_0]$.

Proof: Obviously, $\rho(\mathbf{x}, t) \in C(\overline{\Omega} \times [0, \tau_0])$; by (4.2) and Lemma 4.2, we have $\rho(\mathbf{x}, t) \in C^{\delta, \delta/2}(\overline{\Omega} \times (0, \tau_0])$ and $c(\mathbf{x}, t) \in C^\delta\left([0, \tau_0], C^{1+\delta}(\overline{\Omega})\right)$ for a sufficiently small $\delta > 0$.

We now use [13, Theorem 3.2.2] to deduce that $c(\mathbf{x}, t)$ is also the unique strong solution $(c(\mathbf{x}, t) \in C^1((0, \tau_0], L_p(\Omega)) \cap C((0, \tau_0], \mathscr{D}(A_2^a)))$ of the c-equation in (4.1) satisfying the corresponding boundary and initial conditions. By the aforementioned regularity of ρ and c and by the parabolic Schauder regularity theory (see, e.g., [23]), we have $c(\mathbf{x}, t) \in C^{2+\delta, 1+\delta/2}(\overline{\Omega} \times (0, \tau_0])$; moreover, by (4.10) and [23, Theorem V.5.19], we have

$$
\|\Delta c(\cdot, t)\|_{C(\overline{\Omega})} \leq C_{\beta, \delta_0} t^{-(\delta_0 + \beta)}, \quad \forall t > 0,
\tag{4.12}
$$

provided $\delta_0 + \beta \in (0, \frac{1}{2})$.

To prove the regularity of $\rho(\mathbf{x}, t)$ as stated in (4.11), we define the space $\mathscr{T}_1 := C([0, \tau_0], \mathscr{D}(A_1^b))$ and its subset

$$
\mathscr{S}_1 := \left\{\widehat{\rho}(\cdot, t) \in \mathscr{T}_1 \ \middle| \ \|\widehat{\rho}(\cdot, t) - \rho_0\|_{\mathscr{D}(A_1^b)} \leq 1, \ 0 \leq t \leq \tau_0\right\},
\tag{4.13}
$$

and let

$$
\begin{aligned}
T_2 \widehat{\rho} &= e^{-A_1 t} \rho_0(\cdot) - \int_0^t e^{-A_1(t-s)} \boldsymbol{\nabla} \cdot \boldsymbol{P}\left(\widehat{\rho}(\cdot, s), c(\cdot, s), \nabla c(\cdot, s)\right) ds \\
&\quad + \int_0^t e^{-A_1(t-s)}\left(\rho(\cdot, s) + R(\rho(\cdot, s), c(\cdot, s))\right) ds
\end{aligned}
\tag{4.14}
$$

for any $\widehat{\rho} \in \mathscr{S}_1$. Then, by already established regularity of ρ and c, especially by (4.12), we see that for each fixed $\beta \in (0, \frac{1}{2})$ and a small $\delta_0 > 0$ there exists a constant K such that for all $t \in (0, \tau_0]$ and $\widehat{\rho} \in \mathscr{S}_1$,

$$
\begin{aligned}
&\|\rho(\cdot, t) + R(\rho(\cdot, t), c(\cdot, t))\|_{C(\overline{\Omega})} \leq K, \\
&\|\boldsymbol{\nabla} \cdot \boldsymbol{P}(\widehat{\rho}, c, \nabla c)(\cdot, t)\|_{C(\overline{\Omega})} \leq K(1 + t^{-(\delta_0 + \beta)}).
\end{aligned}
\tag{4.15}
$$

From (4.3), (4.4) and (4.15), it follows that

$$
\begin{aligned}
\|T_2 \widehat{\rho} - \rho_0\|_{\mathscr{D}(A_1^b)} &\leq \|e^{-A_1 t} \rho_0 - \rho_0\|_{\mathscr{D}(A_1^b)} + \int_0^t \|A_1^b e^{-A_1(t-s)}\|_{C(\overline{\Omega}) \to C(\overline{\Omega})} \|\boldsymbol{\nabla} \cdot (\boldsymbol{P}(\overline{\rho}, c, \nabla c))\|_{C(\overline{\Omega})} ds \\
&\quad + \int_0^t \|A_1^b e^{-A_1(t-s)}\|_{C(\overline{\Omega}) \to C(\overline{\Omega})} \|\rho(\cdot, t) + R(\rho(\cdot, s), c(\cdot, s))\|_{C(\overline{\Omega})} ds \\
&\leq \|\left(e^{-A_1 t} - I\right) A_1^b \rho_0\|_{C(\overline{\Omega})} + C_b^* \int_0^t \frac{K(2 + s^{-(\delta_0 + \beta)})}{(t-s)^b} ds \\
&\leq \|\left(e^{-A_1 t} - I\right) A_1^b \rho_0\|_{C(\overline{\Omega})} + C_1 t^{1-b} + C_2 t^{1-b-(\delta_0 + \beta)},
\end{aligned}
$$

and thus for a sufficiently small τ_0, the operator T_2 is a mapping from $\mathscr{S}_1$ to $\mathscr{S}_1$. Similarly, one can prove that for a sufficiently small τ_0, T_2 is also a contracting mapping on $\mathscr{S}_1$. Then, the contracting mapping theorem implies that there exists a unique $\widehat{\rho}(\mathbf{x}, t) \in \mathscr{S}_1$ satisfying

$$
\begin{aligned}
\widehat{\rho}(\cdot, t) &= e^{-A_1 t} \rho_0(\cdot) - \int_0^t e^{-A_1(t-s)} \boldsymbol{\nabla} \cdot \left(\boldsymbol{P}\left(\widehat{\rho}(\cdot, s), c(\cdot, s), \nabla c(\cdot, s)\right)\right) ds \\
&\quad + \int_0^t e^{-A_1(t-s)} \left(\rho(\cdot, s) + R(\rho(\cdot, s), c(\cdot, s))\right) ds, \quad \forall t \in [0, \tau_0].
\end{aligned}
\tag{4.16}
$$

It is easy to see that $\widehat{\rho}(\mathbf{x}, t)$ also satisfies

$$
\begin{aligned}
\widehat{\rho}(\cdot, t) &= e^{-A_1 t} \rho_0(\cdot) - \int_0^t \boldsymbol{\nabla} \cdot \left(e^{-A_1(t-s)} \boldsymbol{P}(\widehat{\rho}(\cdot, s), c(\cdot, s), \nabla c(\cdot, s))\right) ds \\
&\quad + \int_0^t e^{-A_1(t-s)} \left(\rho(\cdot, s) + R(\rho(\cdot, s), c(\cdot, s))\right) ds.
\end{aligned}
\tag{4.17}
$$

On the other hand, note that $\rho(\mathbf{x}, t)$ satisfies (4.17) as well; and if one defines another operator acting on $\widehat{\rho}$ as the RHS of (4.17), one can show that this operator is a contracting mapping in the unit ball in the space $C([0, \tau_0], C(\overline{\Omega}))$ centered at ρ_0 (the proof is similar to the one for T_2). Therefore,

$$
\widehat{\rho}(\mathbf{x}, t) \equiv \rho(\mathbf{x}, t) \quad \text{on} \quad \overline{\Omega} \times [0, \tau_0],
$$

and hence $\rho(\mathbf{x}, t) \in C([0, \tau_0], \mathscr{D}(A_1^b))$. Furthermore, one can prove that, as in Lemma 4.2, there exists a small $\delta > 0$ and $r \in (\frac{1}{2}, b)$ such that $\rho(\mathbf{x}, t) \in C^{\delta}([0, \tau_0], \mathscr{D}(A_1^r))$; hence

$\rho(\mathbf{x}, t) \in C^{\delta}([0, \tau_0], C^{1+\delta}(\overline{\Omega}))$. This together with the already established regularity of c imply that $\boldsymbol{\nabla} \cdot (\boldsymbol{P}(\rho(\mathbf{x}, s), c(\mathbf{x}, s), \nabla c(\mathbf{x}, s))) \in C^{\delta/2}((0, \tau_0], X_1)$; moreover we use (4.15) to obtain

$$\int_0^{\tau_0} \|\boldsymbol{\nabla} \cdot (\boldsymbol{P}(\rho(\cdot, s), c(\cdot, s), \nabla c(\cdot, s)))\|_{X_1} \, ds < \infty.$$

Next, we treat the ρ-component in a way similar to the treatment of the c-component in the beginning of this proof. Namely, we use [13, Theorem 3.2.2] to infer that ρ is a strong solution $(\rho \in C^1((0, \tau_0], C(\overline{\Omega})) \cap C((0, \tau_0], \mathscr{D}(A_1)))$ of the ρ-equation in (4.1) satisfying the corresponding boundary and initial conditions. We now apply the parabolic Schauder regularity theory to conclude that $\rho(\mathbf{x}, t) \in C^{2+\delta, 1+\delta/2}(\overline{\Omega} \times (0, \tau_0])$. ∎

We are now ready to state and prove the following main results on the existence of weak and classical maximal solution of IBVP (4.1).

Theorem 4.1 *Let $\partial\Omega$, $\boldsymbol{P}(\rho, c, \nabla c)$, $R(\rho, c)$ and $S(\rho, c)$ satisfy the assumptions of Lemma 4.1.*

(i) *For any given $\rho_0 \in C(\overline{\Omega})$ and $c_0 \in W^{2,p}(\Omega)$ with $p > d$ and $\frac{\partial c_0}{\partial n}\big|_{\partial\Omega} = 0$ (or more generally, $c_0 \in \mathscr{D}(A_2^a)$ with $\frac{1}{2}(1 + \frac{d}{p}) < a < 1$), there exists a unique maximal solution $(\rho(x, t), c(x, t))$ of (4.5) on some maximal time interval $J_0 := [0, T_{\max})$ with $0 < T_{\max} \leq \infty$. For a small $\delta > 0$, this solution satisfies*

$$\begin{aligned}
&\rho \in C([0, T_{\max}), C(\overline{\Omega})) \cap C^{\delta, \delta/2}(\overline{\Omega} \times (0, T_{\max})), \\
&c \in C([0, T_{\max}), \mathscr{D}(A_2^a)) \cap C^{\delta}([0, T_{\max}), C^{1+\delta}(\overline{\Omega})) \cap C^{2+\delta, 1+\delta/2}(\overline{\Omega} \times (0, T_{\max})).
\end{aligned} \tag{4.18}$$

Furthermore, if $T_{\max} < \infty$, then

$$\limsup_{t \to T_{\max} - 0} \left\{ \|\rho(\cdot, t)\|_{C(\overline{\Omega})} + \|c(\cdot, t)\|_{C(\overline{\Omega})} \right\} = \infty. \tag{4.19}$$

(ii) *Suppose that, in addition to the conditions in (i), $\rho_0 \in C^{1+\alpha}(\overline{\Omega})$ for some $\alpha > 0$ and $\frac{\partial \rho_0}{\partial n}\big|_{\partial\Omega} = 0$ (or equivalently, $\rho_0 \in \mathscr{D}(A_1^b)$ for some $b \in (\frac{1}{2}, 1)$). Then, $(\rho(x, t), c(x, t))$ is also a classical solution of the IBVP (4.1) satisfying*

$$\begin{aligned}
&\rho \in C([0, T_{\max}), \mathscr{D}(A_1^b)) \cap C^{\delta}([0, T_{\max}), C^{1+\delta}(\overline{\Omega})) \cap C^{2+\delta, 1+\delta/2}(\overline{\Omega} \times (0, T_{\max})), \\
&c \in C([0, T_{\max}), \mathscr{D}(A_2^a)) \cap C^{\delta}([0, T_{\max}), C^{1+\delta}(\overline{\Omega})) \cap C^{2+\delta, 1+\delta/2}(\overline{\Omega} \times (0, T_{\max}))
\end{aligned} \tag{4.20}$$

for some small $\delta > 0$.

Proof: The existence and uniqueness of the maximal solution of (4.5) satisfying (4.18) follows directly from Lemmas 4.1 and 4.2 and from the standard extension argument (see [13, Theorem 3.3.4]); the same extension argument also implies that if $T_{\max} < \infty$, then

$$\limsup_{t \to T_{\max} - 0} \left\{ \|\rho(\cdot, t)\|_{C(\overline{\Omega})} + \|c(\cdot, t)\|_{\mathscr{D}(A_2^a)} \right\} = \infty. \tag{4.21}$$

In the following, we shall prove (by contradiction) that if (4.21) is true then (4.19) holds. To this end, we assume that (4.21) is satisfied while (4.19) is not, that is, there exists a constant C_1 such that

$$\|\rho(\cdot, t)\|_{C(\overline{\Omega})} + \|c(\cdot, t)\|_{C(\overline{\Omega})} \leq C_1, \quad \forall t \in [0, T_{\max}). \tag{4.22}$$

Using (4.22) and the estimates similar to those in (4.9), one can easily obtain that there exists a constant C_2 depending only on C_1, $T_{\max}$ and $\|c_0\|_{\mathscr{D}(A_2^a)}$ such that

$$\|c(\cdot, t)\|_{\mathscr{D}(A_2^a)} \leq C_2, \quad \forall t \in [0, T_{\max}),$$

which, together with (4.22), contradicts (4.21). This completes the proof of *(i)*.

It follows from Lemma 4.3 and its proof that if one can prove the non-existence of a $\tau_1 \in (\tau_0, T_{\max})$ such that

$$\limsup_{t \to \tau_1 - 0} \{\|\rho(\cdot, t)\|_{\mathscr{D}(A_1^b)} + \|c(\cdot, t)\|_{\mathscr{D}(A_2^a)}\} = \infty, \tag{4.23}$$

then we have the regularity stated in (4.20) on the maximal interval $[0, T_{\max})$. Suppose such τ_1 exists. We then have the regularity (4.20) on the interval $[0, \tau_1)$, and in addition, (4.2) and (4.18) (which holds on J_0) imply that there exist constants K_1 and K_2 such that

$$\|\rho(\cdot, t)\|_{C(\overline{\Omega})} + \|c(\cdot, t)\|_{C^2(\overline{\Omega})} \leq K_1, \quad \forall t \in [\tau_0, \tau_1] \tag{4.24}$$

and

$$
\begin{aligned}
\|\rho(\cdot, t) + R(\rho(\cdot, t), c(\cdot, t))\|_{C(\overline{\Omega})} &\leq K_2, & \forall t \in [\tau_0, \tau_1], \\
\|\nabla \cdot (\boldsymbol{P}(\rho(\cdot, t), c(\cdot, t), \nabla c(\cdot, t)))\|_{C(\overline{\Omega})} &\leq K_2(1 + \|\rho(\cdot, t)\|_{\mathscr{D}(A_1^b)}), & \forall t \in [\tau_0, \tau_1].
\end{aligned}
\tag{4.25}
$$

Using (4.17), (4.24) and (4.25), we obtain

$$
\begin{aligned}
\|\rho(\cdot, t)\|_{\mathscr{D}(A_1^b)} &\leq \|e^{-A_1(t-\tau_0)}\rho(\cdot, \tau_0)\|_{\mathscr{D}(A_1^b)} + \int_{\tau_0}^{t} \|A_1^b e^{-A_1(t-s)}\|_{C(\overline{\Omega}) \to C(\overline{\Omega})} \|\boldsymbol{\nabla} \cdot (\boldsymbol{P}(\widehat{\rho}, c, \nabla c))\|_{C(\overline{\Omega})} \, ds \\
&\quad + \int_{\tau_0}^{t} \|A_1^b e^{-A_1(t-s)}\|_{C(\overline{\Omega}) \to C(\overline{\Omega})} \|\rho(\cdot, s) + R(\rho(\cdot, s), c(\cdot, s))\|_{C(\overline{\Omega})} \, ds \\
&\leq K_0 K_1 + K_b \int_{\tau_0}^{t} \frac{K_2 \|\rho(\cdot, s)\|_{\mathscr{D}(A_1^b)}}{(t-s)^b} \, ds + K_b \int_{\tau_0}^{t} \frac{2K_2}{(t-s)^b} \, ds, \quad \forall t \in [\tau_0, \tau_1]. \tag{4.26}
\end{aligned}
$$

Finally, we apply the general Gronwall inequality ([13, Lemma 7.1.1]) to (4.26) and conclude that there exists a constant C depending only on K_1, K_2, τ_0 and τ_1 such that

$$\|\rho(\cdot, t)\|_{\mathscr{D}(A_1^b)} \leq C, \quad \forall t \in [\tau_0, \tau_1),$$

which, together with (4.24), contradicts (4.23). This completes the proof of *(ii)* and of Theorem 4.1. ∎

4.3 Global Existence

In this section, we combine the local existence results from §4.2 and the a-priori estimates established in §3 to obtain the existence of the global classical solution of the chemotaxis system with a saturated chemotactic flux.

Theorem 4.2 *Consider the IBVP (3.1) with initial data being nonnegative but not identically equal to zero, and with a saturated chemotactic flux that satisfies (1.3). Let the conditions in Theorem 4.1 be satisfied. Then, the IBVP (3.1) admits a unique global uniformly bounded classical solution; both components of the solution (ρ, c) are positive on $\bar{\Omega} \times (0, \infty)$.*

Proof: First, Theorem 4.1 implies that there exists a unique classical solution of the IBVP (3.1) defined on some maximal existence interval $[0, T_{\max})$. At the same time, Theorem 3.1 ensures that the solution is uniformly bounded there, hence $T_{\max} = \infty$, which means that the classical solution is global. The positivity of the solution follows from the strong maximum principle and Hopf boundary point lemma. ∎

4.3.1 Numerical Examples

We now illustrate the existence of global solutions of (1.2), (1.4) and their stability. In the numerical experiments below, we consider the system (1.2), (1.4) subject to the initial condition

$$\rho(x, y, 0) = 100, \qquad c(x, y, 0) = 500 e^{-500(x^2 + y^2)} \tag{4.27}$$

in the domain $\Omega = \left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right]$ and take

$$\gamma_c = \gamma_\rho = \nu = 1, \ \chi = 10.$$

The parameter s^* in (1.4) varies in the range $[0, 10]$ in different examples.

In Figures 4.1–4.3, we plot time snapshots of the cell density ρ, computed by the numerical scheme, described in Appendix A. We use a uniform grid with $\Delta x = \Delta y = 1/201$ and set the values $s^* = 0$, $s^* = 1$ and $s^* = 10$, respectively. As one can see, in all the cases the numerical solutions develop a spike at the origin and, as expected, the height of the spike depends on the values of the switching parameter s^* (note in the difference in the scale of the vertical axis). For larger values of s^*, the solution stays longer in the regime of the linear chemotactic flux and therefore more cell aggregate in a small neighborhood of the spike. We have performed a careful mesh refinement study showing that the solution does not blow up for any s^* (even larger than 10).
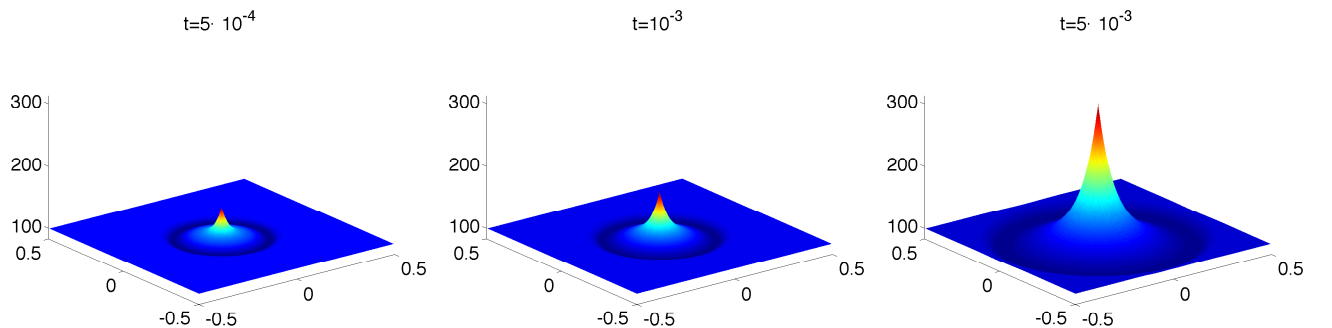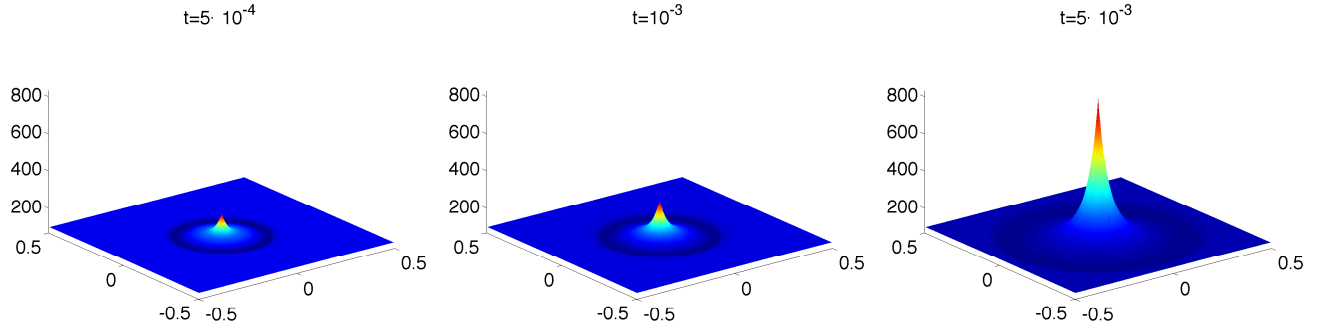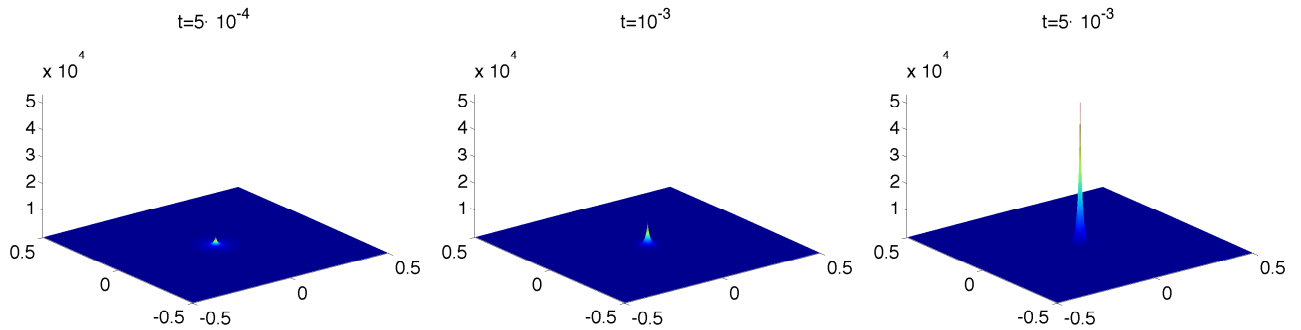

Figure 4.1: Numerical solution (ρ) of (1.2), (1.4), (4.27) with $s^* = 0$.

For comparison, we also computed the solution of the same IVP, but with the linear chemotactic flux, that is $\mathbf{Q}(\nabla c) = \nabla c$. The results, plotted in Figure 4.4, demonstrate the blowing up

Figure 4.2: Numerical solution (ρ) of (1.2), (1.4), (4.27) with $s^* = 1$.

Figure 4.3: Numerical solution (ρ) of (1.2), (1.4), (4.27) with $s^* = 10$.

phenomenon inherent in the original PKS model (1.1). Note that the time snapshots of the solution of the PKS system are given at much smaller times than for the system with the saturated flux, since by the last shown time ($t = 10^{-6}$) the solution has already blown up. To numerically verify this we have performed a mesh refinement study, which clearly shows that, unlike the previous (saturated) case, the height of the spike increases by a factor of 4 when the number of grid cells is doubled. This is consistent with the fact that the magnitude of the "numerical" δ-function is proportional to $1/(\Delta x \Delta y)$.
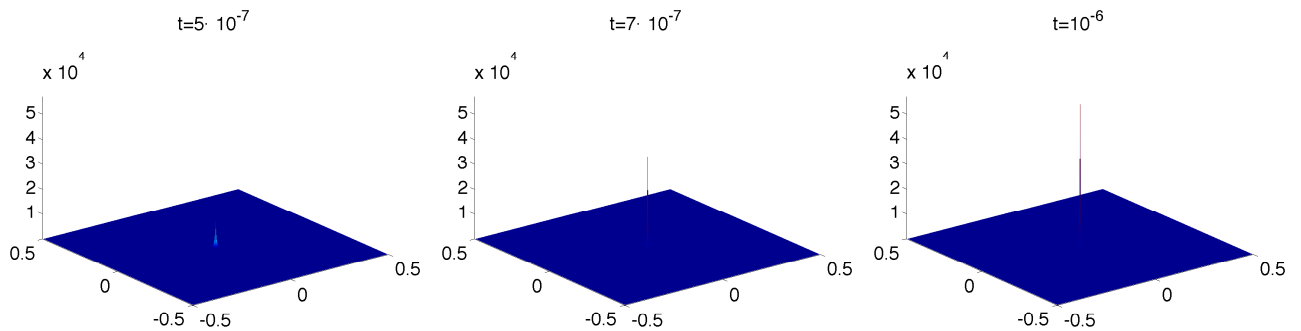
Figure 4.4: Numerical solution (ρ) of (1.1), (4.27).

5 One-Dimensional Steady-State Solutions

In this section, we consider the 1-D version of the system (1.2) subject to the homogeneous Neumann boundary conditions and seek its steady-state solutions satisfying the following boundary value problem (BVP):

$$\begin{cases} \chi(\rho Q(c'))' = \nu\rho'', & x \in (0, L), \\ -c'' + \gamma_c c - \gamma_\rho \rho = 0, & x \in (0, L), \\ \rho'(0) = \rho'(L) = c'(0) = c'(L) = 0, \end{cases} \tag{5.1}$$

where Q is a C^2-smooth bounded increasing function satisfying (compare with (1.3))

$$Q(0) = 0, \quad |Q| \leq 1, \quad Q'(u) > 0, \ \forall u. \tag{5.2}$$

A typical example of such Q is (compare with (1.4))

$$Q(c') = \frac{c'}{\sqrt{1 + (c')^2}}. \tag{5.3}$$

Since in the time-dependent case the total mass of ρ is preserved, the value of

$$m_\rho := \frac{1}{L} \int_0^L \rho(x) \, dx \tag{5.4}$$

is prescribed and, as we will show below, given a $m_\rho > 0$, the system (5.1) has positive solutions for a suitable range of χ.

We shall establish the existence of a solution of (5.1) by using the local bifurcation theory from [11] and the global bifurcation theory for nonlinear Fredholm mappings from [30, 34]. To this end, we write (5.1), (5.4) in the abstract form

$$\mathcal{F}(\rho, c, \chi) = 0, \quad (\rho, c, \chi) \in \mathscr{X} \times \mathscr{X} \times \mathbb{R}, \tag{5.5}$$

where $\mathscr{X} = H_N^2(0, L) = \{f \in H^2(0, L) \mid f'(0) = f'(L) = 0\}$, and

$$\mathcal{F}(\rho, c, \chi) = \begin{pmatrix} -\nu\rho'' + \chi(\rho Q(c'))' \\ -c'' + \gamma_c c - \gamma_\rho \rho \\ \int_0^L \rho(x) \, dx - L m_\rho \end{pmatrix}, \tag{5.6}$$

and state several results that are basic for the bifurcation theory to apply.

Lemma 5.1 *The operator $\mathcal{F}$, defined in (5.6), satisfies the following properties:*

(i) $\mathcal{F}(m_\rho, m_c, \chi) = 0, \quad \forall \chi \in \mathbb{R}, \quad m_c = \dfrac{\gamma_\rho m_\rho}{\gamma_c} = \displaystyle\int_0^L c(x) \, dx.$

(ii) $\mathcal{F} : \mathscr{X} \times \mathscr{X} \times \mathbb{R} \to \mathscr{Y}_0 \times \mathscr{Y} \times \mathbb{R}$, *where* $\mathscr{Y} = L^2(0, L)$ *and* $\mathscr{Y}_0 = \left\{ f \in \mathscr{Y} \ \Big| \ \int_0^L f(x)\, dx = 0 \right\}$,

      is C^1-*smooth.*

(iii) *For any fixed* $(\rho_0, c_0) \in \mathscr{X} \times \mathscr{X}$, *the Frechet derivative is given by*

$$D_{(\rho,c)}\mathcal{F}(\rho_0, c_0, \chi)(\rho, c) = \begin{pmatrix} -\nu\rho'' + \chi\left(\rho Q(c_0') + \rho_0 c' Q'(c_0')\right)' \\ -c'' + \gamma_c c - \gamma_\rho \rho \\ \int_0^L \rho(x)\, dx \end{pmatrix}. \tag{5.7}$$

(iv) $D_{(\rho,c)}\mathcal{F}(\rho_0, c_0, \chi)(\rho, c) : \mathscr{X} \times \mathscr{X} \to \mathscr{Y}_0 \times \mathscr{Y} \times \mathbb{R}$ *is a Fredholm operator with zero index.*

Proof: Properties **(i)**–**(iii)** can be verified by straightforward calculations, which are left to the reader.

Property **(iv)** can be proved as follows. We rewrite (5.7) as

$$D_{(\rho,c)}\mathcal{F}(\rho_0, c_0, \chi)(\rho, c) = T_3(\rho, c) + T_4(\rho, c),$$

where

$$T_3(\rho, c) = \begin{pmatrix} -\nu\rho'' + \chi\left(\rho Q(c_0') + \rho_0 c' Q'(c_0')\right)' \\ -c'' + \gamma_c c - \gamma_\rho \rho \\ 0 \end{pmatrix} \quad \text{and} \quad T_4(\rho, c) = \begin{pmatrix} 0 \\ 0 \\ \int_0^L \rho(x)\, dx \end{pmatrix}.$$

Obviously, $T_4 : \mathscr{X} \times \mathscr{X} \to \mathscr{Y}_0 \times \mathscr{Y} \times \mathbb{R}$ is linear and compact. On the other hand, by Theorem 4.4 from [3] or Remark 2.5 (case 3) from [34], the differential system in T_3 is elliptic and satisfies "Agmon's condition". Then, by Theorem 3.3 and Remark 3.4 from [34], $T_3 : \mathscr{X} \times \mathscr{X} \to \mathscr{Y} \times \mathscr{Y} \times \{0\}$ is a Fredholm operator with zero index, and hence

$$\mathscr{Y} \times \mathscr{Y} \times \{0\} = \mathcal{R}(T_3) \oplus W,$$

where $\mathcal{R}(T_3)$ is the range of T_3 and W is a closed subspace of $\mathscr{Y} \times \mathscr{Y} \times \{0\}$ with $\dim W = \dim \mathcal{N}(T_3) < \infty$ ($\mathcal{N}(T_3)$ is the null space of T_3). Therefore,

$$\mathscr{Y} \times \mathscr{Y} \times \mathbb{R} = \mathcal{R}(T_3) \oplus W \oplus \mathrm{span}\{(0,0,1)\}.$$

Since the first component of $T_3(\rho, c)$ is in $\mathscr{Y}_0$, we have

$$\mathscr{Y}_0 \times \mathscr{Y} \times \mathbb{R} = \mathcal{R}(T_3) \oplus W_0 \oplus \mathrm{span}\{(0,0,1)\},$$

where $W_0 = \left\{ (f, g, r) \in W \mid \int_0^L f(x)\, dx = 0 \right\}$. Also, since $W = W_0 \oplus \mathrm{span}\{(1,0,0)\}$, $\dim W = \dim W_0 + 1$. Thus, the codimension of $\mathcal{R}(T_3)$ in $\mathscr{Y}_0 \times \mathscr{Y} \times \mathbb{R}$ is equal to $\dim W = \dim \mathcal{N}(T_3)$, hence $T_3 : \mathscr{X} \times \mathscr{X} \to \mathscr{Y} \times \mathscr{Y} \times \mathbb{R}$ is a Fredholm operator with zero index. Finally, **(iv)** follows from the compactness of T_4 and the well-known fact that a compact perturbation does not change the Fredholmness and the index of a Fredholm operator. ∎

Equipped with Lemma 5.1, we proceed as follows. By property **(i)**, (m_ρ, m_c, χ) is a constant (trivial) solution of (5.5) for all $\chi \in \mathbb{R}$. We seek now nontrivial solutions of (5.5) bifurcating from these trivial solutions. The necessary condition for bifurcation to occur at (m_ρ, m_c, χ) is

$$\mathcal{N}\big(D_{(\rho,c)}\mathcal{F}(m_\rho, m_c, \chi)\big) \neq \{0\}. \tag{5.8}$$

By **(iii)**, the null space consists of solutions of

$$\begin{cases} -\nu\rho'' + \chi m_\rho Q'(0)c'' = 0, & x \in (0, L), \\ -c'' + \gamma_c c - \gamma_\rho \rho = 0, & x \in (0, L), \\ \rho'(0) = \rho'(L) = c'(0) = c'(L) = 0, \\ \int_0^L \rho(x)\,dx = 0. \end{cases} \tag{5.9}$$

From the first and the third equations, we have $\nu\rho - \chi m_\rho Q'(0)c = \text{const}$ on $[0, L]$. Integrating the second equation, we obtain $\int_0^L \rho(x)\,dx = \int_0^L c(x)\,dx = 0$. Thus, $\rho \equiv \dfrac{\chi m_\rho Q'(0)c}{\nu}$ on $[0, L]$, from where we are led to

$$\begin{cases} -c'' = \left(\dfrac{\gamma_\rho \chi m_\rho Q'(0)}{\nu} - \gamma_c\right)c, & x \in (0, L), \\ c'(0) = c'(L) = 0. \end{cases} \tag{5.10}$$

This BVP will have a nontrivial solution if and only if $\dfrac{\gamma_\rho \chi m_\rho Q'(0)}{\nu} - \gamma_c$ is one of the Neumann eigenvalues for the interval $(0, L)$, that is, $\dfrac{\gamma_\rho \chi m_\rho Q'(0)}{\nu} - \gamma_c = \dfrac{k^2\pi^2}{L^2}$ for some nonnegative integer k. Then, c is the corresponding eigenfunction $\bar{c}_k(x) = \cos\big(\frac{k\pi x}{L}\big)$. Notice that the case $k = 0$ can be excluded because $\int_0^L c(x)\,dx = 0$.

We have thus obtained that (5.8) is satisfied only for

$$\bar{\chi}_k = \frac{\left(\gamma_c + \frac{k^2\pi^2}{L^2}\right)\nu}{\gamma_\rho m_\rho Q'(0)}, \quad \bar{\rho}_k(x) = \frac{\bar{\chi}_k m_\rho Q'(0)}{\nu}\bar{c}_k(x), \quad \bar{c}_k(x) = \cos\left(\frac{k\pi x}{L}\right), \tag{5.11}$$

for which

$$\mathcal{N}\big(D_{(\rho,c)}\mathcal{F}(m_\rho, m_c, \bar{\chi}_k)\big) = \text{span}\,\{(\bar{\rho}_k, \bar{c}_k)\}, \quad k \in \mathbb{N}.$$

Theorem 5.1 *Assume that the function Q satisfies (5.2). Then, for each $k \in \mathbb{N}$, there exists an interval $(-\delta, \delta)$ and continuous functions: $s \in (-\delta, \delta) \to \chi_k(s) \in \mathbb{R}$, $s \in (-\delta, \delta) \to (\rho_k(s), c_k(s)) \in \mathscr{X} \times \mathscr{X}$, such that*

$$\chi_k(0) = \bar{\chi}_k, \quad (\rho_k(s, x), c_k(s, x)) = (m_\rho, m_c) + s\left(\frac{\bar{\chi}_k m_\rho Q'(0)}{\nu}, 1\right)\cos\left(\frac{k\pi x}{L}\right) + o(s),$$

and $(\rho_k(s), c_k(s), \chi_k(s))$ is a solution of (5.5) (and (5.1), (5.4)). Moreover, all nontrivial solutions of (5.5) near the bifurcation point $(m_\rho, m_c, \bar{\chi}_k)$ are on the curve $(\rho_k(s), c_k(s), \chi_k(s))$.

Proof: Our proof is based on Theorem 1.7 from [11]. In the previous discussion, we have checked all but the following "transvesality condition" required by the theorem:

$$\frac{d}{d\chi}\big(D_{(\rho,c)}\mathcal{F}(m_\rho, m_c, \chi)\big)(\bar\rho_k, \bar c_k)\Big|_{\chi=\bar\chi_k} \notin \mathcal{R}\big(D_{(\rho,c)}\mathcal{F}(m_\rho, m_c, \bar\chi_k)\big). \tag{5.12}$$

If this condition fails, then the following BVP

$$\begin{cases} -\nu\rho'' + \bar\chi_k m_\rho Q'(0)c'' = m_\rho Q'(0)\bar c_k'', & x \in (0, L), \\ -c'' + \gamma_c c - \gamma_\rho \rho = 0, & x \in (0, L), \\ \rho'(0) = \rho'(L) = c'(0) = c'(L) = 0, \\ \int_0^L \rho(x)\, dx = 0. \end{cases} \tag{5.13}$$

has a solution (ρ, c). Similarly, as in the discussion for (5.9), we have

$$\rho(x) = \frac{(\bar\chi_k c(x) - \bar c_k)m_\rho Q'(0)}{\nu}, \quad x \in (0, L),$$

which we then substitute into the second equation in (5.13) and obtain

$$\begin{cases} -c'' + \left(\gamma_c - \dfrac{\gamma_\rho \bar\chi_k m_\rho Q'(0)}{\nu}\right)c = \dfrac{\gamma_\rho m_\rho Q'(0)}{\nu}\bar c_k, & x \in (0, L), \\ c'(0) = c'(L) = 0. \end{cases}$$

Since $\bar c_k$ is a solution of the corresponding homogeneous BVP (5.10), we reach a contradiction (to Fredholm Alternative). This completes the proof of (5.12) and hence that of the theorem. ∎

Remark 5.1 It should be observed that $(\rho_1(s, x), c_1(s, x))$ is a monotone solution of (5.5) (and (5.1) with (5.4)), while $(\rho_k(s, x), c_k(s, x))$, $k \geq 2$ are nonmonotone ones.

Next, we wish to extend globally the local bifurcation curves in Theorem 5.1. We shall do so for the first curve ($k = 1$) by using the global bifurcation theory; for the existence of nonmonotone solutions, we prefer to use the reflection (in x) argument and the monotone solutions (that stay on the first bifurcation branch). The following result guarantees the existence of positive monotone solutions of (5.1) with (5.4).

Theorem 5.2 *Assume that the function Q satisfies (5.2). Then, the set of solutions of (5.5) (and (5.1) with (5.4)) contains a closed connected set $\mathcal{P} \subset \mathscr{X} \times \mathscr{X} \times \mathbb{R}$ such that*

(a) *$\mathcal{P}$ contains $(\rho_1(s, x), c_1(s, x), \chi_1(s))$, $\quad s \in (-\delta, \delta)$;*

(b) *$\forall(\rho, c, \chi) \in \mathcal{P}$, ρ and c are positive on $[0, L]$;*

(c) *$\mathcal{P} = \mathcal{P}^+ \bigcup \mathcal{P}^-$, where $\mathcal{P}^\pm$ are closed connected subsets of $\mathcal{P}$ with $\mathcal{P}^+ \bigcap \mathcal{P}^- = \{(m_\rho, m_c, \bar\chi_1)\}$; $\mathcal{P}^+ \backslash \{(m_\rho, m_c, \bar\chi_1)\}$ consists of (ρ, c, χ) with both ρ and c being strictly decreasing on $[0, L]$, while $\mathcal{P}^- \backslash \{(m_\rho, m_c, \bar\chi_1)\}$ consists of (ρ, c, χ) with both ρ and c being strictly increasing on $[0, L]$;*

(d) *$\forall \chi > \bar\chi_1$, there exists $(\rho, c, \chi) \in \mathcal{P}^+$; the same holds for $\mathcal{P}^-$.*
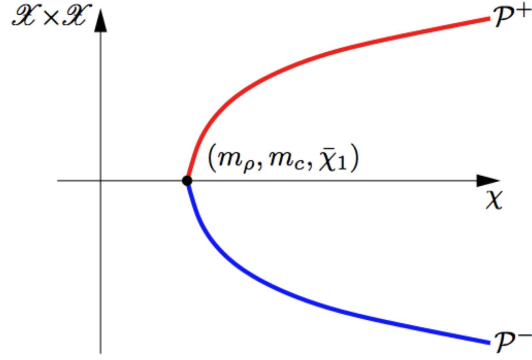
Figure 5.1: A bifurcation curve illustrating Theorem 5.2.

The results of the theorem are schematically presented in Figure 5.1.

Proof: **(a)** follows from [34, Theorem 4.3]: $\mathcal{P}$ is a component (maximal connected subset) of the closure of $\mathcal{S} = \{(\rho, c, \chi) \in \mathscr{X} \times \mathscr{X} \times \mathbb{R} \mid \mathcal{F}(\rho, c, \chi) = 0, \ (\rho, c) \neq (m_\rho, m_c)\}$, containing $(m_\rho, m_c, \bar{\chi}_1)$.

To prove **(b)**, we define $\mathcal{P}_1 = \{(\rho, c) \in \mathscr{X} \times \mathscr{X} \mid \rho > 0, c > 0 \text{ on } [0, L]\}$ and shall show that $\mathcal{P} \subset \mathcal{P}_1 \times \mathbb{R}$. The part of $\mathcal{P}$ near $(m_\rho, m_c, \bar{\chi}_1)$ is obviously contained in $\mathcal{P}_1 \times \mathbb{R}$. Since $\mathcal{P}$ is connected and $\mathcal{P}_1$ is open, one may conclude that if $\mathcal{P} \not\subset \mathcal{P}_1 \times \mathbb{R}$, then there exists $(\rho, c, \chi) \in \mathcal{P} \bigcap \partial(\mathcal{P}_1 \times \mathbb{R})$ such that $\rho, c \geq 0$ on $[0, L]$, and either $\rho = 0$ or $c = 0$ somewhere in $[0, L]$. In the latter case, applying the strong maximum principle and Hopf's boundary point lemma to

$$
\begin{cases}
-c'' + \gamma_c c = \gamma_\rho \rho \geq 0 \quad \text{on} \quad [0, L], \\
c'(0) = c'(L) = 0,
\end{cases}
$$

we have $c \equiv 0$ on $[0, L]$ that contradicts the fact that $\int_0^L c(x)\, dx = Lm_c$. By a similar argument, we obtain that ρ cannot be equal to zero anywhere in $[0, L]$. This completes the proof of **(b)**.

To show **(c)**, let $\mathcal{P}^+$ be the component of $\mathcal{P} \backslash \{(\rho_1(s, x), c_1(s, x), \chi_1(s)) \mid -\delta < s < 0\}$ containing $\{(\rho_1(s, x), c_1(s, x), \chi_1(s)) \mid 0 \leq s < \delta\}$, correspondingly $\mathcal{P}^-$ is the component of $\mathcal{P} \backslash \{(\rho_1(s, x), c_1(s, x), \chi_1(s)) \mid 0 < s < \delta\}$ containing $\{(\rho_1(s, x), c_1(s, x), \chi_1(s)) \mid -\delta < s \leq 0\}$. Then $\mathcal{P} = \mathcal{P}^+ \bigcup \mathcal{P}^-$. Let us introduce the following four subsets of $\mathcal{P}$:

$$
\begin{aligned}
\mathcal{P}_0^+ &:= \mathcal{P}^+ \backslash \{(m_\rho, m_c, \bar{\chi}_1)\}, \quad \mathcal{P}_2^+ = \{(\rho, c) \in \mathscr{X} \times \mathscr{X} \mid \rho' > 0, c' > 0 \text{ on } (0, L)\}, \\
\mathcal{P}_0^- &:= \mathcal{P}^- \backslash \{(m_\rho, m_c, \bar{\chi}_1)\}, \quad \mathcal{P}_2^- = \{(\rho, c) \in \mathscr{X} \times \mathscr{X} \mid \rho' < 0, c' < 0 \text{ on } (0, L)\}.
\end{aligned}
$$

We now prove that $\mathcal{P}_0^+ \subset \mathcal{P}_2^- \times \mathbb{R}$. Since $\mathcal{P}_0^+$ is a connected subset of $\mathscr{X} \times \mathscr{X} \times \mathbb{R}$, we need to show that $\mathcal{P}_0^+ \bigcap (\mathcal{P}_2^- \times \mathbb{R})$ is nonempty (which immediately follows from Theorem 5.1) and is both open and closed with respect to the relative topology of $\mathcal{P}_0^+$.

To show the openness, assume that $(\tilde{\rho}, \tilde{c}, \tilde{\chi}) \in \mathcal{P}_0^+ \bigcap (\mathcal{P}_2^- \times \mathbb{R})$ and the sequence $\{(\tilde{\rho}_k, \tilde{c}_k, \tilde{\chi}_k)\}_{k=1}^\infty \in \mathcal{P}_0^+$ converges to $(\tilde{\rho}, \tilde{c}, \tilde{\chi})$ in the norm of $\mathscr{X} \times \mathscr{X} \times \mathbb{R}$. The elliptic regularity theory implies that this convergence occurs in the norm of $C^2([0, L]) \times C^2([0, L]) \times \mathbb{R}$. We differentiate the second equation in (5.1) and since $\tilde{c}' < 0$ and $\tilde{\rho}' < 0$ in $(0, L)$, we obtain

$$
\begin{cases}
-(\tilde{c}')'' + \gamma_c \tilde{c}' = \gamma_\rho \tilde{\rho}' < 0 \quad \text{on} \quad [0, L], \\
\tilde{c}'(0) = \tilde{c}'(L) = 0,
\end{cases} \tag{5.14}
$$

and thus, Hopf's boundary point lemma gives

$$\tilde{c}''(L) > 0 > \tilde{c}''(0). \tag{5.15}$$

Using the facts that

$$\tilde{\rho}' \equiv \tilde{\chi}\tilde{\rho}Q(\tilde{c}') \quad \text{on } [0, L] \tag{5.16}$$

and $\tilde{\chi} > 0$, we conclude that

$$\tilde{\rho}''(L) > 0 > \tilde{\rho}''(0). \tag{5.17}$$

It then follows from (5.15) and (5.17) that $(\tilde{\rho}_k, \tilde{c}_k, \tilde{\chi}_k) \in \mathcal{P}_2^- \times \mathbb{R}$ for large k, that is, $\mathcal{P}_0^+ \bigcap (\mathcal{P}_2^- \times \mathbb{R})$ is open in $\mathcal{P}_0^+$.

To show that $\mathcal{P}_0^+ \bigcap (\mathcal{P}_2^- \times \mathbb{R})$ is also closed in $\mathcal{P}_0^+$, we now assume that the sequence $\{(\tilde{\rho}_k, \tilde{c}_k, \tilde{\chi}_k)\}_{k=1}^\infty \in \mathcal{P}_0^+ \bigcap (\mathcal{P}_2^- \times \mathbb{R})$ converges to some $(\tilde{\rho}, \tilde{c}, \tilde{\chi}) \in \mathcal{P}_0^+$ in the norm of $\mathscr{X} \times \mathscr{X} \times \mathbb{R}$ (hence, in the norm of $C^2([0, L]) \times C^2([0, L]) \times \mathbb{R}$ as well). Then $\tilde{\rho}' \leq 0$ and $\tilde{c}' \leq 0$ on $[0, L]$. If $\tilde{c}' = 0$ somewhere in $(0, L)$, then applying the strong maximum principle to

$$\begin{cases} -(\tilde{c}')'' + \gamma_c \tilde{c}' = \gamma_\rho \tilde{\rho}' \leq 0 \quad \text{on} \quad [0, L], \\ \tilde{c}'(0) = \tilde{c}'(L) = 0, \end{cases} \tag{5.18}$$

we obtain $\tilde{c}' \equiv 0$, which, in turn, implies that $\tilde{\rho}' \equiv 0$ on $[0, L]$. Then $(\tilde{\rho}, \tilde{c}) \equiv (m_\rho, m_c)$ and $\tilde{\chi}$ is a bifurcation value. Thus, $\tilde{\chi}$ is equal to $\bar{\chi}_m$ for some $m \geq 1$. The value $m = 1$ is impossible because $(\tilde{\rho}, \tilde{c}, \tilde{\chi}) \neq (m_\rho, m_c, \bar{\chi}_1)$. Any of the values $m \geq 2$ is also impossible since by Theorem 5.1, for large k, $(\tilde{\rho}_k, \tilde{c}_k)$ must be some $(\rho_m(s, x), c_m(s, x))$, which is non-monotone in x. Hence,

$$\tilde{c}' < 0 \quad \text{on } (0, L). \tag{5.19}$$

Since $\tilde{\chi}_k > 0 \; \forall k$, $\tilde{\chi} \geq 0$. If $\tilde{\chi} = 0$, then $\tilde{\rho}' \equiv 0$, which implies $\tilde{c}' \equiv 0$ (by (5.18)), and we again reach a contradiction. Therefore, $\tilde{\chi} > 0$ and by (5.16),

$$\tilde{\rho}' < 0 \quad \text{on } (0, L),$$

which, together with (5.19) implies that $\mathcal{P}_0^+ \bigcap (\mathcal{P}_2^- \times \mathbb{R})$ is closed in $\mathcal{P}_0^+$.

We have thus shown that $\mathcal{P}_0^+ \subset \mathcal{P}_2^- \times \mathbb{R}$. Similarly, one can prove that $\mathcal{P}_0^- \subset \mathcal{P}_2^+ \times \mathbb{R}$. This completes the proof of **(c)**.

Finally, we proceed with the proof of **(d)**. By Theorem 4.4 of [34], each of $\mathcal{P}^\pm$ satisfies one of the following three alternatives:

(i) It is not compact in $\mathscr{X} \times \mathscr{X} \times \mathbb{R}$;

(ii) It contains a point (m_ρ, m_c, χ_*) with $\chi_* \neq \bar{\chi}_1$;

(iii) It contains a point $(m_\rho + \tilde{\rho}, m_c + \tilde{c}, \chi)$, where $0 \neq (\tilde{\rho}, \tilde{c}) \in \mathscr{Z}$ with $\mathscr{Z}$ being a closed complement of $\mathcal{N}\big(D_{(\rho,c)}\mathcal{F}(m_\rho, m_c, \bar{\chi}_1)\big) = \text{span}\{(\bar{\rho}_1, \bar{c}_1)\}$ in $\mathscr{X} \times \mathscr{X}$. We take (see (5.11))

$$\mathscr{Z} = \left\{ (\rho, c) \in \mathscr{X} \times \mathscr{X} \; \middle| \; 0 = \int_0^L [\bar{\rho}_1(x)\rho(x) + \bar{c}_1(x)c(x)] \, dx \right. $$

$$\left. = \int_0^L \left[\frac{\bar{\chi}_1 m_\rho Q'(0)}{\nu} \rho(x) + c(x) \right] \cos\left(\frac{\pi x}{L}\right) dx \right\}. \tag{5.20}$$

If alternative *(ii)* occurs, then χ_* is a bifurcation value, which is impossible (this can be proved as in the proof of **(c)** above). If alternative *(iii)* occurs, then we integrate by parts:

$$0 = \int_0^L \left[\frac{\bar{\chi}_1 m_\rho Q'(0)}{\nu} \tilde{\rho}(x) + \tilde{c}(x) \right] \cos\left(\frac{\pi x}{L}\right) dx = -\frac{L}{\pi} \int_0^L \left[\frac{\bar{\chi}_1 m_\rho Q'(0)}{\nu} \tilde{\rho}'(x) + \tilde{c}'(x) \right] \sin\left(\frac{\pi x}{L}\right) dx > 0,$$

and reach a contradiction. Thus, according to alternative *(i)*, $\mathcal{P}^\pm$ are not compact in $\mathscr{X} \times \mathscr{X} \times \mathbb{R}$. This means that they are unbounded in $\mathscr{X} \times \mathscr{X} \times \mathbb{R}$ by the elliptic regularity theory. By Remark 3.1, if χ is bounded, then (ρ, c) is also bounded. Thus, the projection of each of $\mathcal{P}^\pm$ on the χ-axis is unbounded. Since $\mathcal{P}^\pm$ are connected, the projection must be an interval of the form $[a, \infty)$ with $a \leq \bar{\chi}_1$. This completes the proof of **(d)** and thus of Theorem 5.2. ∎

5.1 Behavior of Monotone Solutions of (5.1), (5.4) as $\chi \to \infty$

Let (ρ, c) be a solution of (5.1), (5.4) satisfying $\rho' < 0$ and $c' < 0$ on $(0, L)$. By Theorem 5.2, such a solution is guaranteed to exist for $\chi > \bar{\chi}_1$.

Theorem 5.3 *As $\chi \to \infty$, ρ concentrates at $x = 0$, that is,*

$$\rho(x) \to \rho_\infty(x) = L m_\rho \delta(x) \tag{5.21}$$

in the sense of distribution, and

$$c(x) \to c_\infty(x) = \frac{\sqrt{\gamma_c} \, m_c L}{e^{2\sqrt{\gamma_c} L} - 1} \left(e^{\sqrt{\gamma_c} x} + e^{\sqrt{\gamma_c}(2L - x)} \right) \tag{5.22}$$

uniformly on $[0, L]$.

Proof: Since ρ is decreasing and $\int_0^L \rho(x)\, dx = L m_\rho$, then for any small $\varepsilon > 0$, ρ is uniformly bounded on $[\varepsilon, L]$ as $\chi \to \infty$. Then by Helly's compactness theorem, after passing to a subsequence of $\chi \to \infty$, $\rho(x)$ converges to some function, which we denote by $\rho_\infty(x)$, for all $x \in (0, L]$. If we show that $\rho_\infty(x) = 0, \ \forall x \in (0, L]$, then (5.21) holds without passing to a subsequence.

On the other hand, after integrating the second equation in (5.1), we have

$$c'(x) = -\int_x^L [\gamma_c c(\xi) - \gamma_\rho \rho(\xi)] \, d\xi \tag{5.23}$$

and since $\int_0^L c(x)\, dx = L m_c$, $c(x)$ is bounded in $C^1[0, L]$ as $\chi \to \infty$. Therefore, by the Arzela-Ascoli theorem, after passing to a subsequence of $\chi \to \infty$, $c(x)$ converges to some function, which we denote by $c_\infty(x)$, uniformly on $[0, L]$. If we show that c_∞ is given by (5.22), then the convergence in (5.22) holds without passing to a subsequence.

Next, we integrate the first equation in (5.1) and differentiate the second equation in (5.1) to obtain

$$\begin{cases} (c')'' + \left(\dfrac{\gamma_\rho \chi \rho Q(c')}{\nu c'} - \gamma_c \right) c' = 0 \ \ \text{on } [0, L], \\ c'(0) = c'(L) = 0. \end{cases} \tag{5.24}$$

If there exists $x_0 \in (0, L]$ such that $\rho_\infty(x_0) \neq 0$, then $\left(\frac{\gamma_\rho \chi \rho Q(c')}{\nu c'} - \gamma_c \right) \to \infty$ as $\chi \to \infty$ uniformly on $[0, x_0]$. Then, by the Sturm oscillation theorem applied to (5.24), c' changes sign on $(0, x_0)$ for large χ, which contradicts the monotonicity assumption. We have thus shown that $\rho_\infty \equiv 0$ on $(0, 1]$.

Finally, we integrate (5.23) to obtain

$$c(x) = c(L) + \int_x^L d\eta \int_\eta^L [\gamma_c c(\xi) - \gamma_\rho \rho(\xi)] \, d\xi.$$

Taking the limit in this equation as $\chi \to \infty$ and using the Lebesque dominated convergence theorem, we have

$$c_\infty(x) = c_\infty(L) + \int_x^L d\eta \int_\eta^L \gamma_c c_\infty(\xi) \, d\xi, \quad x \in (0, L].$$

Hence,

$$\begin{cases} c_\infty'' - \gamma_c c_\infty = 0 & \text{on } [0, L], \\ c_\infty'(L) = 0. \end{cases}$$

This and the fact that $\int_0^L c_\infty(x) \, dx = L m_c$ imply (5.22). ■

5.1.1 Numerical Examples

The purpose of this section is to illustrate the statement of Theorem 5.3. To this end, we consider the 1-D version of the system (1.2),

$$\begin{cases} \rho_t + \chi(\rho Q(c_x))_x = \nu \rho_{xx}, \\ c_t = c_{xx} - \gamma_c c + \gamma_\rho \rho, \end{cases} \tag{5.25}$$

with Q given by (5.3) and subject to the homogeneous Neumann boundary conditions and the following initial data:

$$\rho(x, 0) = 1 + 0.05 \bar{\chi}_1 \cos(\pi x), \quad c(x, 0) = 1 + 0.05 \cos(\pi x), \tag{5.26}$$

prescribed on the interval $[0, 1]$. In the numerical experiments below, we choose $\nu = \gamma_c = \gamma_\rho = 1$ (thus by formula (5.11), $\bar{\chi}_1 = 1 + \pi^2$), take either small ($\chi = 1.2\bar{\chi}_1$), intermediate ($\chi = 12\bar{\chi}_1$) or large ($\chi = 1200\bar{\chi}_1$) chemosensitivity constant, and run the simulations until the computed solutions reach their steady states. The resulting monotone steady-state solutions (both ρ- and c-components) are plotted in Figure 5.2. These results clearly illustrate that the studied solutions converge to the corresponding analytical steady states, and as $\chi \to \infty$, ρ concentrates at $x = 0$ and blows up there, while c remains bounded.

5.2 Stability of Bifurcating Solutions Near $(m_\rho, m_c, \bar{\chi}_1)$

We shall now show that the bifurcating solutions of (5.1), (5.4) (mentioned in Theorem 5.1) are asymptotically stable provided the coefficient γ_c is not too large. Before doing so, we need to determine the direction in which the bifurcation curve turns.
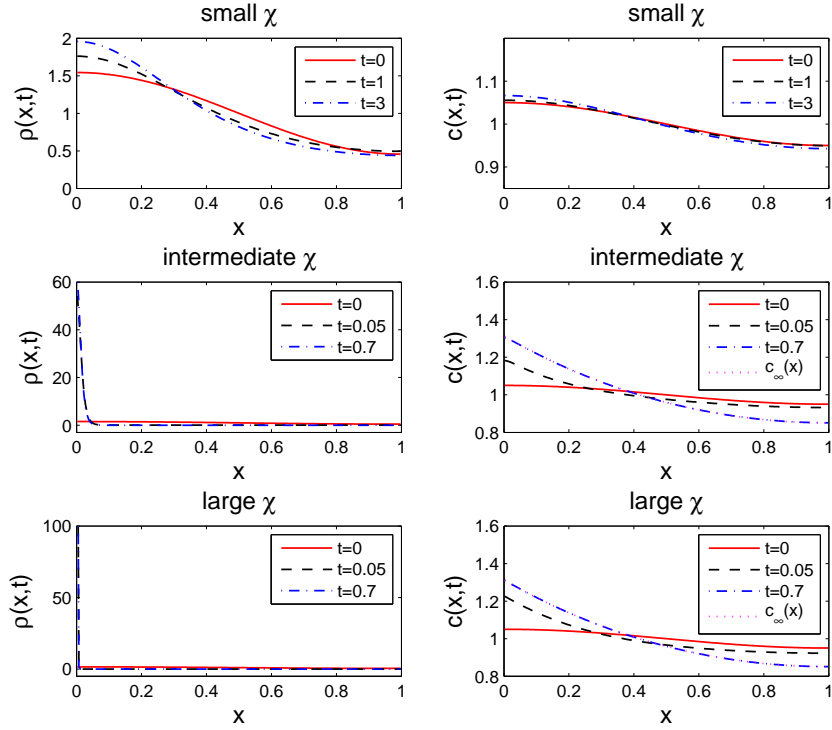
Figure 5.2: Monotone steady states.

By Theorem 1.7 from [11],

$$(\rho_1(s,x), c_1(s,x)) - (m_\rho, m_c) - s\left(\frac{\bar{\chi}_1 m_\rho Q'(0)}{\nu}, 1\right)\cos\left(\frac{\pi x}{L}\right) \in \mathscr{Z} \quad \forall s \in (-\delta, \delta),$$

where $\mathscr{Z}$ is defined in (5.20). Furthermore, if Q is C^5-smooth, then the operator $\mathcal{F}$ defined in (5.6) is C^4-smooth and hence, by Theorem 1.18 of [11], (ρ_1, c_1, χ_1) is a C^3-smooth function of s. Thus, we can write the following expansions:

$$\begin{cases} \rho_1(s,x) = m_\rho + \dfrac{s\bar{\chi}_1 m_\rho Q'(0)}{\nu}\cos\left(\dfrac{\pi x}{L}\right) + s^2\psi_1(x) + s^3\psi_2(x) + o(s^3), \\[2mm] c_1(s,x) = m_c + s\cos\left(\dfrac{\pi x}{L}\right) + s^2\varphi_1(x) + s^3\varphi_2(x) + o(s^3), \\[2mm] \chi_1(s,x) = \bar{\chi}_1 + sK_2 + s^2K_3 + o(s^2), \end{cases} \qquad (5.27)$$

where $(\psi_1, \varphi_1) \in \mathscr{Z}$, $(\psi_2, \varphi_2) \in \mathscr{Z}$, and the $o(s^4)$ in the expansions for ρ_1 and c_1 are with respect to the H^2-norm, and K_2 and K_3 are constants. Since $\int_0^L \rho_1(s,x)\,dx = Lm_\rho$ and $\int_0^L c_1(s,x)\,dx = Lm_c$, $\int_0^L \psi_i(x)\,dx = \int_0^L \varphi_i(x)\,dx = 0$ for $i = 1, 2$. Also note that ψ_i and φ_i satisfy Neumann boundary conditions.

We also make additional assumptions on the chemotactic flux Q (satisfied, for example, by (5.3)):

$$Q''(0) = 0, \quad Q'''(0) < 0, \qquad (5.28)$$

so that its Taylor expansion reads

$$Q(u) = Q'(0)u + \frac{Q'''(0)}{6}u^3 + o(u^3),$$

and substitute (5.27) into the ρ-equation in (5.1) to obtain

$$\nu\left(\frac{\bar{\chi}_1 m_\rho Q'(0)}{\nu}\cos\left(\frac{\pi x}{L}\right) + s\psi_1(x) + s^2\psi_2(x) + o(s^2)\right)'' = (\bar{\chi}_1 + sK_2 + s^2 K_3 + o(s^2))$$

$$\cdot\left[\left(m_\rho + \frac{s\bar{\chi}_1 m_\rho Q'(0)}{\nu}\cos\left(\frac{\pi x}{L}\right) + s^2\psi_1(x) + s^3\psi_2(x) + o(s^3)\right)\right.$$

$$\cdot\left(\cos\left(\frac{\pi x}{L}\right) + s\varphi_1(x) + s^2\varphi_2(x) + o(s^2)\right)'$$

$$\left.\cdot\left(Q'(0) + \frac{Q'''(0)}{6}s^2\left\{\left(\cos\left(\frac{\pi x}{L}\right)\right)' + s\varphi_1'(x) + s^2\varphi_2'(x) + o(s^2)\right\}^2 + o(s^3)\right)\right]'. \quad (5.29)$$

Collecting the $\mathcal{O}(s)$ terms in (5.29), we obtain

$$\frac{\nu}{Q'(0)}\psi_1'' = \frac{\bar{\chi}_1^2 m_\rho Q'(0)}{\nu}\left(\cos\left(\frac{\pi x}{L}\right)\left(\cos\left(\frac{\pi x}{L}\right)\right)'\right)' + K_2 m_\rho\left(\cos\left(\frac{\pi x}{L}\right)\right)'' + \bar{\chi}_1 m_\rho\varphi_1''. \quad (5.30)$$

Substituting (5.27) into the c-equation in (5.1) and using the fact that $\gamma_c m_c = \gamma_\rho m_\rho$, we obtain

$$-\left(\cos\left(\frac{\pi x}{L}\right) + s\varphi_1(x) + s^2\varphi_2(x) + o(s^2)\right)'' + \gamma_c\left(\cos\left(\frac{\pi x}{L}\right) + s\varphi_1(x) + s^2\varphi_2(x) + o(s^2)\right)$$

$$-\gamma_\rho\left(\frac{\bar{\chi}_1 m_\rho Q'(0)}{\nu}\cos\left(\frac{\pi x}{L}\right) + s\psi_1(x) + s^2\psi_2(x) + o(s^2)\right) = 0. \quad (5.31)$$

Collecting the $\mathcal{O}(s)$ terms in (5.31) gives

$$-\varphi_1'' + \gamma_c\varphi_1 - \gamma_\rho\psi_1 = 0. \quad (5.32)$$

Multiplying (5.32) by $\cos\left(\frac{\pi x}{L}\right)$ and integrating by parts, we have

$$\int_0^L\left[\left(\gamma_c + \frac{\pi^2}{L^2}\right)\varphi_1 - \gamma_\rho\psi_1\right]\cos\left(\frac{\pi x}{L}\right)\,dx = 0. \quad (5.33)$$

On the other hand, recall that $(\psi_1, \varphi_1) \in \mathscr{L}$, which together with the first formula in (5.11) leads to

$$\int_0^L\left[\left(\gamma_c + \frac{\pi^2}{L^2}\right)\psi_1 + \gamma_\rho\varphi_1\right]\cos\left(\frac{\pi x}{L}\right)\,dx = 0,$$

which, in turn, together with (5.33) implies

$$\int_0^L\psi_1\cos\left(\frac{\pi x}{L}\right)\,dx = \int_0^L\varphi_1\cos\left(\frac{\pi x}{L}\right)\,dx = 0. \quad (5.34)$$

Now multiplying (5.30) by $\cos\left(\frac{\pi x}{L}\right)$, integrating with respect to x and using (5.34), we obtain

$$0 = \int_0^L \left(\frac{\nu}{Q'(0)} \psi_1'' - \bar{\chi}_1 m_\rho \varphi_1'' \right) \cos\left(\frac{\pi x}{L}\right) dx$$

$$= \frac{\bar{\chi}_1^2 m_\rho Q'(0)}{2\nu} \int_0^L \left(\cos^2\left(\frac{\pi x}{L}\right) \right)'' \cos\left(\frac{\pi x}{L}\right) dx + K_2 m_\rho \int_0^L \left(\cos\left(\frac{\pi x}{L}\right) \right)'' \cos\left(\frac{\pi x}{L}\right) dx$$

$$= -K_2 m_\rho \int_0^L \left[\left(\cos\left(\frac{\pi x}{L}\right) \right)' \right]^2 dx,$$

from which it follows that $K_2 = 0$.

Next, gathering the $\mathcal{O}(s^2)$ terms in (5.29), we have

$$\frac{\nu}{Q'(0)} \psi_2'' = \bar{\chi}_1 \left(\psi_1 \left(\cos\left(\frac{\pi x}{L}\right) \right)' \right)' + \bar{\chi}_1 m_\rho \varphi_2'' + \frac{\bar{\chi}_1 m_\rho Q'''(0)}{6Q'(0)} \left(\left[\left(\cos\left(\frac{\pi x}{L}\right) \right)' \right]^3 \right)'$$

$$+ \frac{\bar{\chi}_1^2 m_\rho Q'(0)}{\nu} \left(\varphi_1' \cos\left(\frac{\pi x}{L}\right) \right)' + K_3 m_\rho \left(\cos\left(\frac{\pi x}{L}\right) \right)''. \qquad (5.35)$$

Note that (5.34) also holds with ψ_1 and φ_1 replaced by ψ_2 and φ_2, respectively. This enables us to eliminate ψ_2 and φ_2 from (5.35) after the multiplication by $\cos\left(\frac{\pi x}{L}\right)$ and integration by parts, so that we obtain

$$0 = \bar{\chi}_1 \int_0^L \left(\psi_1 \left(\cos\left(\frac{\pi x}{L}\right) \right)' \right)' \cos\left(\frac{\pi x}{L}\right) dx + \frac{\bar{\chi}_1 m_\rho Q'''(0)}{6Q'(0)} \int_0^L \left(\left[\left(\cos\left(\frac{\pi x}{L}\right) \right)' \right]^3 \right)' \cos\left(\frac{\pi x}{L}\right) dx$$

$$+ \frac{\bar{\chi}_1^2 m_\rho Q'(0)}{\nu} \int_0^L \left(\varphi_1' \cos\left(\frac{\pi x}{L}\right) \right)' \cos\left(\frac{\pi x}{L}\right) dx + K_3 m_\rho \int_0^L \left(\cos\left(\frac{\pi x}{L}\right) \right)'' \cos\left(\frac{\pi x}{L}\right) dx. \qquad (5.36)$$

Observe that integration by parts yields

$$\int_0^L \left(\psi_1 \left(\cos\left(\frac{\pi x}{L}\right) \right)' \right)' \cos\left(\frac{\pi x}{L}\right) dx = \frac{\pi^2}{2L^2} \int_0^L \psi_1 \cos\left(\frac{2\pi x}{L}\right) dx \qquad (5.37)$$

and

$$\int_0^L \left(\varphi_1' \cos\left(\frac{\pi x}{L}\right) \right)' \cos\left(\frac{\pi x}{L}\right) dx = -\frac{\pi^2}{L^2} \int_0^L \varphi_1 \cos\left(\frac{2\pi x}{L}\right) dx. \qquad (5.38)$$

Multiplying (5.30) by $\cos\left(\frac{2\pi x}{L}\right)$, integrating with respect to x and taking into account that $K_2 = 0$, we have

$$\frac{\nu}{Q'(0)} \int_0^L \psi_1'' \cos\left(\frac{2\pi x}{L}\right) dx = \frac{\bar{\chi}_1^2 m_\rho Q'(0)}{2\nu} \int_0^L \left(\cos^2\left(\frac{\pi x}{L}\right) \right)'' \cos\left(\frac{2\pi x}{L}\right) dx + \bar{\chi}_1 m_\rho \int_0^L \varphi_1'' \cos\left(\frac{2\pi x}{L}\right) dx.$$

Integrating by parts the above integrals and evaluating the first integral on the RHS, yields

$$\frac{\nu}{Q'(0)} \int_0^L \psi_1 \cos\left(\frac{2\pi x}{L}\right) dx = \frac{\bar{\chi}_1^2 m_\rho Q'(0) L}{8\nu} + \bar{\chi}_1 m_\rho \int_0^L \varphi_1 \cos\left(\frac{2\pi x}{L}\right) dx. \tag{5.39}$$

Multiplying (5.32) by $\cos\left(\frac{2\pi x}{L}\right)$ and integrating by parts, we have

$$\gamma_\rho \int_0^L \psi_1 \cos\left(\frac{2\pi x}{L}\right) dx = \left(\gamma_c + \frac{4\pi^2}{L^2}\right) \int_0^L \varphi_1 \cos\left(\frac{2\pi x}{L}\right) dx.$$

This together with (5.39) leads to

$$\begin{aligned}
\int_0^L \varphi_1 \cos\left(\frac{2\pi x}{L}\right) dx &= \frac{\bar{\chi}_1(\gamma_c L^2 + \pi^2) Q'(0) L}{24\pi^2 \nu}, \\
\int_0^L \psi_1 \cos\left(\frac{2\pi x}{L}\right) dx &= \frac{\bar{\chi}_1(\gamma_c L^2 + \pi^2)(\gamma_c L^2 + 4\pi^2) Q'(0)}{24\pi^2 \nu \gamma_\rho L}.
\end{aligned} \tag{5.40}$$

Finally, combining (5.40) with (5.36)–(5.38), evaluating the second and the last integrals in (5.36), and using the relation (5.11), we obtain

$$K_3 m_\rho \pi^2 = \frac{(\gamma_c L^2 + \pi^2)\nu}{8\gamma_\rho Q'(0) L^4} \left[\frac{(\gamma_c L^2 + \pi^2)^2}{3\gamma_\rho^2 m_\rho^2} \left(\frac{2\pi^2}{L^2} - \gamma_c\right) - \frac{\pi^4 Q'''(0)}{Q'(0) L} \right] \tag{5.41}$$

Our assumptions on Q, (5.2) and (5.28), guarantee that the last term on the RHS of (5.41) is positive. Therefore, the cubic polynomial in γ_c inside the bracket in (5.41) has only one root, which we denote by $\gamma_c^* = \gamma_c^*(L, \gamma_\rho, m_\rho, Q'(0), Q'''(0)) > 0$, and we conclude that

$$\operatorname{sgn}(K_3) = \operatorname{sgn}(\gamma_c^* - \gamma_c)$$

Thus, we have proven the following theorem:

Theorem 5.4 *Assume that the function Q is C^5-smooth and satisfies (5.2) and (5.28). Then, the bifurcation curve (studied in Theorem 5.2) at $(m_\rho, m_c, \bar{\chi}_1)$ turns to the right if $\gamma_c \in (0, \gamma_c^*)$ and to the left if $\gamma_c \in (\gamma_c^*, \infty)$. Also, $\chi_1'(0) = 0$ and $\operatorname{sgn}(\chi_1''(0)) = \operatorname{sgn}(\gamma_c^* - \gamma_c)$.*

Two typical bifurcation curves, illustrating the results of Theorem 5.4, are schematically presented in Figure 5.3.

We now study the stability of the bifurcation steady states $(\rho_1(s, x), c_1(s, x))$ for $s \in (-\delta, \delta)$, $\delta > 0$. In order to prove that they are asymptotically stable, we need to show that the real part of any eigenvalue λ of the following eigenvalue problem is positive:

$$\begin{cases}
-\nu\rho'' + \chi_1(s)\left(\rho Q(c_1'(s, x)) + \rho_1 c' Q'(c_1'(s, x))\right)' = \lambda\rho, & x \in (0, L), \\
-c'' + \gamma_c c - \gamma_\rho \rho = \lambda c, & x \in (0, L), \\
\rho'(0) = \rho'(L) = c'(0) = c'(L) = 0, & \\
\int_0^L \rho(x) \, dx = 0.
\end{cases} \tag{5.42}$$

Note that when $s = 0$, 0 is an eigenvalue and the corresponding eigenspace is $\mathcal{N}\big(D_{(\rho,c)}\mathcal{F}(m_\rho, m_c, \bar{\chi}_1)\big) = \operatorname{span}\{(\bar{\rho}_1, \bar{c}_1)\}$. To study small eigenvalues of (5.42) when $s \neq 0$, we need to first use the eigenvalue perturbation result of [10, Corollary 1.13]. To fit into the abstract framework of [10], we define $\mathcal{K} : \mathscr{X} \times \mathscr{X} \to \mathscr{Y}_0 \times \mathscr{Y} \times \mathbb{R}$ by

$$
\mathcal{K}(\rho, c) = \begin{pmatrix} \rho - \dfrac{1}{L} \displaystyle\int_0^L \rho(x)\, dx \\ c \\ 0 \end{pmatrix},
$$

so that $\mathcal{K}$ is linear and bounded, and consider the following eigenvalue problem:

$$
D_{(\rho,c)}\mathcal{F}(\rho_1, c_1, \chi_1(s))(\rho, c) = \lambda \mathcal{K}(\rho, c), \qquad (\rho, c) \in \mathscr{X} \times \mathscr{X}. \tag{5.43}
$$

Obviously, (5.42) and (5.43) are equivalent (see (5.7)).

We first prove that $\lambda = 0$ is a "simple eigenvalue" of the pair $\big(D_{(\rho,c)}\mathcal{F}(m_\rho, m_c, \bar{\chi}_1), \mathcal{K}\big)$, which, according to [10], means that

(i) $D_{(\rho,c)}\mathcal{F}(m_\rho, m_c, \bar{\chi}_1) : \mathscr{X} \times \mathscr{X} \to \mathscr{Y}_0 \times \mathscr{Y} \times \mathbb{R}$ is Fredholm with zero index and has one-dimensional null space (this has been proved in Lemma 5.1);

(ii) $\mathcal{K}(\bar{\rho}_1, \bar{c}_1) \notin \mathcal{R}\big(D_{(\rho,c)}\mathcal{F}(m_\rho, m_c, \bar{\chi}_1)\big)$.

Proof of (ii): Assume that (ii) is not true. Then, there exists $(\rho, c) \in \mathscr{X} \times \mathscr{X}$ such that

$$
\begin{cases}
-\nu\rho'' + \bar{\chi}_1 m_\rho Q'(0)c'' = \bar{\rho}_1, & x \in (0, L), \\
-c'' + \gamma_c c - \gamma_\rho \rho = \bar{c}_1, & x \in (0, L), \\
\rho'(0) = \rho'(L) = c'(0) = c'(L) = 0, \\
\int_0^L \rho(x)\, dx = 0.
\end{cases}
$$

Integrating the first equation in the above system and using the formula for $\bar{\rho}_1$ in (5.11), we obtain

$$
-\nu\rho + \bar{\chi}_1 m_\rho Q'(0)c = -\frac{\bar{\chi}_1 m_\rho Q'(0) L^2}{\nu\pi^2} \cos\Big(\frac{\pi x}{L}\Big),
$$

and hence

$$
\begin{cases}
-c'' - \dfrac{\pi^2}{L^2}c = \dfrac{\gamma_c L^2 + (1 + \nu)\pi^2}{\nu\pi^2} \cos\Big(\dfrac{\pi x}{L}\Big), & x \in (0, L), \\
c'(0) = c'(L) = 0,
\end{cases}
$$

which contradicts Fredholm Alternative. ∎

Now, it follows from Corollary 1.13 in [10] that there exist real-valued and C^1-smooth functions $\chi \to \lambda_1(\chi)$ (χ is in a neighborhood of $\bar{\chi}_1$) and $s \in (-\delta, \delta) \to \lambda_2(s)$, where $\lambda_1(\bar{\chi}_1) = 0$, $\lambda_2(0) = 0$, $\lambda_1(\chi)$ is a real eigenvalue of

$$
D_{(\rho,c)}\mathcal{F}(m_\rho, m_c, \chi)(\rho, c) = \lambda \mathcal{K}(\rho, c), \qquad (\rho, c) \in \mathscr{X} \times \mathscr{X}, \tag{5.44}
$$

and $\lambda_2(s)$ is a real eigenvalue of (5.43). Moreover, for any fixed neighborhood of the origin of the complex plane, $\lambda_1(\chi)$ is the only eigenvalue of (5.44) in that neighborhood (the analogous assertion can be made for $\lambda_2(s)$).

Next, we use the dot-notation for the differentiation with respect to χ and compute $\dot{\lambda}_1(\bar{\chi}_1) = \dfrac{d\lambda_1}{d\chi}\Big|_{\chi=\bar{\chi}_1}$. The eigenfunction of (5.44) corresponding to $\lambda_1(\chi)$ can be written as $(\rho(\chi, x), c(\chi, x))$, which, by [10], depends on χ smoothly and is uniquely determined by

$$(\rho(\chi_1, x), c(\chi_1, x)) = (\bar{\rho}_1(x), \bar{c}_1(x)), \ (\rho(\chi, x), c(\chi, x)) - (\bar{\rho}_1(x), \bar{c}_1(x)) \in \mathscr{Z}.$$

Differentiating (5.44) with respect to χ and then setting $\chi = \bar{\chi}_1$, we obtain

$$\begin{cases} -\nu\dot{\rho}'' + \bar{\chi}_1 m_\rho Q'(0)\dot{c}'' + m_\rho Q'(0)\bar{c}_1'' = \dot{\lambda}_1(\bar{\chi}_1)\bar{\rho}_1, & x \in (0, L), \\ -\dot{c}'' + \gamma_c\dot{c} - \gamma_\rho\dot{\rho} = \dot{\lambda}_1(\bar{\chi}_1)\bar{c}_1, & x \in (0, L), \\ \dot{\rho}'(0) = \dot{\rho}'(L) = \dot{c}'(0) = \dot{c}'(L) = 0. \end{cases}$$

Multiplying the $\dot{\rho}$-equation by $\bar{c}_1$ and the $\dot{c}$-equation by $\nu\pi^2\bar{c}_1/(L^2\gamma_\rho)$, adding the resulting equations, integrating with respect to x over $[0, L]$ and using (5.11), we arrive at

$$-m_\rho Q'(0)\pi^2 = \dot{\lambda}_1(\bar{\chi}_1)\frac{\gamma_c L^2 + (1+\nu)\pi^2}{\gamma_\rho},$$

which implies that $\dot{\lambda}_1(\bar{\chi}_1) < 0$.

Now, by Theorem 1.16 in [10],

$$\lim_{s\to 0} \frac{-s\chi_1'(s)\dot{\lambda}_1(\bar{\chi}_1)}{\lambda_2(s)} = 1.$$

This, together with Theorem 5.4, implies that for $s \in (-\delta, \delta)$, $s \neq 0$,

$$\mathrm{sgn}\lambda_2(s) = \mathrm{sgn}(\gamma_c^* - \gamma_c). \tag{5.45}$$

Theorem 5.5 *Assume all the conditions in Theorem 5.4 hold. For $s \in (-\delta, \delta)$, $s \neq 0$, the steady state $(\rho_1(s, x), c_1(s, x))$ is asymptotically stable within the class of functions (ρ, c) with $\int_0^L \rho(x)\, dx = m_\rho$ if $\gamma_c \in (0, \gamma_c^*)$ and unstable if $\gamma_c \in (\gamma_c^*, \infty)$.*

Proof: The instability in the case of $\gamma_c \in (\gamma_c^*, \infty)$ immediately follows from the negative sign of $\lambda_2(s)$, established in (5.45).

To show the stability in the case of $\gamma_c \in (0, \gamma_c^*)$, we just need to show that given a small neighborhood of the origin of the complex plane, (5.42) does not have an eigenvalue which has a negative real part and which is outside of this neighborhood. This would follow from the standard eigenvalue perturbation theory if we can show that the limit of (5.42) as $s \to 0$, that is, the eigenvalue problem

$$\begin{cases} -\nu\rho'' + \bar{\chi}_1 m_\rho Q'(0)c'' = \lambda\rho, & x \in (0, L), \\ -c'' + \gamma_c c - \gamma_\rho\rho = \lambda c, & x \in (0, L), \\ \rho'(0) = \rho'(L) = c'(0) = c'(L) = 0, \\ \int_0^L \rho(x)\, dx = 0 \end{cases} \tag{5.46}$$

has no nonzero eigenvalues with nonpositive real parts. To this end, we expand ρ and c as follows:

$$\rho(x) = \sum_{k=0}^{\infty} a_k \cos\left(\frac{k\pi x}{L}\right), \quad c(x) = \sum_{k=0}^{\infty} b_k \cos\left(\frac{k\pi x}{L}\right),$$

where $a_0 = 0$ because $\int_0^L \rho(x)\,dx = 0$. Then

$$\begin{cases} \nu k^2\pi^2 a_k - \bar{\chi}_1 m_\rho Q'(0)k^2\pi^2 b_k = \lambda L^2 a_k, \\ k^2\pi^2 b_k + \gamma_c L^2 b_k - \gamma_\rho L^2 a_k = \lambda L^2 b_k, \end{cases} \tag{5.47}$$

and λ is an eigenvalue of (5.46) if and only if there exist $k \geq 0$ and $(a_k, b_k) \neq (0,0)$ such that (5.47) holds, which is equivalent to

$$L^4\lambda^2 - L^2\left((1+\nu)k^2\pi^2 + \gamma_c L^2\right)\lambda + \nu k^2\pi^2(k^2\pi^2 + \gamma_c L^2) - \bar{\chi}_1 m_\rho Q'(0)k^2\pi^2\gamma_\rho L^2 = 0.$$

Thus, any nonzero eigenvalue λ must have a positive real part. ∎

We surmise that if $\gamma_c \in (0, \gamma_c^*)$, then all nontrivial solutions of (5.1), (5.4) on the bifurcation curve $\mathcal{P}$ are asymptotically stable, while if $\gamma_c \in (\gamma_c^*, \infty)$, then the nontrivial solutions of (5.1), (5.4) on $\mathcal{P}$, which are not near the bifurcation point $(m_\rho, m_c, \bar{\chi}_1)$ are also asymptotically stable, see Figure 5.3.
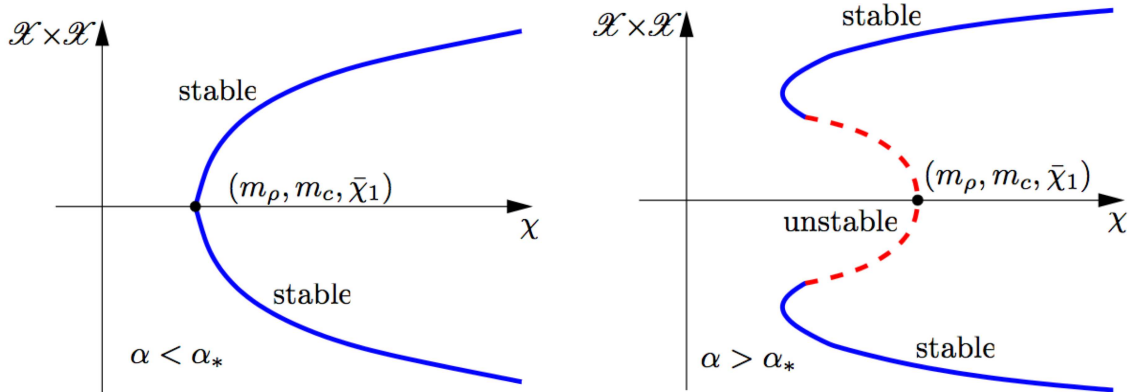
Figure 5.3: Stable (solid line) and unstable (dashed line) parts of typical bifurcation curves.

5.3 Nonmonotone steady states on $(0, L)$

In this section, we provide a few examples of how to construct multi-spike solutions of (5.1), (5.4) using the monotone solutions, discussed above, and the reflection argument. We also numerically demonstrate that these solutions may emerge as steady-state solutions of a time-dependent IBVP for the system (5.25) and conduct their experimental stability study.

Example 1 — Double boundary spike steady states. Given $m_\rho > 0$, let (ρ, c) be a positive decreasing solution of (5.1) on $(0, L/2)$ with $\int_0^{L/2} \rho(x)\,dx = Lm_\rho/2$. Extending (ρ, c) by reflecting it about $x = L/2$, results in the function, which is a solution of (5.1) satisfying (5.4) and increasing on $(L/2, L)$.

Example 2 — Single interior spike steady states. Given $m_\rho > 0$, let (ρ, c) be a positive increasing solution of (5.1) on $(0, L/2)$ with $\int_0^{L/2} \rho(x)\, dx = Lm_\rho/2$. Reflecting this solution with respect to $x = L/2$, we obtain a solution of (5.1) satisfying (5.4) and decreasing on $(L/2, L)$.

Obviously, we can produce steady states with arbitrary many boundary and interior spikes. The stability of each of these non-monotone steady steady states is the same as the stability of the monotone ones (which we use as building blocks), within the class of functions that have the same symmetry as the non-monotone steady state. However, if the perturbation does not have the same symmetry as the corresponding steady-state solution, the limiting (as $t \to \infty$) solution may have a completely different structure. To illustrate these results numerically, we consider a particular example, in which the system (5.25), (5.3) with different values of the chemosensitivity constant χ and subject to the homogeneous Neumann boundary conditions is numerically studied on the interval $[0, 1]$. Here, Q is given by (5.3), and $\nu = \gamma_c = \gamma_\rho = 1$.

We first consider the initial data

$$\rho(x, 0) = 1 + s\bar{\chi}_2 \cos(2\pi x), \quad c(x, 0) = 1 + s \cos(2\pi x),$$

with $\bar{\chi}_2 = 1 + 4\pi^2$ (see (5.11)) and the small parameter $s = \pm 0.01$. We take either the small $\chi = 1.2\bar{\chi}_2$ or the intermediate $\chi = 12\bar{\chi}_2$. Taking $s = 0.01$ leads to the convergence towards the numerical steady-state solution, whose ρ-component contains two boundary spikes, see Figure 5.4. Such solution corresponds to an analytical steady state described in Example 1. Switching the sign of s to $s = -0.01$ leads to the convergence to a different steady state with only one interior spike in the ρ-component of the solution, see Figure 5.5. This corresponds to the single interior spike steady-state solution described in Example 2. We note that in both cases, the obtained non-monotone steady-state solutions are stable under small numerical perturbations present in every numerical computation.
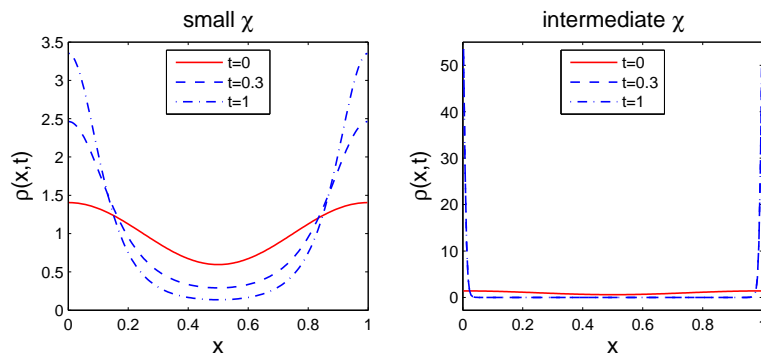

Figure 5.4: Double boundary spike steady states.

We then consider the following initial data,

$$\rho(x, 0) = 1 + 0.01\bar{\chi}_3 \cos(3\pi x), \quad c(x, 0) = 1 + 0.01 \cos(3\pi x), \quad \bar{\chi}_3 = 1 + 9\pi^2,$$

where $\rho(x, 0)$ has two local maxima and two local minima on the interval $[0, 1]$. We take a small chemotaxis sensitivity constant $\chi = 1.2\bar{\chi}_3$ and compute numerical solutions on two different uniform grids with $\Delta x = 1/201$ and $\Delta x = 1/200$. The ρ-component of the solution obtained with $\Delta x = 1/201$ is shown in Figure 5.6. As one can see, this numerical solution preserves
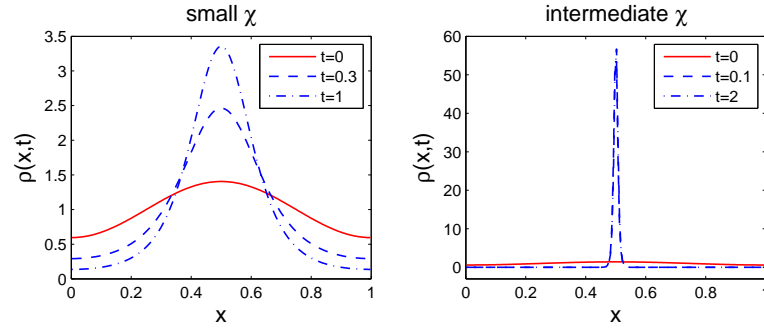
Figure 5.5: Single interior spike steady states.

the symmetry of the initial datum. The other solution (the one computed with $\Delta x = 1/200$) is presented in Figure 5.7. In this case, the solution preserves its initial symmetry for some time (in fact, if one stops the computation at time $t = 1$, one may conclude that the numerical solution had already reached its symmetric steady state by that time). However, at later times, the symmetry gets destroyed and the resulting steady-state solution has only one boundary spike at $x = 1$. The reason why the two numerical solutions are so different is in the fact that 201 is divisible by 3 while 200 is not. Notice that in the latter case, the grid does not correspond to the data symmetry, which leads to the lack of symmetry in numerical perturbations, which in turn causes the break of the solution symmetry for large t.
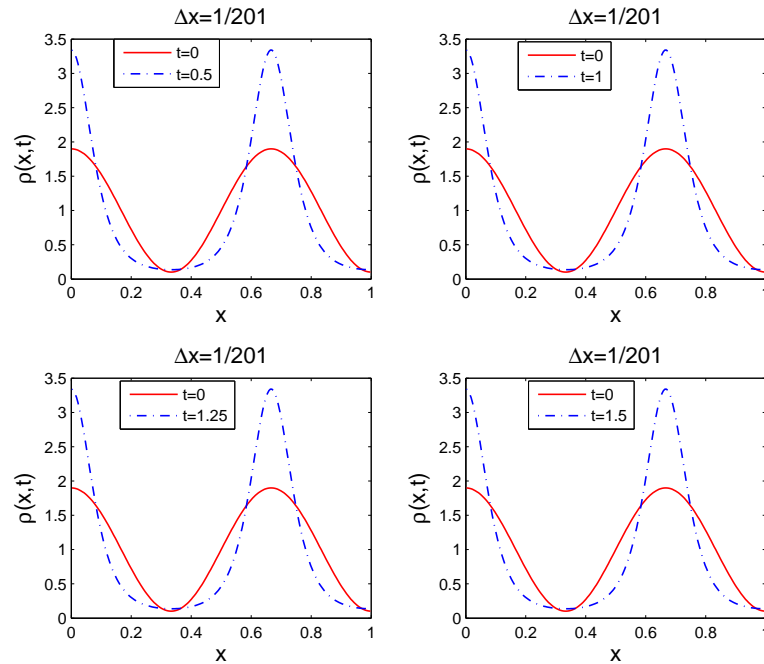
Figure 5.6: Unstable solution computed on a grid with $\Delta x = 1/201$.
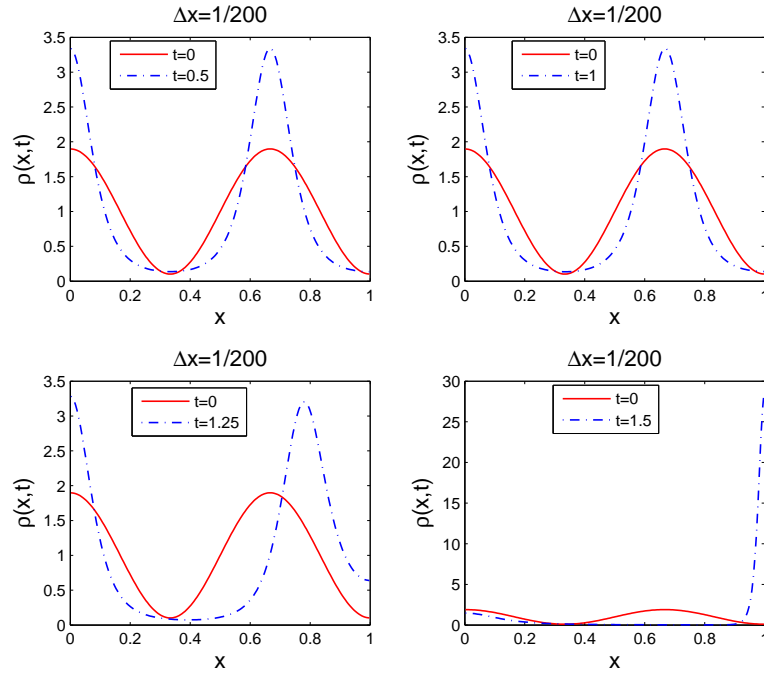
Figure 5.7: Unstable solution computed on a grid with $\Delta x = 1/200$.

6 Multidimensional Steady-State Solutions

In the d-dimensional case, $d \geq 2$, we can also obtain a local bifurcation result, which is an analogue of Theorem 5.1. This can be done by slightly modifying the preliminaries and the proof of Theorem 5.1. First, we take $\mathscr{X} = \{u \in W^{2,p}(\Omega) \mid \frac{\partial u}{\partial \mathbf{n}}|_{\partial \Omega} = 0\}$ and $\mathscr{Y} = L^p(\Omega)$ with a fixed $p > d$. Then, $\mathcal{N}\big(D_{(\rho,c)}\mathcal{F}(m_\rho, m_c, \chi)\big) \neq \{0\}$ if and only if $\frac{\gamma_\rho \chi m_\rho Q'(0)}{\nu} - \gamma_c$ is one of the *positive* Neumann eigenvalues, which we denote by μ_k, $k = 1, 2, \ldots$. Next, let $\bar{c}_k(x)$ be a Neumann eigenfunction associated with μ_k,

$$\bar{\rho}_k(x) = \frac{\bar{\chi}_k m_\rho Q'(0)}{\nu}\bar{c}_k(x) \quad \text{and} \quad \bar{\chi}_k = \frac{(\gamma_c + \mu_k)\nu}{\gamma_\rho m_\rho Q'(0)}.$$

Then, Theorem 5.1 with $\cos\big(\frac{k\pi x}{L}\big)$ replaced by $\bar{c}_k(x)$ holds for any $k \geq 1$ such that the Neumann eigenvalue corresponding to μ_k is the 1-D one.

For instance, in the special 2-D case with $\Omega = (0,2) \times (0,1)$, the positive Neumann eigenvalues are (the number of repetition equal to the dimension of the eigenspace):

$$\frac{\pi^2}{4}, \qquad \pi^2, \qquad \pi^2, \qquad \frac{5\pi^2}{4}, \qquad 2\pi^2, \qquad \frac{9\pi^2}{4}, \qquad \cdots$$
$$\updownarrow \qquad \updownarrow \qquad \updownarrow \qquad \updownarrow \qquad \updownarrow \qquad \updownarrow \qquad (6.1)$$
$$\cos\Big(\frac{\pi x}{2}\Big), \quad \cos(\pi x), \quad \cos(\pi y), \quad \cos\Big(\frac{\pi x}{2}\Big)\cos(\pi y), \quad \cos(\pi x)\cos(\pi y), \quad \cos\Big(\frac{3\pi x}{2}\Big), \quad \cdots$$

In this special case, every Neumann eigenvalue, repeated or not, gives rise to a bifurcation point, since a repeated eigenvalue produces bifurcating steady states depending on only one spatial variable.

In the remaining part of this section, we perform an extensive numerical study of 2-D steady-state solutions. We consider the IBVP (3.1) with $\nu = \gamma_c = \gamma_\rho = 1$, $\mathbf{Q}$ given by (1.5), and choose different values of χ and different sets of initial data.

First, we numerically study the case corresponding to the Neumann eigenvalue $\mu_3 = \frac{5\pi^2}{4}$ (notice that for smaller eigenvalues the corresponding steady states are quasi 1-D, see (6.1)). We take $\chi = 10\mu_3$ and the initial data

$$\rho(x, y, 0) = 1 - 0.01\mu_3 \cos\left(\frac{\pi x}{2}\right) \cos(\pi y), \qquad c(x, y, 0) = 1 - 0.01 \cos\left(\frac{\pi x}{2}\right) \cos(\pi y), \qquad (6.2)$$

prescribed in the domain $\Omega = [0, 2] \times [0, 1]$ (see Figure 6.1). The solution of this IBVP is expected to develop two spikes at two opposite corners of Ω, $(2, 0)$ and $(0, 1)$. Indeed, the numerical steady state computed on the uniform grid with $\Delta x = \Delta y = 1/100$, shown in Figure 6.1, meets the expectations.
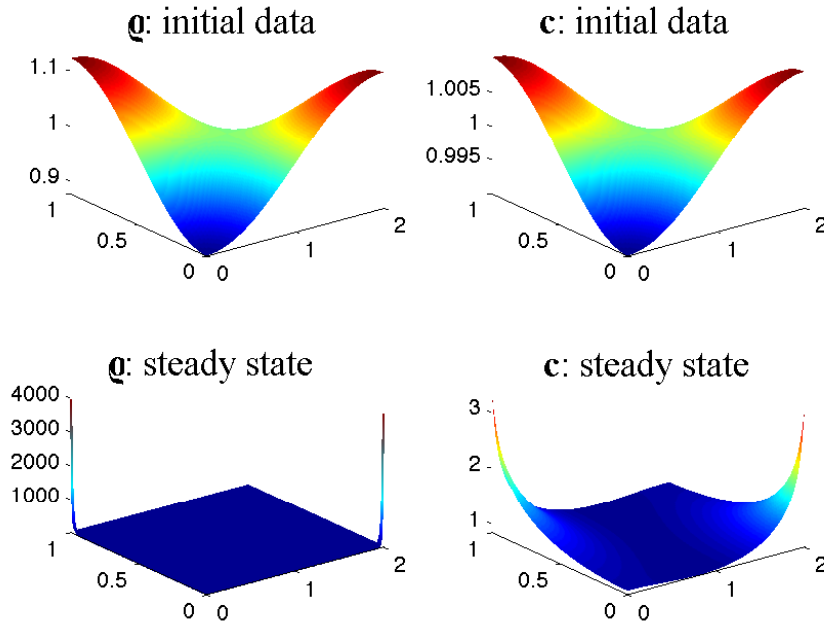

Figure 6.1: Double corner spike emerging out of the symmetric initial data (6.2).

Next, we modify the initial data to

$$\rho(x, y, 0) = 1 - 0.01\mu_3 \cos\left(\frac{\pi x}{2}\right) \sin(\pi y), \qquad c(x, y, 0) = 1 - 0.01 \cos\left(\frac{\pi x}{2}\right), \qquad (6.3)$$

and consider the IBVP (3.1), (1.5), (6.3) on the same domain $\Omega = [0, 2] \times [0, 1]$. The shape of the initial data, shown in Figure 6.2, suggests that now the solution is expected to develop only one spike at $(2, 0.5)$, which is the middle of the edge of Ω. The obtained numerical steady-state solution, computed on the uniform grid with $\Delta x = \Delta y = 1/100$, confirms the conjecture, see Figure 6.2.
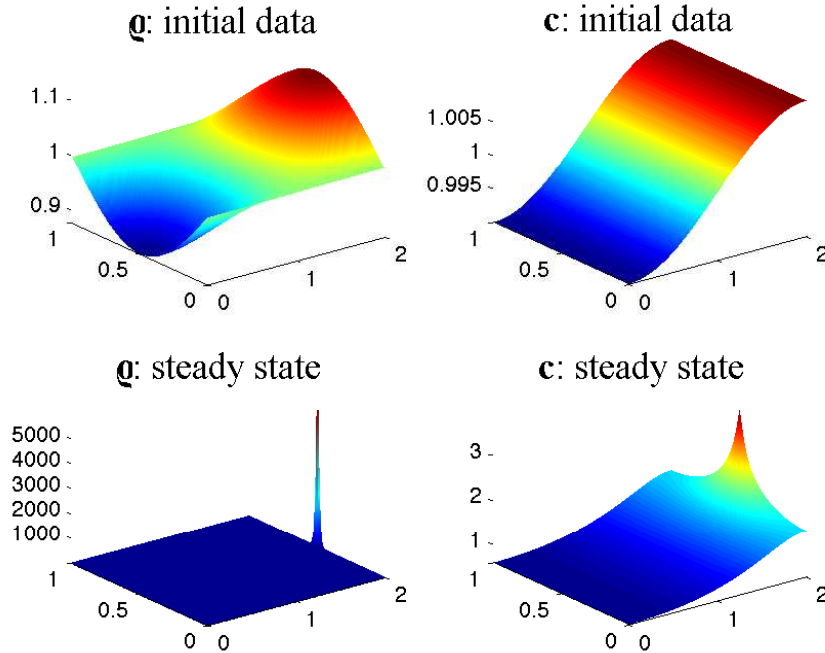
Figure 6.2: Single boundary spike emerging out of the initial data (6.3).

In the next experiment, we take $\chi = 100$ and consider the IBVP (3.1), (1.5) in the same domain $\Omega = [0, 2] \times [0, 1]$, but subject to different initial conditions,

$$\rho(x, y, 0) \equiv c(x, y, 0) = 100 + \cos\left(\frac{\pi x}{2}\right)\sin(\pi y), \tag{6.4}$$

shown in Figure 6.3. In this case, one can also expect the solution to develop spikes at the locations of initial local extrema. However, the numerical solution, computed on the uniform grid with $\Delta x = \Delta y = 1/100$, behaves in an unpredictable way: it develops one large interior spike and two smaller spikes at the opposite sides of Ω, and an additional smaller spike at the middle of the third side of Ω, see Figure 6.3. It is instructive to compare the above solution with another numerical solution, computed on a slightly finer uniform grid with $\Delta x = 2/201$ and $\Delta y = 1/101$. The large spike seems to be located at the same interior point (its height is slightly different though), but we now have four additional smaller spikes: two of them are located at the opposite sides of Ω, and two (even smaller) ones emerge at the corners $(2, 0)$ and $(2, 1)$. This five-spike steady-state solution is presented in Figure 6.4. The obtained results demonstrate that small numerical perturbations of initial data may lead to quite different solution structures.

The last example suggests that steady states may have quite complicated spiky structures. To further investigate this, we numerically solve the IBVP (3.1), (1.5) with $\chi = 50$ in a larger square domain $\Omega = [0, 10] \times [0, 10]$ with the initial data

$$\rho(x, y, 0) = 1 + \sigma, \quad c(x, y, 0) = 0, \tag{6.5}$$

where σ is a random variable uniformly distributed in $[-0.1, 0.1]$. The numerical solution develops a stable multi-spike structure, see the obtained numerical steady state plotted in Figure 6.5.
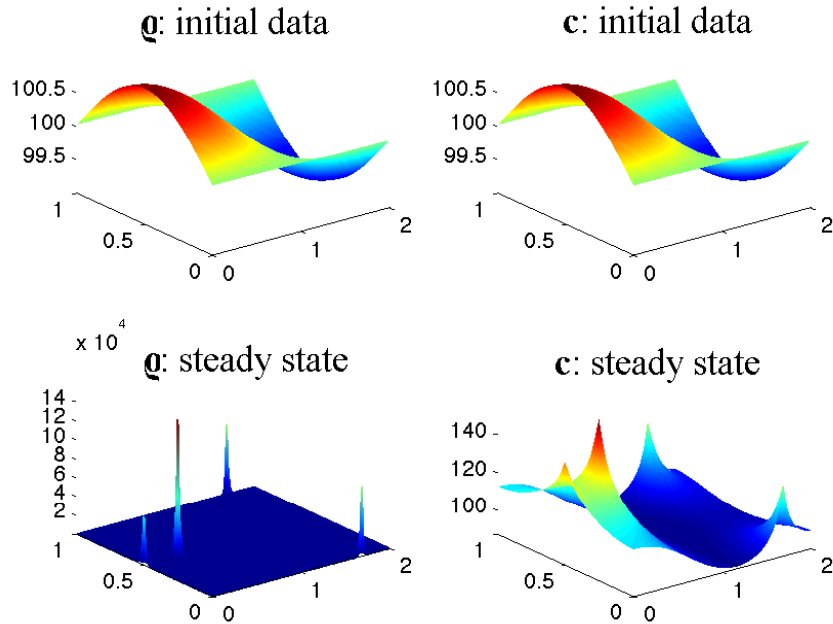
Figure 6.3: Four-spike structure emerging out of the initial data (6.4) using $\Delta x = \Delta y = 1/100$.

This result suggests that the studied chemotaxis system with a saturated chemotactic flux can be used to model solutions with multiple spikes (both interior and boundary ones) appearing in real biological systems.

Remark 6.1 We would like to point out that our numerical experiments, in which the multi-spike solutions arise, provide only preliminary results. A further rigorous analysis of the 2-D chemotaxis system with a saturated chemotactic flux is required to fully understand the development, evolution and stability of such complicated solutions.
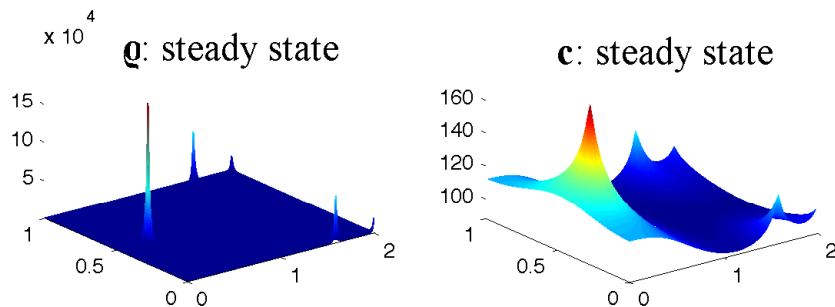
Figure 6.4: Five-spike structure emerging out of the initial data (6.4) using $\Delta x = 2/201$, $\Delta y = 1/101$.
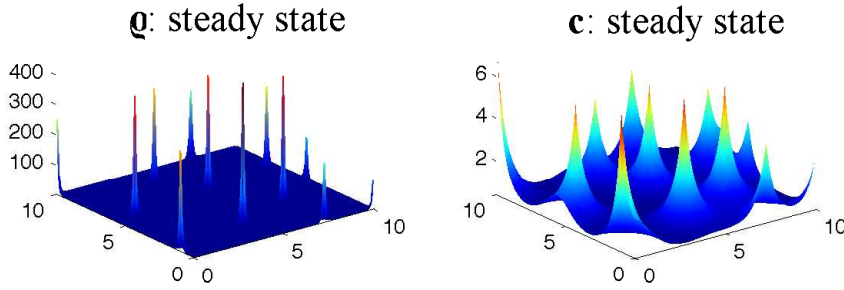
Figure 6.5: Multi-spike structure emerging out of the random initial data (6.5).

Appendix A

The numerical results presented in this paper are obtained using a second-order positivity preserving upwind scheme, which is a straightforward extension of the finite-volume method developed in [7]. In this section, we briefly describe a 2-D version of the scheme for the chemotaxis system (1.2).

We introduce a Cartesian mesh consisting of the uniform cells $C_{j,k} = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \times [y_{k-\frac{1}{2}}, y_{k+\frac{1}{2}}]$ of the size $\Delta x \Delta y$ centered at (x_j, y_k). The computed quantities are the cell averages of cell density ρ,

$$\bar{\rho}_{j,k}(t) := \frac{1}{\Delta x \Delta y} \iint\limits_{C_{j,k}} \rho(x, y, t)\, \Delta x \Delta y,$$

and point values of the chemoattractant concentration c, $c_{j,k}(t) = c(x_j, y_k, t)$, which are evolved in time according to the semi-discrete scheme:

$$
\begin{cases}
\begin{aligned}
\frac{d\bar{\rho}_{j,k}}{dt} =& -\frac{H^x_{j+\frac{1}{2},k} - H^x_{j-\frac{1}{2},k}}{\Delta x} - \frac{H^y_{j,k+\frac{1}{2}} - H^y_{j,k-\frac{1}{2}}}{\Delta y} \\
&+ \nu \left(\frac{\bar{\rho}_{j-1,k} - 2\bar{\rho}_{j,k} + \bar{\rho}_{j+1,k}}{(\Delta x)^2} + \frac{\bar{\rho}_{j,k-1} - 2\bar{\rho}_{j,k} + \bar{\rho}_{j,k+1}}{(\Delta y)^2} \right),
\end{aligned} \\[2ex]
\frac{dc_{j,k}}{dt} = \frac{c_{j-1,k} - 2c_{j,k} + c_{j+1,k}}{(\Delta x)^2} + \frac{c_{j,k-1} - 2c_{j,k} + c_{j,k+1}}{(\Delta y)^2} - \gamma_c c_{j,k} + \gamma_\rho \bar{\rho}_{j,k}.
\end{cases}
\tag{A.1}
$$

Here, $H^x_{j+\frac{1}{2},k}$ and $H^y_{j,k+\frac{1}{2}}$ are the following upwind numerical fluxes:

$$H^x_{j+\frac{1}{2},k} = \chi \rho_{j+\frac{1}{2},k} Q_1\left(\frac{c_{j+1,k} - c_{j,k}}{\Delta x}\right), \quad H^y_{j,k+\frac{1}{2}} = \chi \rho_{j,k+\frac{1}{2}} Q_2\left(\frac{c_{j,k+1} - c_{j,k}}{\Delta y}\right), \tag{A.2}$$

where Q_1 and Q_2 are the components of the vector function $\mathbf{Q}$ (see (1.3)) and

$$
\rho_{j+\frac{1}{2},k} = \begin{cases} \rho^E_{j,k}, & \text{if } Q_1\left(\dfrac{c_{j+1,k} - c_{j,k}}{\Delta x}\right) > 0, \\ \rho^W_{j,k}, & \text{otherwise}, \end{cases}
\qquad
\rho_{j,k+\frac{1}{2}} = \begin{cases} \rho^N_{j,k}, & \text{if } Q_2\left(\dfrac{c_{j,k+1} - c_{j,k}}{\Delta y}\right) > 0, \\ \rho^S_{j,k}, & \text{otherwise}. \end{cases}
$$

The point values $\rho^{E(W,N,S)}_{j,k}$ are obtained using the piecewise linear reconstruction

$$\widetilde{\rho}(x, y) = \bar{\rho}_{j,k} + (\rho_x)_{j,k}(x - x_j) + (\rho_y)_{j,k}(y - y_k), \qquad (x, y) \in C_{j,k},$$

with the slopes $(\rho_x)_{j,k}$ and $(\rho_y)_{j,k}$ calculated using the minmod2 limiter:

$$(\rho_x)_{j,k} = \text{minmod}\left(2\frac{\bar{\rho}_{j+1,k} - \bar{\rho}_{j,k}}{\Delta x}, \frac{\bar{\rho}_{j+1,k} - \bar{\rho}_{j-1,k}}{2\Delta x}, 2\frac{\bar{\rho}_{j,k} - \bar{\rho}_{j-1,k}}{\Delta x}\right),$$

$$(\rho_y)_{j,k} = \text{minmod}\left(2\frac{\bar{\rho}_{j,k+1} - \bar{\rho}_{j,k}}{\Delta y}, \frac{\bar{\rho}_{j,k+1} - \bar{\rho}_{j,k-1}}{2\Delta y}, 2\frac{\bar{\rho}_{j,k} - \bar{\rho}_{j,k-1}}{\Delta y}\right),$$

where the minmod function is defined by

$$\text{minmod}(z_1, z_2, \ldots, z_m) := \begin{cases} \min(z_1, z_2, \ldots, z_m), & \text{if } z_i > 0 \ \ \forall i = 1, \ldots, m, \\ \max(z_1, z_2, \ldots, z_m), & \text{if } z_i < 0 \ \ \forall i = 1, \ldots, m, \\ 0, & \text{otherwise.} \end{cases}$$

Thus,

$$\rho_{j,k}^{\text{E}} = \widetilde{\rho}(x_{j+\frac{1}{2}}, y_k), \ \ \rho_{j,k}^{\text{W}} = \widetilde{\rho}(x_{j-\frac{1}{2}}, y_k), \ \ \rho_{j,k}^{\text{N}} = \widetilde{\rho}(x_j, y_{k+\frac{1}{2}}), \ \ \rho_{j,k}^{\text{S}} = \widetilde{\rho}(x_j, y_{k-\frac{1}{2}}).$$

Remark A.2 Notice that in the above formulae, the quantities $\bar{\rho}_{j,k}$, $c_{j,k}$, $H^x_{j+\frac{1}{2},k}$, $H^y_{j,k+\frac{1}{2}}$, $\rho_{j+\frac{1}{2},k}$, $\rho_{j,k+\frac{1}{2}}$, $\rho_{j,k}^{\text{E (W,N,S)}}$, $(\rho_x)_{j,k}$, $(\rho_y)_{j,k}$ and the function $\widetilde{\rho}(x,y)$ depend on time, but we suppress this dependence for brevity.

Remark A.3 The semi-discrete scheme (A.1) is a system of time-dependent ODEs, which has to be integrated numerically using a stable and accurate ODE solver. In this paper, we have used a third-order strong stability preserving (SSP) Runge-Kutta method from [12]. The efficiency of the fully discrete method can be improved by applying an SSP implicit-explicit Runge-Kutta method (see, e.g., [15] and references therein), as discussed in [7].

Remark A.4 The 1-D version of the numerical method used in the paper can be easily reduced from the 2-D scheme described above.

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