Semi-discrete central schemes for balance laws. Application to the Broadwell model.

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ABSTRACT. *We discuss applications of semi-discrete central schemes for systems of balance laws. We distinguish between two generic cases of stiff and nonstiff source terms. In the nonstiff case, the main advantage of semi-discrete central schemes is their universality. Since no (approximate) Riemann problem solver or characteristic field decomposition is involved, no operator splitting is required. It allows one to eliminate splitting errors, which may be very significant, especially for quasi-stationary solutions. In the stiff case, operator splitting or an implicit-explicit ODE solver has to be implemented in order to preserve efficiency of the scheme. Our numerical experiments demonstrate that the designed method, based on the semi-discrete central scheme from [KUR 00a], performs extremely well in both regimes.*

KEYWORDS: *Balance laws, conservation laws, high-resolution semi-discrete central schemes, stiff source term, Broadwell model.*

1. Introduction

We consider the one-dimensional (1-D) system of balance laws,

$$u_t + f(u)_x = \frac{1}{\varepsilon} S(u,x,t), \qquad u \in \mathbb{R}^N, \quad \varepsilon > 0, \tag{1}$$

subject to the initial data, $u(x,0) = u_0(x)$. Balance laws arise in several hydrodynamical models (including shallow water equations), gravitational flows, multi-phase models, chemotaxis models and other applications. If the parameter $\varepsilon << 1$, the source term is stiff, and the system [1] is of relaxation type. Stiff balance laws are of a special interest, since they are used to model reacting flows, combustion, detonation, absorption, semiconductors, magnetohydrodynamics, traffic flows, and many other phenomena.

In this paper, we study finite-volume methods for balance laws. In particular, we focus on Godunov-type central schemes, which are not tied to a special eigenstructure of the problem, and therefore can be used as a "black-box-solver" for complicated systems. The prototype of Godunov-type central schemes is the first-order Lax-Friedrichs scheme [LAX 71]. Its *staggered* second-order generalization was proposed in [NES 90]. For higher-order extensions of the staggered central schemes we refer the reader to [LEV 00] and the references therein.

The major drawback of staggered central schemes is their relatively large numerical dissipation, which can be decreased when (one-sided) local speeds of propagation are utilized for more precise estimate of the width of the Riemann fans. This leads to nonstaggered central schemes, developed in [KUR 00b, KUR 01, KUR 00a] for homogeneous conservation laws. It should be noticed that unlike the staggered central schemes, these new schemes admit a particularly simple semi-discrete form, briefly described in §2. This feature is especially advantageous when the central schemes are applied to convection-diffusion equations, [KUR 00b, KUR 01], or nonstiff balance laws, [KUR 02], since no operator splitting is required in these cases.

The paper is focused on the application of semi-discrete central schemes to hyperbolic systems with relaxation. As a test-problem, which admits both stiff and nonstiff regimes, the Broadwell model [BRO 64] is considered. It is quite a challenging task to design a numerical method, which is capable to treat both regimes of the model efficiently and accurately (see [CAF 97]). Staggered central schemes for the Broadwell model, which achieve this goal, have been recently introduced in [LIO 00, PAR 02a]. The nonstaggered semi-discrete framework, however, is much more convenient than the fully-discrete staggered one, especially in the stiff case. In §3, we design a second-order semi-discrete scheme for the Broadwell model. The numerical experiments, presented in §4, confirm a very high resolution of the proposed method.

2. Semi-discrete central schemes – a brief overview

In this section we give a brief description of the semi-discrete central schemes, developed in [KUR 00b, KUR 01, KUR 00a, KUR 02].

We first introduce a uniform spatial grid, $x_\alpha = \alpha\Delta x$. The integration of system [1] over the control volume $I_j := [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ results in an equivalent semi-discrete form of [1],

$$\frac{d}{dt}\bar{u}_j(t) + \frac{f\Big(u(x_{j+\frac{1}{2}},t)\Big) - f\Big(u(x_{j-\frac{1}{2}},t)\Big)}{\Delta x} = \frac{1}{\varepsilon\Delta x}\int\limits_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} S\Big(u(x,t),x,t\Big)\,dx, \quad [2]$$

where $\bar{u}_j(t)$ denotes the cell average over I_j, $\bar{u}_j(t) := \frac{1}{\Delta x}\int_{I_j} u(x,t)\,dx$. Semi-discrete finite-volume schemes are then obtained by approximating the fluxes at $x_{j\pm\frac{1}{2}}$, and by the application of an appropriate quadrature for computing the source average, $\bar{S}_j(t)$, on the right-hand side of [2]:

$$\frac{d}{dt}\bar{u}_j(t) = -\frac{H_{j+\frac{1}{2}}(t) - H_{j-\frac{1}{2}}(t)}{\Delta x} + \frac{1}{\varepsilon}\bar{S}_j(t), \qquad [3]$$

where $H_{j+\frac{1}{2}}$ is a numerical flux.

In this paper, we use a family of central-upwind schemes from [KUR 00a], whose numerical fluxes can be presented in the following form,

$$H_{j+\frac{1}{2}}(t) = \frac{a^+_{j+\frac{1}{2}} f(u^-_{j+\frac{1}{2}}) - a^-_{j+\frac{1}{2}} f(u^+_{j+\frac{1}{2}})}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + a^+_{j+\frac{1}{2}} a^-_{j+\frac{1}{2}} \left[\frac{u^+_{j+\frac{1}{2}} - u^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} - \frac{q_{j+\frac{1}{2}}}{2}\right]. \qquad [4]$$

Here, $u^{\pm}_{j+\frac{1}{2}} := p_{j+\frac{1}{2}\pm\frac{1}{2}}(x_{j+\frac{1}{2}})$ are the intermediate values of the piecewise polynomial interpolant, $\sum_j p_j(x,t)\chi_j$, reconstructed at each time step from the previously computed cell averages, $\{\bar{u}_j(t)\}$. The functions $\{p_j(\cdot,t)\}$ are polynomials of a given degree, and χ_j is the characteristic function of the interval I_j.

The one-sided *local* speeds of propagation, $a^{\pm}_{j+\frac{1}{2}}$, are determined by

$$\begin{aligned} a^+_{j+\frac{1}{2}} &= \max\left\{\lambda_N\left(\frac{\partial f}{\partial u}(u^-_{j+\frac{1}{2}})\right), \lambda_N\left(\frac{\partial f}{\partial u}(u^+_{j+\frac{1}{2}})\right), 0\right\}, \\ a^-_{j+\frac{1}{2}} &= \min\left\{\lambda_1\left(\frac{\partial f}{\partial u}(u^-_{j+\frac{1}{2}})\right), \lambda_1\left(\frac{\partial f}{\partial u}(u^+_{j+\frac{1}{2}})\right), 0\right\}, \end{aligned} \qquad [5]$$

with $\lambda_1 < \ldots < \lambda_N$ being the N eigenvalues of the Jacobian $\partial f/\partial u$.

Finally, $q_{j+\frac{1}{2}} := q(u^{\pm}_{j+\frac{1}{2}}, a^{\pm}_{j+\frac{1}{2}})$ represents an additional degree of freedom, which may be used to further decrease the numerical dissipation, attributed to the original central-upwind scheme from [KUR 01], where $q_{j+\frac{1}{2}}$ was set to be zero. We refer the reader to [KUR 00a] for details.

REMARK. — We would like to emphasize that one of the main advantages of the above semi-discrete central scheme is its simplicity and universality. Indeed, the 1-D system of balance laws can be solved component-wise since no (approximate) Riemann problem solvers were utilized. Moreover, this allows one to use the scheme [3]–[5] without operator splitting. For example, in [KUR 02], such unsplit scheme (with $q_{j+\frac{1}{2}} = 0$) was applied to the Saint-Venant system of shallow water equations [SAI 1871] with the source term due to the nonflat bottom elevation. The most delicate point in this application was to preserve a balance between the flux gradients and the source term, when the solution is quasi-stationary. This was achieved by the use of a special quadrature for the source average in [3], see [KUR 02] for details.

REMARK. — The (formal) order of the numerical flux [4] is determined by the (formal) order of a piecewise linear reconstruction used in the computation of the intermediate values. The non-oscillatory nature of the computed solution is typically guaranteed by using a non-oscillatory reconstruction. In this paper, we use the two-parameter family of piecewise linear, second-order reconstructions from [LIE 02].

REMARK. — The semi-discrete scheme [3]–[5] forms a system of ODEs, which should be solved by a stable ODE solver of an appropriate order. When the system [1] in nonstiff, explicit methods can be efficiently used. The situation is much more delicate for stiff systems, in which case explicit methods may be inefficient. Alternative implicit approaches are discussed below.

3. Stiff problems: the Broadwell model

In this section, we design a second-order semi-discrete central scheme for stiff systems of balance laws ($\varepsilon << 1$). As an example, we consider the Broadwell model, [BRO 64], that describes a two-dimensional (2-D) gas as composed of particles of only four velocities with a binary collision law and spatial variation in only one direction. When looking for 1-D solutions of the 2-D gas, the evolution equations of the model are given by

$$\left\{\begin{array}{rcl} f_t + f_x &=& \frac{1}{\varepsilon}(h^2 - fg), \\ h_t &=& -\frac{1}{\varepsilon}(h^2 - fg), \\ g_t - g_x &=& \frac{1}{\varepsilon}(h^2 - fg), \end{array}\right. \qquad [6]$$

where ε is the mean free path, f, h, and g denote the mass densities of gas particles with speeds 1,0, and -1 respectively.

The fluid dynamics variables are density, $\rho := f + 2h + g$, and momentum, $m := f - g$. We also define $z := f + g$, and rewrite system [6] in the equivalent form as,

$$\left\{\begin{array}{l} \rho_t + m_x = 0, \\ m_t + z_x = 0, \\ z_t + m_x = \frac{1}{2\varepsilon}(\rho^2 + m^2 - 2\rho z). \end{array}\right. \qquad [7]$$

If $\varepsilon \to 0$, z is given by a local Maxwellian distribution, $z = (\rho^2 + m^2)/2\rho$, and in the limit system [7] becomes the following model Euler equations,

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ (\rho v)_t + \Big(\dfrac{\rho + \rho v^2}{2}\Big)_x = 0, \end{cases}$$

with velocity $v := m/\rho$.

Our goal is to construct a semi-discrete central scheme for system [7] that will perform in the stiff case ($\varepsilon \sim 0$) as good as in the nonstiff one ($\varepsilon = \mathcal{O}(1)$). The scheme [3]–[5] can be applied directly. The hyperbolic part of the system is linear, and therefore the local speeds are $a^{\pm}_{j+\frac{1}{2}} = \pm 1$ for all j. The numerical flux [4] is then reduced to

$$H_{j+\frac{1}{2}}(t) = \frac{f(u^+_{j+\frac{1}{2}}) + f(u^-_{j+\frac{1}{2}})}{2} - \frac{1}{2}\Big[(u^+_{j+\frac{1}{2}} - u^-_{j+\frac{1}{2}}) - q_{j+\frac{1}{2}}\Big],$$

where $u = (\rho, m, z)^T$. Since all the fields are linear, we may take $q_{j+\frac{1}{2}} = (u^+_{j+\frac{1}{2}} - u^-_{j+\frac{1}{2}})/2$ without risking oscillations (see [KUR 00a] for details). The resulting flux is then given by

$$H_{j+\frac{1}{2}}(t) = \frac{f(u^+_{j+\frac{1}{2}}) + f(u^-_{j+\frac{1}{2}})}{2} - \frac{1}{4}\Big[u^+_{j+\frac{1}{2}} - u^-_{j+\frac{1}{2}}\Big]. \qquad [8]$$

Finally, the spatial source average, $\bar{S}(t)$, in [3] is computed using the midpoint quadrature.

As a result of the semi-discretization [3],[8], we obtain a system of ODEs, needed to be solved by a stable and accurate ODE solver. The efficiency of the method can be ensured if one uses an implicit-explicit Runge-Kutta-type solver (see [ASC 75, PAR 02b]). An alternative approach, realized in the numerical experiments, is to use the second-order Strang operator splitting, [STR 68]: we treat the hyperbolic and the relaxation parts of [3] separately. A second-order modified Euler method is used for the hyperbolic evolution, while the stiff relaxation is resolved using the fifth-order implicit RADAU5 solver from [HAI 96].

4. Numerical experiments

We solve the Broadwell model in the fluid dynamics variables, [7], by the method described in §3. The Riemann initial data, taken from [CAF 97], are

$$\begin{array}{ll} (\rho, m, z) = (2, 1, 1), & \text{for} \quad x < 0.2, \\ (\rho, m, z) = (1, 0.13962, 1), & \text{for} \quad x > 0.2. \end{array} \qquad [9]$$

We test two versions of the semi-discrete schemes (abbreviated by SD1 and SD2). The only difference between them is in the reconstruction of the intermediate values $u^{\pm}_{j+\frac{1}{2}}$, required by the flux, [8]. The SD1 scheme employs the second-order UNO reconstruction ([HAR 87, NES 90]), applied to the fluid dynamics variables, ρ, m, and z. To improve the resolution of contact waves, we perform a piecewise linear reconstruction in the original, characteristic variables, f, h, and g (it is important that all the three fields are linear), as it was suggested in [KUR 00a]. In addition, we use a very compressive limiter from [LIE 02], where the two-parameter family of piecewise linear reconstructions is introduced. The SD2 scheme corresponds to the most compressive choice of these parameters ($\theta = 2$ and $\tau = -0.25$), see [LIE 02] for details.

Remark. — The use of the compressive limiters does not provide a satisfactory solution, if they applied to the fluid dynamics variables. In this case, the computed solutions (not presented in this paper) are overcompressed.

In Figures 1a–c, we present the solutions of [7],[9], computed by the SD1 and SD2 schemes in three different regimes: nonstiff ($\varepsilon = 1$), stiff ($\varepsilon = 10^{-8}$), and the intermediate one (where $\varepsilon = 0.02$ is proportional to Δx). The reference solution is computed by the SD2 scheme with 4000 grid points. As one can see in Figure 1a, SD2 provides much sharper resolution of the contact discontinuity. The solutions obtained by the SD1 scheme are not shown in Figures 1b and 1c, since they look very similar to the solutions obtained by SD2.

In Figure 1d, we show the numerical solutions of [7] subject to the different initial data, taken from [CAF 97] as well (we set $\varepsilon = 10^{-8}$),

$$\begin{array}{ll} (\rho, m, z) = (1, 0, 1), & \text{for} \quad x < 0.5, \\ (\rho, m, z) = (0.2, 0, 1), & \text{for} \quad x > 0.5. \end{array} \qquad [10]$$

In this case, the SD2 scheme overcompresses the rarefaction wave (when a large number of grid points is used). Therefore, the reference solution is computed by the SD1 scheme. At the same time, when a small number of grid points is used, the SD2 scheme resolves the rarefaction wave much better than SD1.

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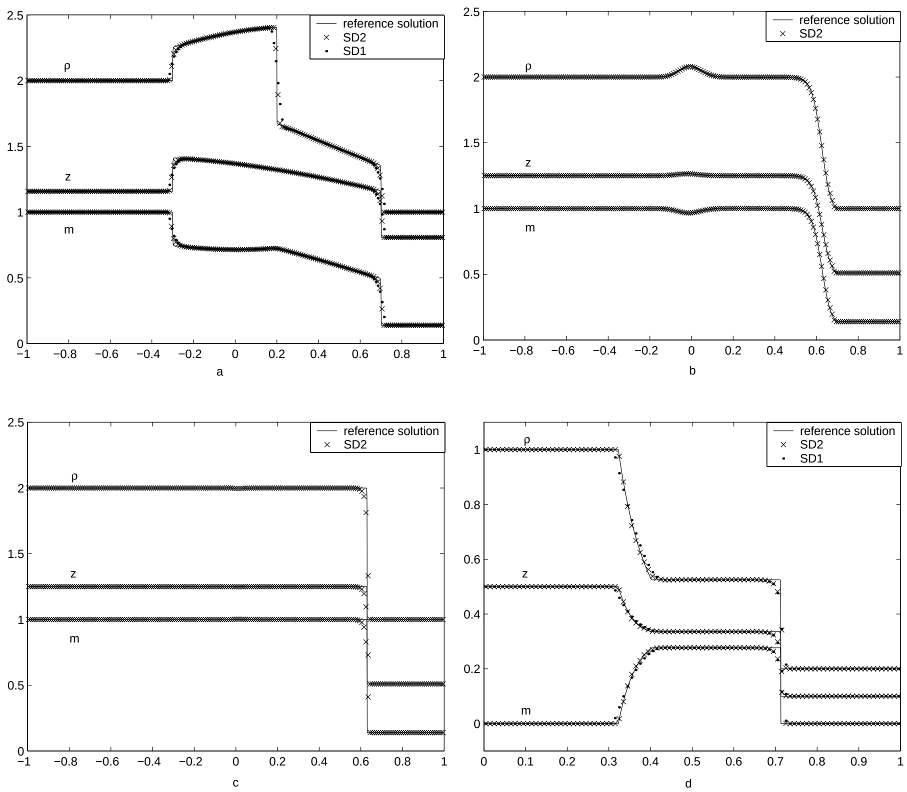

Figure 1. *The numerical solution of the problem [7],[9] with (a)* $\varepsilon = 1$*, (b)* $\varepsilon = 0.02$*, (c)* $\varepsilon = 10^{-8}$*. 200 grid points are used, the final time is 0.5. (d) shows the computed solution of the problem [7],[10] with* $\varepsilon = 10^{-8}$ *and 100 grid points at the final time 0.25. The CFL condition is* $\Delta t = \Delta x/2$.

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