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ERROR ESTIMATES FOR THE STAGGERED LAX–FRIEDRICHS SCHEME ON UNSTRUCTURED GRIDS*

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Abstract. Staggered grid finite volume methods (also called *central schemes*) were introduced in one dimension by Nessyahu and Tadmor in 1990 in order to avoid the necessity of having information on solutions of Riemann problems for the evaluation of numerical fluxes. We consider the general case in multidimensions and on general staggered grids which have to satisfy only an overlap assumption. We interpret the staggered Lax–Friedrichs scheme as a three-step method consisting of a prolongation step onto a finer *intersection grid*, a finite volume step with an arbitrarily good numerical flux (e.g., Godunov flux) on the *intersection grid*, followed by an averaging step such that the calculation of numerical fluxes reduces to evaluations of the continuous flux. Using this point of view, we prove an a posteriori error estimate and an a priori error estimate in the L^1-norm in space and time which is of order $h^{1/4}$, where h is a mesh-size parameter. Hence, we recover for the staggered Lax–Friedrichs scheme the same order of convergence as for upwind finite volume methods on a fixed grid.

AMS subject classifications. 65M15, 75M12, 35L65

Key words. staggered Lax–Friedrichs scheme, finite volume method, error estimates, conservation law

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1. Introduction.

In this paper we prove an a priori and an a posteriori error estimate for the first order Lax–Friedrichs scheme on general staggered grids for scalar multidimensional conservation laws.

Staggered finite volume schemes were introduced by Nessyahu and Tadmor in 1990 [19]. The main advantage of these schemes is that no information about solutions to local Riemann problems is needed. Using staggered grids one can replace the upwind fluxes with central differences. The price one has to pay is the occurrence of excessive numerical viscosity since the resulting scheme can be interpreted as a Lax–Friedrichs scheme. Therefore, a higher order scheme of MUSCL type in one spatial dimension is proposed in [19]. Numerical experiments in [19] show the good performance of the algorithm. Later, in [2, 4] the central schemes (of second order) have been generalized to multidimensional schemes on unstructured grids and in [13, 2] on two-dimensional tensor product structured grids. In [4] a primal and a dual mesh are used with time evolution performed alternately on either of both meshes.

Convergence of finite volume schemes on a fixed grid has been proven in [7, 16, 5]. In the case of staggered unstructured grids in multidimensions, there exist only a few convergence results. In [3] convergence of a second order central scheme on special two-dimensional grids has been proven for a linear conservation law. Convergence of the first order Lax–Friedrichs scheme on the same special staggered grids for nonlinear scalar problems has been proven in [11].

We prove an a priori and a posteriori error estimate on general staggered grids for the first order Lax–Friedrichs scheme for scalar nonlinear problems in any spatial dimension; see Theorems 3.3 and 3.5. We allow that at each time step one may have

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a new grid. The only *additional* condition on the sequence of staggered grids which we need is an *overlap* assumption of consecutive grids; see Assumption 3.1.

Our main idea for proving the error estimates is to reinterpret the Lax–Friedrichs scheme on staggered grids such that the error analysis, which is known in the case of a fixed grid, can be used and modified; see Definition 2.14.

In order to perform a time evolution we need to construct a relationship between two grids: the grid at the actual time level and the grid at the next level. A natural candidate for this job is the intersection grid. Then, the staggered Lax–Friedrichs scheme can be decomposed into three steps: a prolongation step of the actual values to values on the intersection grid; a time evolution step on the intersection grid with an arbitrary numerical flux, e.g., the Godunov flux (which has to be consistent, conservative, monotone, and Lipschitz; see Assumption 2.5); and an averaging step from the intersection grid to the grid on the next time level. Our point is that due to the properties of the numerical flux (conservation and consistency) and, thanks to the overlap condition on two consecutive grids, the overall scheme (consisting of the three steps described above) reduces to the staggered Lax–Friedrichs scheme.

The technique for proving error estimates for nonlinear hyperbolic equations is quite old and goes back to Kruzkov [17] and Kuznetsov [18], where entropy inequalities are used to establish error estimates. We employ this technique as well, as it is formulated in [8] and [5]. Our a posteriori error estimate is given in the spirit of Kröner and Ohlberger [15] which takes into account the domain of dependence of the error. In that sense, the error estimate is *local*. In [15] an adaptive strategy is developed which can be easily generalized to the situation we consider here.

We want to mention that one can return to nonstaggered grids from staggered grids, retaining the simplicity of the central schemes, by using a projection step; see [19, 12]. The error analysis of these schemes, as well as the error analysis of higher order schemes on staggered grids, constitutes the subject of further ongoing research.

The paper is organized as follows. In section 2 we formulate the problem and introduce the finite volume discretization on a fixed grid and on staggered grids to show the analogy and difference between these two approaches. In section 3 we state our a priori and a posteriori error estimates and give our ideas of how to prove these results. The rest of the paper is devoted to the proofs.

2. Finite volume discretization on staggered grids. In this section we present our model problem. Then, we define the finite volume scheme on a fixed mesh and on staggered meshes. Finally, we give a reinterpretation of the scheme on staggered meshes and link both methods.

2.1. Continuous problem. We consider the following nonlinear scalar hyperbolic conservation law with some initial condition:

$$\left\{ \begin{array}{rcl} \partial_t u(x,t) + \operatorname{div} F(x,t,u(x,t)) & = & 0, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}^+, \\ u(x,0) & = & u_0(x), \quad x \in \mathbb{R}^d, \end{array} \right. \tag{2.1}$$

where $F : \mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^d$, $d \geq 1$, $(x,t,s) \mapsto F(x,t,s)$.

We make the following assumptions about the data.

ASSUMPTION 2.1.

1. $u_0 \in L^\infty(\mathbb{R}^d) \cap BV_{\mathrm{loc}}(\mathbb{R}^d)$ *with* $U_m \leq u_0 \leq U_M$ *almost everywhere.*
2. $F \in C^1(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^d)$, $(x,t,s) \mapsto F(x,t,s)$ *and* $\partial F/\partial s$ *is locally Lipschitz continuous.*
3. $\operatorname{div}_x F(x,t,s) = \sum_{i=1}^d \partial_{x_i} F(x,t,s) = 0$ *for all* $(x,t,s) \in \mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}$.

4. *For all compact $K \subset \mathbb{R}$ there exists a constant $0 < V_K < +\infty$ such that*

$$\left|\frac{\partial F}{\partial s}(x,t,s)\right| \le V_K \quad \textit{for almost all } (x,t,s) \in \mathbb{R}^d \times \mathbb{R}^+ \times K.$$

DEFINITION 2.2. *We say that $u \in L^\infty(\mathbb{R}^d \times]0,\infty[)$ is an* entropy weak solution *to* (2.1) *if for all $\kappa \in \mathbb{R}$ and all $\varphi \in C_0^1(\mathbb{R}^d \times \mathbb{R}^+;\mathbb{R}^+)$*

$$\int_{\mathbb{R}^d\times\mathbb{R}^+} \Big\{|u(x,t)-\kappa|\partial_t\varphi(x,t) + [F(x,t,u(x,t)\top\kappa) - F(x,t,u(x,t)\bot\kappa)]\nabla\varphi(x,t)\Big\}\mathrm{d}t\mathrm{d}x$$

$$+\int_{\mathbb{R}^d} |u_0(x)-\kappa|\varphi(x,0)\mathrm{d}x \ge 0 \tag{2.2}$$

holds, where $a\top b := \max\{a,b\}$ and $a\bot b := \min\{a,b\}$.

Existence and uniqueness of entropy weak solutions to (2.1) have been proven by Kruzkov in 1970 under somewhat stronger assumptions [17]. A proof for the case of Assumption 2.1 can be found in [5, Thm. 2].

2.2. Finite volume discretization on a fixed grid. Let $\mathcal{T} = (T_i)_{i\in I}$, $I \subset \mathbb{N}$, be a partition of $\mathbb{R}^d$ consisting of polygonal cells and any common $(d-1)$-dimensional interface of two cells lies in a hyperplane.

NOTATION 2.3.
- *For $i \in I$ let $N(i)$ denote the set of all indices $j \in I$ of cells having a common interface of dimension $d-1$ with T_i.*
- *For $i \in I$ and $j \in N(i)$ let S_{ij} denote the common interface and n_{ij} the unit outer normal to S_{ij} with respect to T_i.*

This notation is illustrated in Figure 2.1 for the two-dimensional case ($d = 2$).

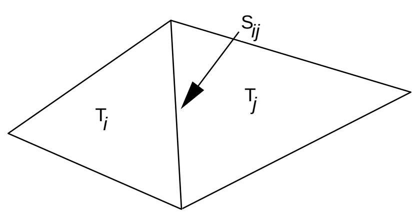


FIG. 2.1. *Illustration of Notation* 2.3.

The time axis $\mathbb{R}^+$ is partitioned into intervals $[t^n, t^{n+1}[$ of length $k^n := t^{n+1} - t^n$. The idea of finite volume schemes is to integrate the partial differential equation in (2.1) over a cell T_i and to apply Gauss's formula. The resulting integrals have to be approximated. For more details we refer to the book of Kröner [14, p. 158].

The first order finite volume scheme is defined as follows.

DEFINITION 2.4 (first order finite volume scheme). *Let $i \in I$ and $n \in \mathbb{N}$. Then*

$$u_i^0 := \frac{1}{|T_i|}\int_{T_i} u_0(x)\mathrm{d}x, \tag{2.3}$$

$$u_i^{n+1} := u_i^n - \frac{k^n}{|T_i|}\sum_{j\in N(i)} g_{ij}^n(u_i^n, u_j^n), \tag{2.4}$$

and

$$u_h(x,t) := u_i^n \quad \textit{for} \quad t^n \le t < t^{n+1} \quad \textit{and} \quad x \in T_i. \tag{2.5}$$

Here, g_{ij}^n denotes a so-called *numerical flux* which is supposed to lie in $C^0(\mathbb{R}^2;\mathbb{R})$, where $i \in I$, $j \in N(i)$, $n \in \mathbb{N}$.

In order to prove error estimates between u_h, defined in (2.5), and the entropy solution to (2.1), one needs the following assumptions on the mesh and the *numerical fluxes* g_{ij}^n.

ASSUMPTION 2.5.

1. *The mesh $\mathcal{T}$ has to satisfy the following assumption: There exists $\alpha > 0$ (regularity of the mesh) such that for all $i \in I$*

$$\alpha h_i^d \leq |T_i|,$$
$$|\partial T_i| \leq \frac{1}{\alpha} h_i^{d-1},$$

where $h_i := \operatorname{diam}(T_i)$. Set

$$h := \max_{i \in I} h_i.$$

2. *The numerical fluxes $g_{ij}^n \in C^0(\mathbb{R}^2;\mathbb{R})$ with $i \in I$, $j \in N(i)$, $n \in \mathbb{N}$, have to satisfy the following properties:*
   (a) (monotony) *g_{ij}^n is nondecreasing with respect to its first argument and nondecreasing with respect to its second argument.*
   (b) (conservation)

$$g_{ij}^n(v,w) = -g_{ji}^n(w,v) \quad \textit{for all } v,w \in \mathbb{R}.$$

   (c) *g_{ij}^n is locally Lipschitz continuous; i.e., for every compact interval $I \subset \mathbb{R}$ there exists a constant $L_g > 0$ depending only on I and F such that for all $u_1, u_2, v_1, v_2 \in I$*

$$|g_{ij}^n(u_1,v_1) - g_{ij}^n(u_2,v_2)| \leq L_g |S_{ij}| \left[|u_1 - u_2| + |v_1 - v_2| \right].$$

   (d) (consistency)

$$g_{ij}^n(s,s) = \frac{1}{k^n} \int_{t^n}^{t^{n+1}} \int_{S_{ij}} F(\gamma,t,s) \cdot n_{ij} \mathrm{d}\gamma \mathrm{d}t \quad \textit{for all } s \in \mathbb{R}.$$

Examples for such numerical fluxes can be found, for instance, in [14, 5]. The following result has been proven in [5, Thm. 4].

THEOREM 2.6. *Let Assumptions* 2.1 *and* 2.5 *hold. Assume that the following CFL-condition is met:*

$$k^n \leq \frac{(1-\xi)\alpha^2 h}{2L_g} \quad \textit{with } \xi \in]0,1[. \tag{2.6}$$

Let u be the unique entropy weak solution to (2.1) *and u_h its first order finite volume approximation which is defined in Definition* 2.4. *If $u_0 \in BV_{\mathrm{loc}}(\mathbb{R}^d)$, the following error estimate holds: For any compact set $E \subset \mathbb{R}^d \times \mathbb{R}^+$, there exists a constant $K > 0$ depending only on E, F, u_0, L_g, α, and ξ such that*

$$\int_E |u(x,t) - u_h(x,t)| \mathrm{d}x\mathrm{d}t \leq K h^{1/4}. \tag{2.7}$$

2.3. Finite volume discretization on unstructured staggered grids. Staggered grid finite volume methods have been introduced by Nessyahu and Tadmor [19] for the one-dimensional case. Later, this method was generalized to two-dimensional Cartesian grids in [13, 2] and to special two-dimensional unstructured grids in [4].

The idea of staggered grid finite volume schemes is to change consecutive grids in such a way that the time evolution corresponds to an integration over the complete Riemann fan. To be more precise, in order to define a new value at time level t^{n+1} one has to evaluate fluxes across edges of the grid at time level t^n and, by using the staggered grid, values of the previous time level are uniquely given on these edges (in contrast to the case of a single grid where one has to use the values on the left and the right of the edge; see Definition 2.4). Hence, no Riemann solvers are needed. The main drawback of using staggered grids is the large amount of numerical dissipation which is compensated by the use of higher order methods. However, as a first step we consider in this paper only the first order case.

We present in this section the extension of the (first order) staggered Lax–Friedrichs scheme on unstructured grids in two spatial dimensions as it was introduced in [4] for the special case $F(x,t,s) = f(s)$.

Let $\mathcal{T}_h$ be a triangular partition of $\mathbb{R}^2$ which is conforming (has no hanging nodes) and regular; see Assumption 2.5.1. We use the following notation.

NOTATION 2.7.

1. *The set of vertices of triangles in* $\mathcal{T}_h$ *is* $(a_i)_{i\in I_v}$ *with* $I_v \subset \mathbb{N}$. *The set of indices of vertices which are direct neighbors to* a_i *is denoted by* $N_v(i)$. *Note that the line segment* $a_i a_j$ *is an edge of a triangle in* $\mathcal{T}_h$.
2. *For* $i \in I_v$ *and* $j \in N_v(i)$ *the midpoint of the line segment* $a_i a_j$ *is denoted by* M_{ij}. *The center of gravity of the triangle which is to the left of the (oriented) line* $a_i a_j$ *is denoted by* G_{ij}^+ *and the one to the right by* G_{ij}^- *(see Figure* 2.2*).*

From $\mathcal{T}_h$ two additional grids are built as follows.

For the first grid, finite volume cells are the barycentric cells C_i, obtained by joining the midpoints M_{ij} of the sides originating at node a_i to the centroids $G_{ij}^\pm$ of the triangles of $\mathcal{T}_h$ which meet at a_i (see Figure 2.2). For the second grid, the finite volume cells are the quadrilaterals L_{ij} obtained by joining two vertices a_i, a_j to the centroids G_{ij}^+, G_{ij}^- of the two triangles of $\mathcal{T}_h$ of which $a_i a_j$ is a side. This construction is illustrated in Figure 2.2.

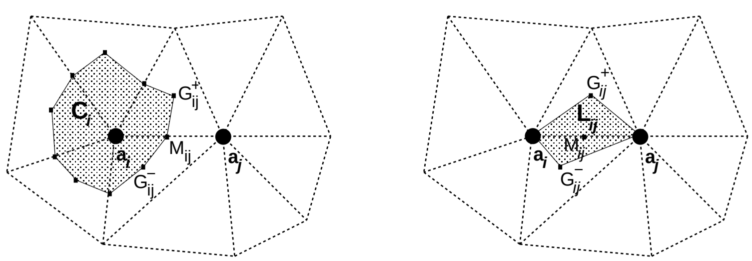


FIG. 2.2. *Construction of special staggered meshes from a triangular mesh. Left: grid consisting of barycentric cells; right: grid consisting of quadrilaterals.*

Since we need the fluxes across edges in a finite volume scheme we have to introduce further notation for the (scaled) normals of the barycentric cells, which are denoted by $\eta_{ij}^\pm$, and normals for quadrilaterals, denoted by $\mu_{ij}^1, \ldots, \mu_{ij}^4$. All normals have the length of the corresponding edge. This notation is illustrated in Figure 2.3. Furthermore, we need the following abbreviation.

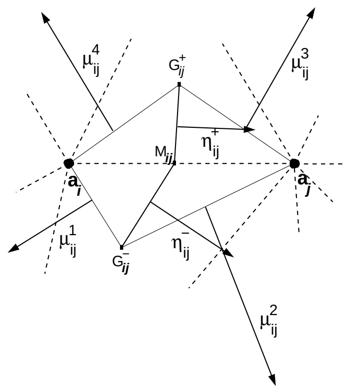


FIG. 2.3. *Normals on barycentric cells ($\eta_{ij}^{\pm}$) and quadrilaterals ($\mu_{ij}^1, \ldots, \mu_{ij}^4$).*

NOTATION 2.8. *Define for $i \in I_v$ and $j \in N_v(i)$*

$$\theta_{ij} := \mu_{ij}^2 + \mu_{ij}^3. \tag{2.8}$$

REMARK 2.9. *Note that $\theta_{ij} = -(\mu_{ij}^1 + \mu_{ij}^4) = \eta_{ij}^+ + \eta_{ij}^-$.*

Now, we are in position to state the finite volume algorithm on these staggered grids (recall that the special case $F(x,t,s) = f(s)$ was considered in [4]). The time axis is partitioned into equidistant intervals of length Δt.

DEFINITION 2.10. *For $i \in I_v$ initial values are given on the barycentric mesh:*

$$u_i^0 = \frac{1}{|C_i|} \int_{C_i} u_0(x)\mathrm{d}x. \tag{2.9}$$

Let $n \in \mathbb{N}$ be an even positive integer and let $(u_i^n)_{i \in I_v}$ be values given on the mesh consisting of barycentric cells.

First step. *For $i \in I_v$ and $j \in N_v(i)$ define values on the mesh consisting of quadrilaterals by*

$$u_{ij}^{n+1} = \frac{1}{2}(u_i^n + u_j^n) - \frac{\Delta t}{|L_{ij}|}\big(f(u_i^n) - f(u_j^n)\big)\theta_{ij}. \tag{2.10}$$

Second step. *For $i \in I_v$ define values on the mesh consisting of barycentric cells by*

$$u_i^{n+2} = \sum_{j \in N_v(i)} \left[\frac{|L_{ij}|}{2|C_i|} u_{ij}^{n+1} - \frac{\Delta t}{|C_i|} f(u_{ij}^{n+1})\theta_{ij}\right]. \tag{2.11}$$

The discrete approximation u_h is defined by

$$u_h(x,t) = \begin{cases} u_i^n & \text{if } (x,t) \in C_i \times [t^n, t^{n+1}[, \\ u_{ij}^{n+1} & \text{if } (x,t) \in L_{ij} \times [t^{n+1}, t^{n+2}[. \end{cases} \tag{2.12}$$

In a paper by Haasdonk, Kröner, and Rohde [11, Thm. 4.1] the following convergence result has been proven.

THEOREM 2.11. *Let $F(x,t,s) = f(s) \in (C^1(\mathbb{R}))^2$, $u_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ with $M := \|u_0\|_\infty$. Let $(\mathcal{T}_{h_k})_{k \in \mathbb{N}}$ be a family of shape regular triangulations satisfying*

$$ah^2 \leq |T| \leq bh^2 \quad \textit{for all } T \in \mathcal{T}_{h_k}$$

uniformly in k *with* $0 < a \leq b$*; the sequence of mesh sizes is supposed to tend to zero, i.e.,* $h_k \to 0$ *for* $k \to \infty$*. Let* β*,* γ *be constants with the relation* $0 < \gamma < \beta < a/4$*. Let* $(\Delta t_k)_{k\in\mathbb{N}}$ *be a sequence of time steps such that for all* $k \in \mathbb{N}$ *the following CFL-condition holds:*

$$\gamma \leq \frac{\Delta t_k}{h_k} \max_{s\in[-M,M], i=1,2} |f_i'(s)| \leq \beta. \tag{2.13}$$

Then the sequence of approximations $(u_{h_k})_k$*, which is defined in Definition* 2.10 *(with* $h = h_k$ *and* $\Delta t = \Delta_k^t$*) converges strongly in* $L^p_{\text{loc}}(\mathbb{R}^2 \times \mathbb{R}^+)$ *for all* $1 \leq p < \infty$ *towards the unique entropy weak solution of* (2.1)*.*

REMARK 2.12. *We prove below an a priori error estimate in a more general framework. See Theorem* 3.5*, which covers the case treated in this theorem.*

2.4. General staggered grid finite volume schemes in multidimensions. We generalize the staggered grid finite volume method introduced in section 2.3 (see [1] for a generalization on tetrahedral grids) to general unstructured grids in arbitrary space dimensions. Of course, the grids have to satisfy certain conditions which will be specified later. The key idea is to reinterpret the finite volume scheme on staggered grids which is defined in Definition 2.10. As a special case, we recover the staggered grid schemes introduced by Arminjon et al. in [2, 4] for two dimensions and in [1] for three dimensions.

To present this general framework we need more notation.

Here, we consider again an arbitrary partition of the time axis

$$0 = t^0 < t^1 < \cdots < t^n < t^{n+1} < \cdots, \quad k^n := t^{n+1} - t^n.$$

Let $(\mathcal{T}_h^n)_{n\in\mathbb{N}}$ be a sequence of finite volume grids of $\mathbb{R}^d$. Here n corresponds to the time level. That is, for every time step we may have a different grid. We use the following notation, which is a bit involved since the grids vary from time step to time step.

NOTATION 2.13.

- $\mathcal{T}_h^n = (T_i^n)_{i\in I^n}$*,* $I^n \subset \mathbb{N}$*, consists of polyhedrons of finite diameter. Set* $h_i^n := \operatorname{diam}(T_i^n)$ *and* $h_n := \max_{i\in I^n} h_i^n$*.*
- *Define the intersection grid of* $\mathcal{T}_h^n \cap \mathcal{T}_h^{n+1}$ *by* $\mathcal{T}_h^{n,n+1} = (T_i^{n,n+1})_{i\in I^{n,n+1}}$*, where there exist unique indices* $k \in I^n$ *and* $l \in I^{n+1}$ *with* $T_i^{n,n+1} = T_k^n \cap T_l^{n+1} \neq \emptyset$*.*
- *For* $i \in I^{n,n+1}$ *let* $\mathcal{N}^{n,n+1}(i)$ *denote the set of all neighboring indices* $j \in I^{n,n+1}$ *with* $S_{ij}^{n,n+1} := \overline{T}_i^{n,n+1} \cap \overline{T}_j^{n,n+1} \neq \emptyset$ *and* $S_{ij}^{n,n+1}$ *being a* $(d-1)$*-dimensional set. We allow* hanging nodes *so that* $S_{ij}^{n,n+1}$ *may consist of a finite number of line segments for* $d = 2$ *and bounded planes for* $d = 3$ *(in general, of a bounded set contained in a* $(d-1)$*-dimensional hyperplane) since the finite volumes are polyhedrons, i.e.,*

$$S_{ij}^{n,n+1} = \bigcup_{l=1}^{L(i,j)} S_{ijl}^{n,n+1}.$$

We denote by $\nu_{ijl}^{n,n+1}$ *the outer normal to* $S_{ijl}^{n,n+1}$ *of length* $|S_{ijl}^{n,n+1}|$*. Set*

$$\nu_{ij}^{n,n+1} := \sum_{l=1}^{L(i,j)} \nu_{ijl}^{n,n+1}.$$

- *(Link between $\mathcal{T}_h^{n+1}$ and $\mathcal{T}_h^{n,n+1}$.) For $i \in I^{n+1}$ set*

$$J^{n,n+1}(i) := \{j \in I^{n,n+1} | \, T_j^{n,n+1} \subset T_i^{n+1}\}.$$

For $i \in I^{n+1}$ and $j \in J^{n,n+1}(i)$ set

$$r_{ij}^{n,n+1} := \frac{|T_j^{n,n+1}|}{|T_i^{n+1}|}.$$

- *We need a link from $\mathcal{T}_h^n$ to $\mathcal{T}_h^{n,n+1}$ as well. For $i \in I^n$ set*

$$K^{n,n+1}(i) := \{j \in I^{n,n+1} | \, T_j^{n,n+1} \subset T_i^n\}.$$

We illustrate this notation with several figures. Figure 2.4 shows the construction of $\mathcal{T}_h^{n+1}$ in the case that $\mathcal{T}_h^n$ consists of barycentric cells and $\mathcal{T}_h^{n+1}$ of quadrilaterals, as was the case in Definition 2.10. Figure 2.5 shows the definition of the index set $J^{n,n+1}(i)$ if $i \in I^{n+1}$ is given. Finally, Figure 2.6 shows the index set $K^{n,n+1}(i)$ if $i \in I^n$ is given.

Using this notation the finite volume scheme reads on these staggered grids as follows.

Definition 2.14. *Define an approximation u_h to the solution of (2.1) by the following scheme.*

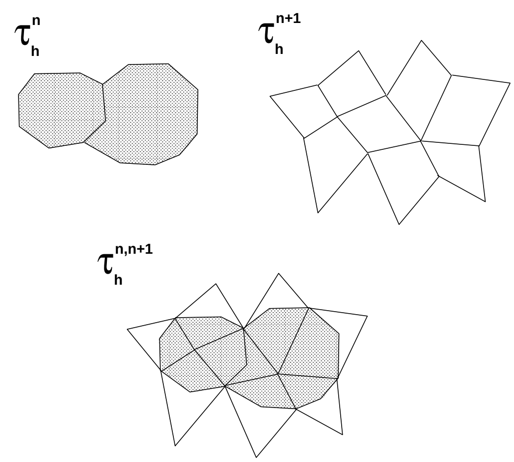


Fig. 2.4. *Definition of $\mathcal{T}_h^{n,n+1}$.*

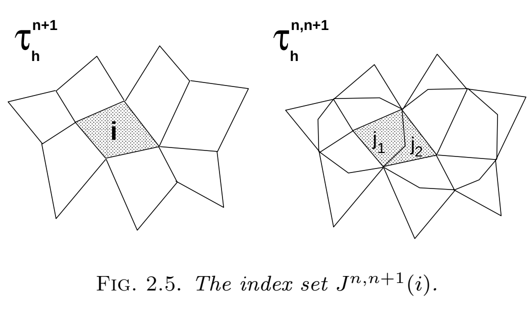


Fig. 2.5. *The index set $J^{n,n+1}(i)$.*

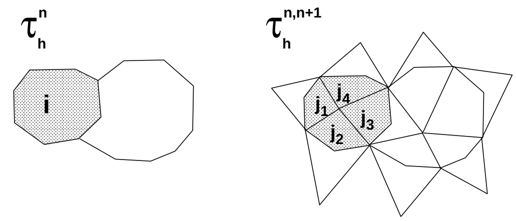


FIG. 2.6. *The index set* $K^{n,n+1}(i)$.

For $i \in I^0$ *set*

$$u_i^0 := \frac{1}{|T_i^0|}\int_{T_i^0} u_0(x)\mathrm{d}x. \tag{2.14}$$

Let $n \in \mathbb{N}$ *and* u_i^n, $i \in I^n$ *be given. Define* u_i^{n+1}, $i \in I^{n+1}$, *by the following algorithm.*

(i) Prolongation to $\mathcal{T}_h^{n,n+1}$.
For $i \in I^{n,n+1}$ *there exists by construction a uniquely defined index* $j \in I^n$ *such that* $i \in K^{n,n+1}(j)$. *Set*

$$v_i^n := u_j^n. \tag{2.15}$$

(ii) Time evolution on $\mathcal{T}_h^{n,n+1}$.
For $i \in I^{n,n+1}$ *set*

$$v_i^{n+1} := v_i^n - \frac{k^n}{|T_i^{n,n+1}|}\sum_{l\in\mathcal{N}^{n,n+1}(i)} g_{il}^{n,n+1}(v_i^n,\, v_l^n), \tag{2.16}$$

where $g_{il}^{n,n+1}$ *is some monotone, consistent, conservative Lipschitz flux which may vary from time step to time step (see Assumption* 2.5.2*).* k^n *is the time step which has to meet a CFL-condition, to be given below.*

(iii) Averaging to $\mathcal{T}_h^{n+1}$.
For $i \in I^{n+1}$ *set*

$$u_i^{n+1} = \sum_{j\in J^{n,n+1}(i)} r_{ij}^{n,n+1} v_j^{n+1}. \tag{2.17}$$

Set

$$u_h(x,t) = u_i^n \quad \textit{for} \quad t \in [t^n, t^{n+1}[\quad \textit{and} \quad x \in T_i^n. \tag{2.18}$$

Note that one would implement the overall algorithm and not the single steps. Due to the conservation of the numerical fluxes g_{il} the fluxes across interior edges of $T_i^{n+1} = \cup_{j\in J^{n,n+1}(i)} T_j^{n,n+1}$ cancel. Hence, fluxes across outer edges of T_i^{n+1} remain. If one assumes that each part of an edge (or face) of elements in $\mathcal{T}_h^{n+1}$ is contained in the interior of an element in $\mathcal{T}_h^n$, then these fluxes across outer edges reduce to evaluations of the function F (see (2.1)) which is due to the consistency of the numerical fluxes.

To be more precise, we introduce more notation.

NOTATION 2.15 (link between $\mathcal{T}_h^n$ and $\mathcal{T}_h^{n+1}$). *Let* $i \in I^{n+1}$ *be given. Denote by* $\tilde{K}^{n,n+1}(i)$ *those indices* $j \in I^n$ *which correspond to elements having a nonempty* *intersection with* T_i^{n+1}*; denote by* $\tilde{K}_\partial^{n,n+1}(i)$ *those indices* $j \in I^n$ *which correspond to elements intersecting the boundary of* T_i^{n+1}*; denote by* $\tilde{\nu}_{ij}^{n,n+1}$ *the scaled outer normal associated with those indices (compare the construction of the normal in Notation* 2.13*); and denote the unit outer normal by*

$$\tilde{n}_{ij}^{n,n+1} = \frac{\tilde{\nu}_{ij}^{n,n+1}}{|\tilde{\nu}_{ij}^{n,n+1}|}.$$

The set $\tilde{K}_\partial^{n,n+1}(i)$ of indices is illustrated in Figure 2.7.

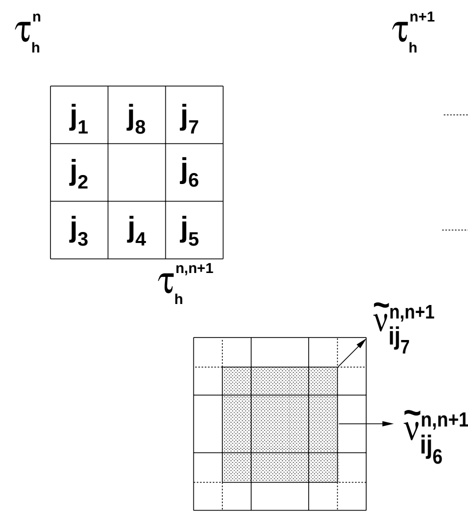


FIG. 2.7. *For* $i \in I^{n+1}$ *the set of indices* $\tilde{K}_\partial^{n,n+1}(i) \subset I^n$ *equals* $j_1, \ldots, j_8$.

Using this notation, we can define an overall algorithm which one would use in implementations.

DEFINITION 2.16. *Define an approximation* u_h *to the solution of* (2.1) *by the following scheme.*

For $i \in I^0$ *set*

$$u_i^0 := \frac{1}{|T_i^0|} \int_{T_i^0} u_0(x)\mathrm{d}x. \tag{2.19}$$

For given values u_i^n, $i \in I^n$, *define values* u_i^{n+1}, $i \in I^{n+1}$, *by*

$$u_i^{n+1} = \sum_{j \in \tilde{K}^{n,n+1}(i)} \tilde{r}_{ij}^{n,n+1} u_j^n - \frac{k^n}{|T_i^{n+1}|} \sum_{j \in \tilde{K}_\partial^{n,n+1}(i)} F_{ij}^{n,n+1}(u_j^n), \tag{2.20}$$

with

$$\tilde{r}_{ij}^{n,n+1} := \frac{|T_j^n \cap T_i^{n+1}|}{|T_i^{n+1}|}$$

and (see also Assumption 2.5*)*

$$F_{ij}^{n,n+1}(s) := \frac{1}{k^n} \int_{t^n}^{t^{n+1}} \int_{\tilde{S}_{ij}^{n,n+1}} F(\gamma, t, s) \cdot \tilde{n}_{ij} \mathrm{d}\gamma \mathrm{d}t.$$

Set

$$u_h(x,t) = u_i^n \quad \text{for} \quad t \in [t^n, t^{n+1}[\quad \text{and} \quad x \in T_i^n. \tag{2.21}$$

PROPOSITION 2.17. *Assume that the numerical fluxes in* (2.16) *satisfy Assumption* 2.5 *and that the sequence of staggered grids satisfies the condition that for all* $n \in \mathbb{N}$, *all* $i \in I^{n+1}$, *and all* $j \in \tilde{K}^{n,n+1}(i)$,

$$\partial T_i^{n+1} \cap \partial T_j^n \quad \text{has dimension at most } d-2 \tag{2.22}$$

holds. Then, the algorithms defined in Definitions 2.14 *and* 2.16 *generate the same numerical approximation.*

Proof. We simply have to use the properties of the numerical fluxes (see Assumption 2.5) and the *overlap assumption* (2.22). The proof follows by induction over $n \in \mathbb{N}$. For $n = 0$ the assertion follows by construction. Let $n > 0$ be given and let $i \in I^{n+1}$; then

$$\begin{aligned} u_i^{n+1} &\overset{(2.17)}{=} \sum_{j \in J^{n,n+1}(i)} r_{ij}^{n,n+1} v_j^{n+1}, \\ &\overset{(2.16)}{=} \sum_{j \in J^{n,n+1}(i)} r_{ij}^{n,n+1} v_j^n - \frac{k^n}{|T_i^{n+1}|} \sum_{j \in J^{n,n+1}(i)} \sum_{l \in \mathcal{N}^{n,n+1}(j)} g_{jl}^{n,n+1}(v_j^n, v_l^n), \\ &\overset{(2.15)}{=} \sum_{j \in \tilde{K}^{n,n+1}(i)} \tilde{r}_{ij}^{n,n+1} u_j^n - \frac{k^n}{|T_i^{n+1}|} \sum_{j \in J^{n,n+1}(i)} \sum_{l \in \mathcal{N}^{n,n+1}(j)} g_{jl}^{n,n+1}(v_j^n, v_l^n), \\ &\overset{\text{Ass. 2.5, (2.22)}}{=} \sum_{j \in \tilde{K}^{n,n+1}(i)} \tilde{r}_{ij}^{n,n+1} u_j^n - \frac{k^n}{|T_i^{n+1}|} \sum_{j \in \tilde{K}_\partial^{n,n+1}(i)} F_{ij}^{n,n+1}(u_j^n), \end{aligned}$$

where we have used in the last step that fluxes across interior edges of $\mathcal{T}_h^{n,n+1}$ in T_i^{n+1} vanish due to the conservation of the numerical fluxes and that fluxes across outer edges reduce to F-evaluations thanks to the overlap condition (2.22) and the consistency of the numerical fluxes. This concludes the proof. □

REMARK 2.18.

1. *This result shows that the staggered Lax–Friedrichs scheme in Definition* 2.16 *can be viewed as an upwind finite volume scheme with an arbitrary good numerical flux (e.g., with Godunov flux) on a finer grid (the intersection grid* $\mathcal{T}_h^{n,n+1}$*) followed by a projection step.*

2. *In the error analysis we will use both points of view of the staggered Lax–Friedrichs scheme. The latter one in Definition* 2.16 *is used to establish an estimate for the entropy dissipation (see Proposition* 4.2*) while the algorithm in its decomposed form (see Definition* 2.14*) is used to prove a continuous entropy estimate for the approximate solution. In this proof (see Theorem* 5.2 *below) we can use parts of the argumentation of the corresponding result for the single-grid case (i.e.,* $\mathcal{T}_h^n \equiv \mathcal{T}_h$ *for each time step* $n \in \mathbb{N}$*).*

One main demand on numerical methods for conservation laws is that these methods be mass conservative. This property is satisfied by the method in Definition 2.14 as well.

PROPOSITION 2.19 (conservation). *Let* u_h *be given in Definition* 2.14. *Then the total mass is conserved, i.e.,*

$$\int_{\mathbb{R}^d} u_h(x,t)\mathrm{d}x = \int_{\mathbb{R}^d} u_0(x)\mathrm{d}x \quad \text{for all} \quad t > 0. \tag{2.23}$$

Proof. By definition we have

$$\int_{\mathbb{R}^d} u_h(x,0)\mathrm{d}x = \sum_{i\in I^0} |T_i^0| u_i^0 = \int_{\mathbb{R}^d} u_0(x)\mathrm{d}x.$$

The assertion follows now by induction if we prove

$$\sum_{i\in I^{n+1}} |T_i^{n+1}| u_i^{n+1} = \sum_{i\in I^n} |T_i^n| u_i^n.$$

This is a simple consequence of the definition of u_h and is also due to the conservation of the numerical fluxes.

$$\begin{aligned}
\sum_{i\in I^{n+1}} |T_i^{n+1}|\, u_i^{n+1} &= \sum_{i\in I^{n+1}} \sum_{j\in I^{n,n+1}(i)} |T_j^{n,n+1}|\, v_j^{n+1} \qquad \text{(see (2.17))}\\
&= \sum_{j\in I^{n,n+1}} |T_j^{n,n+1}|\, v_j^{n+1}\\
&= \sum_{j\in I^{n,n+1}} |T_j^{n,n+1}|\, v_j^{n} \qquad \text{(conservation of the fluxes)}\\
&= \sum_{i\in I^{n}} \sum_{j\in K^{n,n+1}} |T_j^{n,n+1}|\, v_j^{n}\\
&= \sum_{i\in I^{n}} \sum_{j\in K^{n,n+1}} |T_j^{n,n+1}|\, u_i^{n} \qquad \text{(see (2.15))}\\
&= \sum_{i\in I^{n}} |T_i^{n}|\, u_i^{n}. \qquad \square
\end{aligned}$$

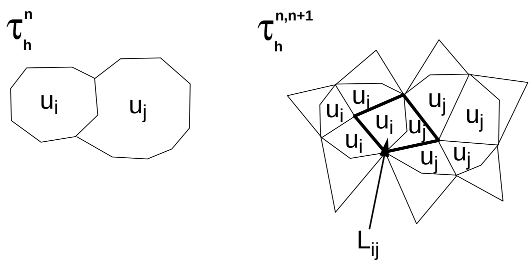
In the rest of this section we show that the staggered grid finite volume scheme defined in [4, 11], which we have stated in Definition 2.10, can be rewritten in the form introduced in Definition 2.16. In order to do so, we simply have to identify the corresponding index sets. However, since our notation is a bit involved, we give some insight into the proof of Proposition 2.17 by showing that (2.10) and (2.11) can be rewritten in the form given in Definition 2.14. The main technical difficulty is the identification of index sets. Therefore, we illustrate a more instructive "proof" with the following figures.

We show that the first step of the algorithm in Definition 2.10 (see (2.10)) can be rewritten in the framework of Definition 2.14; i.e., assume that $n \in \mathbb{N}$ is even and that values $(u_i^n)_{i\in I}$ are given on barycentric cells; see Figure 2.2. We have to show that the three steps in Definition 2.14 lead to values $(u_{ij}^{n+1})_{i\in I, j\in N_v(i)}$ given by

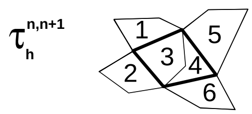
$$u_{ij}^{n+1} = \frac{1}{2}(u_i^n + u_j^n) - \frac{\Delta t}{|L_{ij}|}\big(f(u_i^n) - f(u_j^n)\big)\theta_{ij}.$$

We use the same grids as in Figure 2.4.

Prolongation (see (2.15)). Values $(u_i^n)_{i\in I^n}$ are trivially prolongated onto the intersection grid $\mathcal{T}_h^{n,n+1}$:



Time evolution on $\mathcal{T}_h^{n,n+1}$. For the time evolution on $\mathcal{T}_h^{n,n+1}$ we need the index set $I^{n,n+1}$. We assume an enumeration which is depicted in the following figure:



Identifying $(v_i^n)_{i\in I^{n,n+1}}$ with the corresponding values $(u_i^n)_{i\in I^n}$ we get (using the definition of the normals in Figure 2.3 and noting that $F(x,t,s)=f(s)$), with $i\in I_v$ and $j\in N_v(i)$,

$$v_3^{n+1} = u_i^n - \frac{\Delta t}{|L_{ij}\cap C_i|}\Big(f(u_i^n)(\mu_{ij}^4+\mu_{ij}^1) + g_{34}^n(u_i^n,u_j^n)\Big),$$
$$v_4^{n+1} = u_j^n - \frac{\Delta t}{|L_{ij}\cap C_j|}\Big(f(u_j^n)(\mu_{ij}^3+\mu_{ij}^2) + g_{43}^n(u_j^n,u_i^n)\Big).$$

Averaging to $\mathcal{T}_h^{n+1}$. In the situation we consider here, (2.17) reads

$$u_{ij}^{n+1} = \frac{|L_{ij}\cap C_i|}{|L_{ij}|}u_i^n + \frac{|L_{ij}\cap C_j|}{|L_{ij}|}u_j^n + \frac{\Delta t}{|L_{ij}|}\Big[g_{34}^n(u_i^n,u_j^n) + g_{43}^n(u_j^n,u_i^n)$$
$$+ f(u_i^n)(\mu_{ij}^4+\mu_{ij}^1) + f(u_j^n)(\mu_{ij}^3+\mu_{ij}^2)\Big].$$

Using the conservation of the numerical flux (see Assumption 2.5), the definition of θ_{ij} (see (2.8) and Remark 2.9), and that by construction

$$\frac{|L_{ij}\cap C_i|}{|L_{ij}|} = \frac{|L_{ij}\cap C_j|}{|L_{ij}|} = \frac{1}{2},$$

we end up with (2.11), which is what we wanted to show.

The proof that the second step (2.12) in Definition 2.10 fits into the framework of Definition 2.14 is similar and is left to the reader.

3. Main results and ideas of proofs. In this section we state an a posteriori and an a priori error estimate between the entropy weak solution of (2.1) and the numerical approximation defined in Definition 2.14 in the L^1-norm in space and time. Then we explain the guideline of the proofs of these results.

First, we need additional assumptions on the CFL-number and on the overlap between two consecutive finite volume grids.

ASSUMPTION 3.1.

1. Regularity of the meshes. *Assume that* $(\mathcal{T}_h^n)_{n\in\mathbb{N}}$ *and* $(\mathcal{T}_h^{n,n+1})_{n\in\mathbb{N}}$ *satisfy Assumption* 2.5.1 *uniformly in* n*; i.e., the mesh regularity parameter* α *does not depend on* h_n *or* $h_{n,n+1}$ *for all* $n \in \mathbb{N}$.
2. CFL-condition. *For* $n \in \mathbb{N}$ *the time step* k^n *is chosen such that for all* $i \in I^{n,n+1}$

$$k^n V_{[U_m,U_M]} \le \frac{1}{2}(1-\xi)\alpha^2 h_i^{n,n+1}, \quad \xi \in]0,1[, \tag{3.1}$$

where $V_{[U_m,U_M]}$ *is defined in Assumption* 2.1*. In particular, we have*

$$k^n V_{[U_m,U_M]} \frac{|\partial T_i^{n,n+1}|}{|T_i^{n,n+1}|} \le \frac{1}{2}(1-\xi).$$

3. Inverse CFL-condition. *There exists a constant* $\eta > 0$ *independent of* $(h_{n,n+1})_{n\in\mathbb{N}}$ *and* $(k^n)_{n\in\mathbb{N}}$ *such that for all* $n \in \mathbb{N}$ *and all* $i \in I^{n,n+1}$*, the following estimate holds:*

$$\eta \le \frac{k^n V_{[U_m,U_M]}}{\alpha^2 h_i^{n,n+1}}. \tag{3.2}$$

4. Overlap condition. *There exists a constant* $C_{\text{ov}} > 0$ *such that for all* $n \in \mathbb{N}$, $i \in I^{n+1}$, $j \in \tilde{K}^{n,n+1}(i)$*, the following estimate holds:*

$$C_{\text{ov}} \le \frac{|T_j^n \cap T_i^{n+1}|}{|T_i^{n+1}|} \le 1. \tag{3.3}$$

Furthermore, suppose that for all $n \in \mathbb{N}$*, for all* $i \in I^{n+1}$*, and for all* $j \in \tilde{K}^{n,n+1}(i)$,

$$\partial T_i^{n+1} \cap \partial T_j^n \quad \textit{has dimension at most } d-2. \tag{3.4}$$

Note that the overlap condition states that the mesh sizes of $\mathcal{T}_h^{n,n+1}$ and $\mathcal{T}_h^{n+1}$ should be comparable and that by (3.4)

$$\frac{|T_j^n \cap T_i^{n+1}|}{|T_i^{n+1}|} < 1.$$

First, we state the a posteriori error estimate in space and time which is the analogue to the single grid result given in Kröner and Ohlberger [15, Thm. 2.11].

NOTATION 3.2. *Let* $\omega = V_{[U_m,U_M]}$ *and* $R, T > 0$ *be given. Set*

$$I_0 := \left\{ n \in \mathbb{N} |\ 0 \le t^n \le \min\left\{ \frac{R+1}{\omega}, T \right\} \right\},$$
$$N_0 := \max\{n |\ n \in I_0\},$$
$$D \equiv D_{R+1}(x_0) := \{(x,t) \in \mathbb{R}^d \times \mathbb{R}^+ |\ |x - x_0| + \omega t < R+1\},$$

and for $n \in I_0$,

$$I_D^{n+1} := \{ i \in I^{n+1} |\ T_i^{n+1} \times \{t^n\} \subset D_{R+1}(x_0) \}.$$

The oriented set of edges in $\mathcal{T}_h^{n,n+1}$ *is defined as*

$$\mathcal{E}^{n,n+1} := \{(j,l)|\ j \in I^{n,n+1},\ l \in \mathcal{N}^{n,n+1}(j),\ \textit{and}\ v_j^n > v_l^n\}.$$

The set of edges contained in D *is denoted by* $\mathcal{E}_D^{n,n+1}$.

Furthermore, we need the oriented set of edges $S_{jl}^n = \partial(T_j^n \cap T_l^n)$ *in* $\mathcal{T}_h^n$ *which is defined as*

$$\mathcal{E}^n := \{(j,l)|\ j \in I^n,\ l \in \mathcal{N}^n(j),\ \textit{and}\ u_j^n > u_l^n\}.$$

The set of edges contained in D *is denoted by* $\mathcal{E}_D^n$. *Finally, we need portions of edges in* $\mathcal{E}^n$ *which are contained in* T_i^{n+1}, *where* $i \in I^{n+1}$. *For* $i \in I^{n+1}$ *set*

$$\tilde{\mathcal{E}}^{n,n+1}(i) := \{(j,l) \in \mathcal{E}^n | \tilde{S}_{jl}^{n,n+1}(i) := S_{jl}^n \cap T_i^{n+1} \neq \emptyset\}.$$

Note that I_D^{n+1} contains those elements which are contained in $D_{R+1}(x_0)$ at time t^n since those elements are used to evolve in time from t^n to t^{n+1}.

THEOREM 3.3 (a posteriori error estimate). *Assume that Assumptions* 2.1 *and* 2.5 *and the CFL-condition* (3.1) *hold. Let* $K \subset\subset \mathbb{R}^d \times \mathbb{R}^+$, $\omega = V_{[U_m,U_M]}$, *and choose* $T, R > 0$ *and* $x_0 \in \mathbb{R}^d$ *such that* $T \in\ ]0, R/\omega[$ *and*

$$K \subset \bigcup_{0\le t\le T} B_{R-\omega t}(x_0) \times \{t\}.$$

Then we have

$$\int_K |u - u_h| \le T \left[\int_{|x-x_0|<R+1} |u_0 - u_h(.,0)| + aQ + \sqrt{bcQ} \right], \tag{3.5}$$

where

$$\begin{aligned}
&a := 2\omega + \frac{1}{T} + 2,\\
&b := 4 + 2^{d+2},\\
&c := \|u\|_{\mathrm{BV}} \left[2\left(2\omega + \frac{1}{T}\right) + V_{[U_m,U_M]}(8 + 2^{d+5}) \right] + \|u_0\|_{\mathrm{BV}} \left[2^{d+4} V_{[U_m,U_M]} + 1\right]\\
&\qquad + 2V_{[U_m,U_M]} \max\{U_m, U_M\} \left[|B_{R+1}(0)| - |B_R(0)|\right] T,
\end{aligned}$$

and

$$\begin{aligned}
Q = \frac{1}{2} &\sum_{n=0}^{N_0} \sum_{i\in I_D^{n+1}} h_i^{n+1} |T_i^{n+1}| \sum_{j,l\in\tilde{K}^{n,n+1}(i)} \tilde{r}_{ij}^{n,n+1} \tilde{r}_{il}^{n,n+1} |u_j^n - u_l^n|\\
&+ \sum_{n=0}^{N_0-1} k^n \sum_{i\in I_D^{n+1}} |T_i^{n+1}| \left| u_i^{n+1} - \sum_{j\in\tilde{K}^{n,n+1}(i)} \tilde{r}_{ij}^{n,n+1} u_j^n \right|\\
&+ 6V_{[U_m,U_M]} \sum_{n=0}^{N_0} k^n \sum_{i\in I_D^{n+1}} (h_i^{n+1} + k^n) \sum_{(j,l)\in\tilde{\mathcal{E}}^{n,n+1}(i)} |\tilde{S}_{jl}^{n,n+1}(i)|\, |u_j^n - u_l^n|\\
&+ C_{F,R,T,u_0} \sum_{n=0}^{N_0} \sum_{(j,l)\in\mathcal{E}_D^{n,n+1}} k^n |S_{jl}^{n,n+1}| \left[\ \mathrm{diam}(S_{jl}^{n,n+1}) + k^n\right]^2,
\end{aligned}$$

where C_{F,R,T,u_0} *is defined in* (5.12). *See also Lemma* 6.7.

REMARK 3.4.

1. *Note that the domain of dependence of the error is taken into account since the set K is embedded into the characteristic cone $D_{R+1}(x_0)$.*
2. *If $F(x,t,s) = f(s)$, then one can choose $C_{F,R,T,u_0} = 0$ (see (5.12) in the proof of Theorem 5.2 below).*

From the a posteriori error estimate one gets the following a priori error estimate.

THEOREM 3.5 (a priori error estimate). *Let Assumptions 2.1, 2.5, and 3.1 hold. Let u be the unique entropy solution to (2.1) and let u_h be given by Definition 2.14. Then for all compact $K \subset\subset \mathbb{R}^d \times \mathbb{R}^+$ there exists a constant $C_e > 0$, which depends only on K, F, u_0, and the geometrical constants given above such that*

$$\int_K |u_h(x,t) - u(x,t)| \mathrm{d}x\mathrm{d}t \leq C_e h^{1/4}, \tag{3.6}$$

where

$$h := \sup_{0 \leq n \leq N_0} \max\{h_n, h_{n,n+1}\}.$$

The proof of this result relies on a technique introduced by Kruzkov [17] which has been used by Kuznetsov [18] to prove an error estimate for first order finite volume schemes in one space dimension. The key idea is to use the concept of entropy (see Definition 2.2) and the so-called technique of *doubling the variables*. This can be done on an abstract level and is contained in Approximation Lemma 3.6 below.

Then all the conditions in this approximation lemma have to be verified. The main difficulty is to prove a continuous entropy inequality for the discrete solution.

For a simple introduction concerning error estimation for hyperbolic problems we refer to Cockburn [6].

We will employ the following approximation lemma [5, Lem. 10] in which error terms are expressed with the help of some positive Radon measures. For $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}^d \times \mathbb{R}^+$ we denote by $\mathcal{M}(\Omega)$ the set of positive Radon measures, i.e., the set of positive continuous linear forms on $C_0^0(\Omega)$. For $\mu \in \mathcal{M}(\Omega)$ we set

$$\langle \mu, g \rangle = \int_\Omega g \mathrm{d}\mu, \quad g \in C_0^0(\Omega).$$

APPROXIMATION LEMMA 3.6. *Assume (2.1) and $u_0 \in BV_{\mathrm{loc}}(\mathbb{R}^d)$. Let $\tilde{u} \in L^\infty(\mathbb{R}^d \times \mathbb{R}^+)$. Assume that there exist measures $\mu \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^+)$ and $\mu_0 \in \mathcal{M}(\mathbb{R}^d)$ such that for all $\kappa \in \mathbb{R}$ and all $\phi \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^+, \mathbb{R}^+)$*

$$\begin{aligned} \int_{\mathbb{R}^d \times \mathbb{R}^+} \Big\{ |\tilde{u}(x,t) - \kappa| \phi_t(x,t) + \big[F(x,t,\tilde{u}(x,t) \top \kappa) - F(x,t,\tilde{u}(x,t) \bot \kappa)\big] \\ \times \nabla\phi(x,t) \Big\} \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^d} |u_0(x) - \kappa| \phi(x,0) \mathrm{d}x \\ \geq - \int_{\mathbb{R}^d \times \mathbb{R}^+} \big[|\phi_t(x,t)| + |\nabla\phi(x,t)|\big] \mathrm{d}\mu(x,t) - \int_{\mathbb{R}^d} \phi(x,0) \mathrm{d}\mu_0(x). \end{aligned} \tag{3.7}$$

Let u be the unique entropy solution to (2.1); i.e., for all $\kappa \in \mathbb{R}$ and all $\phi \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^+, \mathbb{R}^+)$ the following estimate holds:

$$\begin{aligned} \int_{\mathbb{R}^d \times \mathbb{R}^+} \Big\{ |u(x,t) - \kappa| \partial_t \varphi(x,t) + [F(x,t,u(x,t) \top \kappa) - F(x,t,u(x,t) \bot \kappa)] \nabla\varphi(x,t) \Big\} \mathrm{d}t\mathrm{d}x \\ + \int_{\mathbb{R}^d} |u_0(x) - \kappa| \varphi(x,0) \mathrm{d}x \geq 0. \end{aligned} \tag{3.8}$$

Then, for all compact sets $E \subset \mathbb{R}^d \times \mathbb{R}^+$, there exist positive $C_{E,F,u_0} > 0$, R, and T which depend only on E, F, and u_0 such that the following error estimate holds:

$$\int_E |\tilde{u}(x,t) - u(x,t)| \mathrm{d}x\mathrm{d}t \leq C_{E,F,u_0}\Big(\mu_0(B_R(0)) + \mu(B_R(0)\times[0,T]) + [\mu(B_R(0)\times[0,T])]^{1/2}\Big). \tag{3.9}$$

In order to apply this lemma we have to prove an L^∞-bound for the discrete solution (see Proposition 4.1 below) and we have to prove inequality (3.7) for the approximate solution (see Theorem 5.2) which follows from a discrete entropy inequality. Then, we have to control the behavior of the measures which are involved. To do this, we need some regularity estimates of the discrete solution, namely, BV-regularity estimates. One can prove a strong BV-regularity result for the solution of (2.1) (see Theorem 3 in [5]):

$$\|u\|_{BV_{\mathrm{loc}}(\mathbb{R}^d\times\mathbb{R}^+)} \leq C(F,u_0).$$

In the case of $F(x,t,s) = f(s)$ this can be sharpened as follows (see [10, Thm. 2.3.1]): For all $t > 0$

$$|u(.,t)|_{\mathrm{BV}(\mathbb{R}^d)} \leq |u_0|_{\mathrm{BV}(\mathbb{R}^d)}.$$

Unfortunately, so far it is not possible to prove such a strong BV-estimate for the discrete approximation on arbitrary unstructured grids. However, it is sufficient to prove a *weak* BV-estimate which is paid for by a loss in the convergence order (see Proposition 4.4 below).

The proofs of these results are given in the rest of the paper and then the proofs of Theorems 3.3 and 3.5 consist of putting together these results and applying Lemma 6.1 below which is a refined version of Approximation Lemma 3.6.

4. Stability results. We prove a maximum principle and an estimate of the entropy production from which weak BV-estimates of the approximate solution can be deduced.

PROPOSITION 4.1 (maximum principle). *Let $u_0 \in L^\infty(\mathbb{R}^d)$ with $U_m \leq u(x) \leq U_M$ almost everywhere. Let u_h be defined in Definition* 2.14 *and suppose that Assumption* 2.5 *holds. Then*

$$U_m \leq u_i^n \leq U_M \quad \forall n \in \mathbb{N},\ \forall i \in I^n, \tag{4.1}$$

and

$$\|u_h\|_{L^\infty(\mathbb{R}^d\times[0,T])} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)} \quad \forall T > 0. \tag{4.2}$$

Proof. This is a proof by induction over $n \in \mathbb{N}$. Note that due to the convex combination in (2.17) we have

$$\min_{j\in J^{n,n+1}(i)} v_j^{n+1} \leq u_i^{n+1} \leq \max_{j\in J^{n,n+1}(i)} v_j^{n+1}.$$

For the v-values on $\mathcal{T}_h^{n,n+1}$ a maximum principle is well known under condition (3.1); see [5, Lem. 1]. Taking into account that the first step of the algorithm (2.15) is just a trivial prolongation, where the function is not changed, concludes the proof. ∎

The next result is a key result for proving the a priori error estimate. This result is a straightforward generalization of [11, Prop. 4.4 and Lem. 4.8]. For the sake of completeness we include its proof here.

Unfortunately, we cannot work with the Kruzkov entropies but have to deal with quadratic entropies. Let $U(x) = x^2$ and Φ denotes the associated entropy flux, i.e.,

$$\partial_s \Phi(x,t,.) = U'(.)\partial_s F(x,t,.) \quad \text{a.e.}$$

We set for $i \in I^{n+1}$ and $j \in \tilde{K}_\partial^{n,n+1}(i)$

$$\Phi_{ij}^{n,n+1}(s) := \frac{1}{k^n} \int_{t^n}^{t^{n+1}} \int_{\tilde{S}_{ij}^{n,n+1}} \Phi(\gamma,t,s) \cdot \tilde{n}_{ij}^{n,n+1} \mathrm{d}\gamma \mathrm{d}t. \tag{4.3}$$

PROPOSITION 4.2 (entropy production estimate). *Assume that Assumptions* 2.1, 2.5*, and* 3.1 *hold where* ξ *in* (3.1) *has to fulfill the restriction* $\xi \in \,]0, C_{\mathrm{ov}}[$*. Then for* $i \in I^{n+1}$ *the following estimate holds:*

$$U(u_i^{n+1}) - \sum_{j\in\tilde{K}^{n,n+1}(i)} \tilde{r}_{ij}^{n,n+1} U(u_j^n) - \frac{k^n}{|T_i^{n+1}|} \sum_{j\in\tilde{K}_\partial^{n,n+1}(i)} \Phi_{ij}^{n,n+1}(u_j^n)$$
$$+ \xi^2 \sum_{j\in\tilde{K}^{n,n+1}(i)} \sum_{l\in\tilde{K}^{n,n+1}(i)} U(u_j^n - u_l^n) \le 0. \tag{4.4}$$

Proof. Let $n \in \mathbb{N}$ and $i \in I^{n+1}$ be given. Enumerate the values u_j^n, $j \in \tilde{K}^{n,n+1}(i)$, by size. To be more precise, set $m := |\tilde{K}^{n,n+1}(i)|$ and define a bijection $\alpha : \{1,\dots,m\} \to \tilde{K}^{n,n+1}(i)$ with $u_l := u_{i\alpha(l)}^n$ and

$$u_1 \le u_2 \le \dots \le u_m.$$

Analogously, set $r_l := \tilde{r}_{i\alpha(l)}^{n,n+1}$, and for $\alpha(l) \in \tilde{K}_\partial^{n,n+1}(i)$ we set $F_l(s) = F_{i\alpha(l)}^{n,n+1}(s)$ and $\Phi_l = \Phi_{i\alpha(l)}^{n,n+1}(s)$. To simplify notation set

$$\delta_l := \begin{cases} 1 & \alpha(l) \in \tilde{K}_\partial^{n,n+1}(i), \\ 0 & \text{else.} \end{cases}$$

Using this notation we can rewrite (2.20) as follows:

$$u_i^{n+1} = \sum_{l=1}^m \left[r_l u_l - \frac{k^n}{|T_i^{n+1}|} F_l(u_l)\delta_l \right] =: q(u_1,\dots,u_m). \tag{4.5}$$

Set

$$p(t_1,\dots,t_m) := U(q(t_1,\dots,t_m)) - \sum_{l=1}^m \left[r_l U(t_l) - \frac{k^n}{|T_i^{n+1}|} \Phi_l(t_l)\delta_l \right] + \xi^2 \sum_{j,l=1}^m U(t_l - t_j).$$

We have to prove that

$$p(u_1,\dots,u_m) \le 0.$$

Clearly, p is differentiable. Define values $P_1, \dots, P_m \in \mathbb{R}^m$ as follows:

$$\begin{aligned} P_1 &:= (u_1, \dots, u_1), \\ P_2 &:= (u_1, u_2, \dots, u_2), \\ &\vdots \quad \vdots \\ P_{m-1} &:= (u_1, \dots, u_{m-1}, u_{m-1}), \\ P_m &:= (u_1, \dots, u_m). \end{aligned}$$

We prove that

(i) $p(P_1) = 0$;

(ii) for $k = 2, \dots, m$ and for all $x \in I(P_{k-1}, P_k)$ we have

$$\nabla P(x)(P_k - P_{k-1}) \leq 0,$$

where $I(P_{k-1}, P_k)$ denotes the straight line between the points P_{k-1} and P_k. Assertion (i) follows directly. For proving (ii) let $l \in \{1, \dots, m\}$ be given. Using $\partial_s \Phi(x, t, .) = U'(.)\partial_s F(x, t, .)$ a.e. and $U'(s) = 2s$ we calculate

$$\partial_l p(t_1, \dots, t_m) \\ = 2 \sum_{k=1}^{m} \left\{ \left[r_l - \frac{k^n}{|T_i^{n+1}|} F_l'(t_l)\delta_l \right] \left[r_l - \frac{k^n}{|T_i^{n+1}|} F_l'(\eta_{kl})\delta_l \right] - \xi^2 \right\}_1 (t_k - t_l),$$

where we used that

$$q(t_1, \dots, t_m) - t_l = \sum_{k=1}^{m} \left[r_k(t_k - t_l) - \frac{k^n}{|T_i^{n+1}|} F_k'(\eta_{kl})\delta_k(t_k - t_l) \right]$$

and η_{kl} is a value between t_k and t_l. Using the CFL-condition (3.1) and $\xi \in]0, C_{\text{ov}}[$ one gets that $\{\dots\}_1 \geq 0$. Finally, with the definition of the values P_k it is now easy to verify (ii), which concludes the proof. □

REMARK 4.3. *Using this entropy production estimate we are in position to prove weak BV-estimates. These estimates are different from the corresponding results on a fixed grid in the following respect. First, these results are weak BV-estimates since half an order of convergence is lost from optimality* [20]. *The estimates are stronger than those on fixed grids* (*cf., e.g.,* [5, *Lem.* 2]) *since those have the following form (with notation obviously changed for the case of a fixed grid):*

$$\sum_{n=0}^{N_0} k^n \sum_{(j,l) \in \mathcal{E}_D} C_{jl}(u_j^n, u_l^n)|u_j^n - u_l^n| \leq C h^{-1/2},$$

where

$$C_{jl}(u, v) = \frac{g_{jl}(u, u) - 2g_{jl}(u, v) - g_{jl}(v, v)}{u - v}.$$

To make our point, in our BV-estimate we have a factor 1 *instead of* C_{jl} *in front of the differences of values* $(u_i^n)_{i \in I^n}$. *This factor* 1 *is essentially needed since it appears naturally in the continuous entropy estimate for the approximate solution (see Theorem* 5.2 *below) which is due to the averaging step* (2.17) *in the numerical scheme in*

Definition 2.14. *It is possible to generalize the result of Chainais-Hillairet* [5, *Lem.* 2] *to the situation we consider here but it is not possible to estimate the factors* C_{jl} uniformly *from below.*

PROPOSITION 4.4 (weak local space BV-estimate). *Assume that the conditions of Proposition* 4.2 *hold. Then there exists a constant* $C > 0$ *depending only on the characteristic cone* D*, data* F, u_0*, the geometry parameter* α*, and the CFL-condition parameter* ξ *such that*

$$\sum_{n=0}^{N_0} k^n \sum_{i\in I_D^{n+1}} |T_i^{n+1}| \sum_{j,l\in \tilde{K}^{n,n+1}(i)} |u_j^n - u_l^n| \le Ch^{1/2}, \tag{4.6}$$

where

$$h = \max_{n=0,\dots,N_T} h_n. \tag{4.7}$$

Proof. First, we note that for $i \in I^{n+1}$ by Assumption 3.1

$$|\tilde{K}^{n,n+1}(i)| = \sum_{j\in\tilde{K}^{n,n+1}(i)} \frac{|T_j^n \cap T_i^{n+1}|}{|T_j^n \cap T_i^{n+1}|} \le \frac{1}{\min_{j\in\tilde{K}^{n,n+1}(i)} |T_j^{n,n+1}|} |T_i^{n+1}| \le \frac{1}{C_{\mathrm{ov}}}.$$

Using Cauchy's inequality we see that

$$\begin{aligned}
&\sum_{n=0}^{N_0} k^n \sum_{i\in I_D^{n+1}} |T_i^{n+1}| \sum_{j,l\in \tilde{K}^{n,n+1}(i)} |u_j^n - u_l^n| \\
&\le C_{\mathrm{ov}}^{-1/2} \sum_{n=0}^{N_0} k^n \sum_{i\in I_D^{n+1}} |T_i^{n+1}| \left(\sum_{j,l\in \tilde{K}^{n,n+1}(i)} |u_j^n - u_l^n|^2 \right)^{1/2} \\
&\le C_{\mathrm{ov}}^{-1/2} \left[\sum_{n=0}^{N_0} k^n \sum_{i\in I_D^{n+1}} |T_i^{n+1}| \right]^{1/2} \\
&\qquad \times \left[\sum_{n=0}^{N_0} k^n \sum_{i\in I_D^{n+1}} |T_i^{n+1}| \sum_{j,l\in \tilde{K}^{n,n+1}(i)} |u_j^n - u_l^n|^2 \right]^{1/2}
\end{aligned}$$

$$\le \left(\frac{|D_{R+1}(x_0)|}{C_{\mathrm{ov}}} \right)^{1/2} \left[\sum_{n=0}^{N_0} k^n \sum_{i\in I_D^{n+1}} |T_i^{n+1}| \sum_{j,l\in \tilde{K}^{n,n+1}(i)} |u_j^n - u_l^n|^2 \right]^{1/2}. \tag{4.8}$$

In order to estimate the last factor we multiply (4.4) by T_i^{n+1} and sum over $i \in I_D^{n+1}$ and get

$$\begin{aligned}
&\|u_h(.,t^{n+1})\|_{L^2(D_{R+1}\cap(B_{R+1}\times\{t^{n+1}\}))} - k^n \sum_{i\in I_D^{n+1}} \sum_{j\in \tilde{K}_\partial^{n,n+1}(i)} \Phi_{ij}^{n,n+1}(u_j^n) \\
&+ \xi^2 \sum_{i\in I_D^{n+1}} |T_i^{n+1}| \sum_{j,l\in\tilde{K}^{n,n+1}(i)} |u_j^n - u_l^n|^2 \\
&\qquad \le \|u_h(.,t^n)\|_{L^2(D_{R+1}\cap(B_{R+1}\times\{t^{n+1}\}))} \\
&\qquad \le \|u_h(.,t^n)\|_{L^2(D_{R+1}\cap(B_{R+1}\times\{t^{n}\}))}.
\end{aligned}$$

Let us consider the term containing the numerical entropy fluxes $\Phi_{ij}^{n,n+1}$. Obviously, these fluxes are conservative in the sense of Assumption 2.5, and hence the sum over all edges (i,j) with $i \in I_D^{n+1}$ and $j \in \tilde{K}_\partial^{n,n+1}(i)$ reduces to the sum over all edges lying on the boundary of the characteristic cone $D_{R+1} \cap (B_{R+1} \times \{t^{n+1}\})$. The number of such edges is bounded by $C(\alpha)h^{d-1}$. Furthermore, since we have the maximum principle (see Proposition 4.1) we know that there exists a constant $C > 0$ depending only on D, F, u_0, U such that

$$|\Phi_{ij}^{n,n+1}(s)| \leq C|S_{ij}^{n,n+1}| \quad \forall s \in [U_m, U_M].$$

Using this we can continue our estimate and get

$$\begin{aligned}\|u_h(.,t^{n+1})\|_{L^2(D_{R+1}\cap(B_{R+1}\times\{t^{n+1}\}))} + \xi^2 \sum_{i\in I_D^{n+1}} |T_i^{n+1}| \sum_{j,l\in\tilde{K}^{n,n+1}(i)} |u_j^n - u_l^n|^2 \\ \leq \|u_h(.,t^n)\|_{L^2(D_{R+1}\cap(B_{R+1}\times\{t^n\}))} + Ck^n.\end{aligned}$$

Summing over $n \in \{0,\ldots,N_0\}$ we end up with

$$\sum_{n=0}^{N_0} \sum_{i\in I_D^{n+1}} |T_i^{n+1}| \sum_{j,l\in\tilde{K}^{n,n+1}(i)} |u_j^n - u_l^n|^2 \leq \frac{1}{\xi^2}\|u_0\|_{L^2(B_{R+1}(x_0))} + C.$$

Using this and applying the CFL-condition (3.1) in inequality (4.8) concludes the proof. ◻

Finally, we are in position to prove a weak time BV-estimate.

Proposition 4.5. *Let the assumptions of Proposition* 4.4 *hold. Then there exists a constant $C > 0$ depending only on the characteristic cone D, data F, u_0, the geometry parameter α, and the CFL-condition parameter ξ such that*

$$\sum_{n=0}^{N_0-1} k^n \sum_{i\in I_R^{n+1}} |T_i^{n+1}| \left|u_i^{n+1} - \sum_{j\in\tilde{K}^{n,n+1}(i)} \tilde{r}_{ij}^{n,n+1} v_j^n\right| \leq Ch^{1/2} \tag{4.9}$$

Proof. The assertion of this proposition follows from Proposition 4.4. Note that by definition of the index sets $J^{n,n+1}(i)$ and $\tilde{K}^{n,n+1}(i)$ (Notations 2.13 and 2.15) we have

$$\sum_{j\in J^{n,n+1}(i)} r_{ij} v_j^n = \sum_{j\in\tilde{K}^{n,n+1}(i)} \tilde{r}_{ij}^{n,n+1} u_j^n.$$

Using the definition of u_i^{n+1} (see (2.17)) and the Lipschitz continuity of the numerical fluxes (Assumption 2.5) we get

$$\begin{aligned}&\left|u_i^{n+1} - \sum_{j\in\tilde{K}^{n,n+1}(i)} \tilde{r}_{ij}^{n,n+1} u_j^n\right| \\ &\leq \sum_{j\in J^{n,n+1}(i)} r_{ij}^{n,n+1} |v_j^{n+1} - v_j^n| \\ &\leq \sum_{j\in J^{n,n+1}(i)} r_{ij}^{n,n+1} \frac{k^n}{|T_j^{n,n+1}|} \sum_{l\in\mathcal{N}^{n,n+1}(j)} |g_{jl}^{n,n+1}(v_j^n, v_l^n) - g_{jl}^{n,n+1}(v_j^n, v_j^n)|\end{aligned}$$

$$\le \sum_{j\in J^{n,n+1}(i)} r_{ij}^{n,n+1} \frac{k^n}{|T_j^{n,n+1}|} L_g \sum_{l\in\mathcal{N}^{n,n+1}(j)} |S_{jl}^{n,n+1}|\,|v_j^n - v_l^n|$$
$$\le \frac{1}{2}(1-\xi) \sum_{j,l\in J^{n,n+1}(i)} |v_j^n - v_l^n|$$
$$= \frac{1}{2}(1-\xi) \sum_{j,l\in \tilde{K}^{n,n+1}(i)} |u_j^n - u_l^n|,$$

where we have used (3.3) and the CFL-condition (3.1). □

REMARK 4.6. *Note that we have estimated in the previous proof the Lipschitz constant of the numerical fluxes by*

$$L_g \le V_{[U_m,U_M]}.$$

Since in practice no numerical flux is needed and since we use these numerical fluxes just as a tool for our analysis, we have freedom to choose the numerical flux. The aforementioned inequality holds, for example, if $g_{jl}^{n,n+1}$ is chosen as Godunov's flux; see, e.g., [5, *section* 2.1].

5. Continuous entropy estimate for the approximate solution. In this section we prove a continuous entropy estimate for the approximate solution. Therefore, we need a *discrete entropy inequality* for the approximate solution. For the concept of entropy solutions and discrete entropy inequalities we refer to [14, 21, 9].

LEMMA 5.1 (discrete entropy inequalities). *Let u_h be given in Definition* 2.14 *and assume that Assumptions* 2.1 *and* 2.5 *and the CFL-condition* (3.1) *hold. For $j,l \in I^{n,n+1}$ define a numerical entropy flux $G_{jl}^{n,n+1}$ by*

$$G_{jl}^{n,n+1}(u,v,\kappa) := g_{jl}^{n,n+1}(u\top\kappa, v\top\kappa) - g_{jl}^{n,n+1}(u\bot\kappa, v\bot\kappa), \tag{5.1}$$

where

$$a\top b := \max\{a,b\} \quad \textit{and} \quad a\bot b := \min\{a,b\}$$

and $g_{jl}^{n,n+1}$ is the numerical flux in (2.16).

Then the following discrete entropy inequalities hold for all $n \in \mathbb{N}$:

$$|T_j^{n+1}| \frac{|v_j^{n+1}-\kappa| - |v_j^n - \kappa|}{k^n} + \sum_{l\in\mathcal{N}^{n,n+1}(j)} G_{jl}^{n,n+1}(v_j^n, v_l^n, \kappa) \le 0 \quad \forall j \in I^{n,n+1}, \tag{5.2}$$

$$\frac{1}{k^n}\left[|T_i^{n+1}|\,|u_i^{n+1}-\kappa| - \sum_{j\in J^{n,n+1}(i)} |T_j^{n,n+1}|\,|v_j^n-\kappa|\right] + \sum_{j\in J^{n,n+1}(i)} \sum_{l\in\mathcal{N}^{n,n+1}(j)} G_{jl}^{n,n+1}(v_j^n, v_l^n, \kappa) \le 0 \quad \forall i \in I^{n+1}. \tag{5.3}$$

Proof. Assertion (5.2) follows from the monotony of the second step in Definition 2.14, which is due to the monotony of the numerical fluxes $g_{jl}^{n,n+1}$ (see Assumption 2.5). The details can be found in [5, Lem. 3]. Assertion (5.3) follows from (5.2) using that u_i^{n+1} is a convex combination of $(v_j^{n+1})_{j\in J^{n,n+1}(i)}$; see (2.17). □

For $\Omega = \mathbb{R}^d$ or $\mathbb{R}^d \times \mathbb{R}^+$ we denote by $\mathcal{M}(\Omega)$ the set of positive Radon measures on Ω. The following theorem is an adaptation of [9, Thm. 6.1] to our case of varying grids.

THEOREM 5.2. *Assume that Assumptions* 2.1 *and* 2.5 *and the CFL-condition* (3.1) *hold. Let* u_h *be given by Definition* 2.14. *Then there exist measures* $\mu_h \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^+)$ *and* $\mu_0 \in \mathcal{M}(\mathbb{R}^d)$ *such that for all* $\kappa \in \mathbb{R}$ *and all* $\phi \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R}^+)$ *the following estimate holds:*

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^d} \big(|u_h(x,t)-\kappa|\phi_t(x,t) + \big[F(x,t,u_h(x,t)\top\kappa) - F(x,t,u_h(x,t)\bot\kappa)\big] \times\nabla\phi(x,t)\big)\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^d}|u_0(x)-\kappa|\phi(x,0)\mathrm{d}x$$

$$\geq -\int_{\mathbb{R}^d\times\mathbb{R}^+}\left(|\phi_t(x,t)| + |\nabla\phi(x,t)|\right)\mathrm{d}\mu_h(x,t) - \int_{\mathbb{R}^d}\phi(x,0)\mathrm{d}\mu_0(x). \tag{5.4}$$

The measures are defined as follows. Let $\psi \in C_0^0(\mathbb{R}^d)$. *Then*

$$\langle\mu_0,\,\psi\rangle \equiv \int_{\mathbb{R}^d}\psi(x)\mathrm{d}\mu_0(x) := \int_{\mathbb{R}^d}|u_h(x,0)-u_0(x)|\psi(x,0)\mathrm{d}x. \tag{5.5}$$

The measure μ_h *is given by* $\mu_h = \mu_{h1}+\mu_{h2}+\mu_{h3}+\mu_{h4}$, *where the four parts are defined as (with* $\psi \in C_0^0(\mathbb{R}^d\times\mathbb{R}^+)$*)*

$$\langle\mu_{h1},\,\psi\rangle := \frac{1}{2}\sum_{n\in\mathbb{N}}\sum_{i\in I^{n+1}}\sum_{j,l\in\tilde{K}^{n,n+1}(i)}|u_j^n - u_l^n| \times\frac{h_i^{n+1}}{|T_i^{n+1}|}\int_{T_l^n\cap T_i^{n+1}}\int_{T_j^n\cap T_i^{n+1}}\int_0^1\psi(x+\theta(y-x),t^{n+1})\mathrm{d}\theta\mathrm{d}y\mathrm{d}x, \tag{5.6}$$

$$\langle\mu_{h2},\,\psi\rangle := \sum_{n\in\mathbb{N}}\sum_{i\in I^{n+1}}\left|u_i^{n+1} - \sum_{j\in\tilde{K}^{n,n+1}(i)}\tilde{r}_{ij}^{n,n+1}u_j^n\right| \times\int_{t^n}^{t^{n+1}}\int_{T_i^{n+1}}\int_0^1\psi(x,t+\theta(t^{n+1}-t))\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}t, \tag{5.7}$$

$$\langle\mu_{h3},\,\psi\rangle := 2\sum_{n\in\mathbb{N}}k^n\sum_{i\in I^{n+1}}\sum_{j\in J^{n,n+1}(i)}\sum_{l\in\mathcal{N}^{n,n+1}(j)}\left|g_{jl}^{n,n+1}(v_j^n,v_l^n) - g_{jl}^{n,n+1}(v_j^n,v_j^n)\right| \times\frac{h_i^{n+1}}{k^n\,|T_i^{n+1}|\,|T_j^{n,n+1}|}\int_{t^n}^{t^{n+1}}\int_{T_i^{n+1}}\int_{T_j^{n,n+1}}\int_0^1\psi(x+\theta(y-x),t)\mathrm{d}\theta\mathrm{d}x\mathrm{d}y\mathrm{d}t, \tag{5.8}$$

$$\langle\mu_{h4},\,\psi\rangle := 2\sum_{n\in\mathbb{N}}k^n\sum_{(j,l)\in\mathcal{E}^{n,n+1}}\Big\{\left|g_{jl}^{n,n+1}(v_j^n,v_l^n) - g_{jl}^{n,n+1}(v_j^n,v_j^n)\right|\langle\mu_{jl}^{n,n+1},\,\psi\rangle + \left|g_{jl}^{n,n+1}(v_j^n,v_l^n) - g_{jl}^{n,n+1}(v_l^n,v_l^n)\right|\langle\mu_{lj}^{n,n+1},\,\psi\rangle\Big\} + 4C_{F,\phi,u_0}\sum_{n\in\mathbb{N}}\sum_{(j,l)\in\mathcal{E}^{n,n+1}}\langle\lambda_{jl}^{n,n+1},\psi\rangle, \tag{5.9}$$

where

$$\langle \mu_{jl}^{n,n+1}, \psi\rangle := \frac{h_j^{n,n+1}+k^n}{(k^n)^2|T_j^{n,n+1}|\,|S_{jl}^{n,n+1}|}\int_{t^n}^{t^{n+1}}\int_{t^n}^{t^{n+1}}\int_{T_j^{n,n+1}}\int_{S_{jl}^{n,n+1}}\int_0^1$$

$$\psi(x+\theta(y-x), t+\theta(s-t))\mathrm{d}\theta\mathrm{d}\gamma(y)\mathrm{d}x\mathrm{d}s\mathrm{d}t, \tag{5.10}$$

$$\langle \lambda_{jl}^{n,n+1}, \psi\rangle := \frac{(\operatorname{diam}(S_{jl}^{n,n+1})+k^n)^2}{k^n|S_{jl}^{n,n+1}|}\int_{t^n}^{t^{n+1}}\int_{t^n}^{t^{n+1}}\int_{S_{jl}^{n,n+1}}\int_{S_{jl}^{n,n+1}}\int_0^1$$

$$\psi(\xi+\theta(\gamma-\xi), t+\theta(s-t))\mathrm{d}\theta\mathrm{d}\xi\mathrm{d}\gamma\mathrm{d}x\mathrm{d}t. \tag{5.11}$$

Proof. Let $\phi\in C_0^\infty(\mathbb{R}^d\times\mathbb{R}^+;\mathbb{R}^+)$ be given. Set

$$T_{10} := -\int_{\mathbb{R}^+}\int_{\mathbb{R}^d}|u_h(x,t)-\kappa|\phi_t(x,t)\mathrm{d}x\mathrm{d}t - \int_{\mathbb{R}^d}|u_0(x)-\kappa|\phi(x,0)\mathrm{d}x,$$

$$T_{20} := -\int_{\mathbb{R}^+}\int_{\mathbb{R}^d}\big[F(x,t,u_h(x,t)\top\kappa)-F(x,t,u_h(x,t)\bot\kappa)\big]\nabla\phi(x,t)\mathrm{d}x\mathrm{d}t.$$

Let us consider the first term T_{10}. Using the definition of u_h in (2.18) we get

$$\begin{aligned}
T_{10} &= -\sum_{n\in\mathbb{N}}\sum_{i\in I^n}|u_i^n-\kappa|\int_{T_i^n}\big[\phi(x,t^{n+1})-\phi(x,t^n)\big]\,\mathrm{d}x - \int_{\mathbb{R}^d}|u_0(x)-\kappa|\phi(x,0)\mathrm{d}x\\
&= \sum_{n\in\mathbb{N}}\left\{\sum_{i\in I^{n+1}}|u_i^{n+1}-\kappa|\int_{T_i^{n+1}}\phi(x,t^{n+1})\mathrm{d}x - \sum_{i\in I^n}|u_i^n-\kappa|\int_{T_i^n}\phi(x,t^{n+1})\mathrm{d}x\right\}\\
&\quad+\sum_{i\in I^0}|u_i^0-\kappa|\int_{T_i^0}\phi(x,t^0)\mathrm{d}x - \int_{\mathbb{R}^d}|u_0(x)-\kappa|\phi(x,0)\mathrm{d}x\\
&=: T_{11}+T_{12}.
\end{aligned}$$

Using the triangle inequality we simply have that

$$\begin{aligned}
T_{12} &= \sum_{i\in I^0}|u_i^0-\kappa|\int_{T_i^0}\phi(x,0)\mathrm{d}x - \int_{\mathbb{R}^d}|u_0(x)-\kappa|\phi(x,0)\mathrm{d}x\\
&\le \int_{\mathbb{R}^d}|u_h(x,0)-u_0(x)|\phi(x,0)\mathrm{d}x\\
&= \langle\mu_0,\phi(.,0)\rangle.
\end{aligned}$$

Now, let us consider the term T_{11}. We note that using the first step of Definition 2.14 (see (2.15)) we have that

$$\begin{aligned}
\sum_{i\in I^n}|u_i^n-\kappa|\int_{T_i^n}\phi(x,t^{n+1})\mathrm{d}x &= \sum_{i\in I^n}|u_i^n-\kappa|\sum_{j\in K^{n,n+1}(i)}\int_{T_j^{n,n+1}}\phi(x,t^{n+1})\mathrm{d}x\\
&= \sum_{j\in I^{n,n+1}}|v_j^n-\kappa|\int_{T_j^{n,n+1}}\phi(x,t^{n+1})\mathrm{d}x\\
&= \sum_{i\in I^{n+1}}\sum_{j\in J^{n,n+1}(i)}|v_j^n-\kappa|\int_{T_j^{n,n+1}}\phi(x,t^{n+1})\mathrm{d}x.
\end{aligned}$$

Using this and inserting an additional term which is due to the definition of u_i^{n+1} in (2.17) we get

$$
\begin{aligned}
T_{11} &= \sum_{n\in\mathbb{N}} \left\{ \sum_{i\in I^{n+1}} |u_i^{n+1}-\kappa| \int_{T_i^{n+1}} \phi(x,t^{n+1})\mathrm{d}x - \sum_{i\in I^n} |u_i^n-\kappa| \int_{T_i^n} \phi(x,t^{n+1})\mathrm{d}x \right\} \\
&= \sum_{n\in\mathbb{N}} \left[\sum_{i\in I^{n+1}} |u_i^{n+1}-\kappa| - \sum_{j\in J^{n,n+1}(i)} \frac{|T_j^{n,n+1}|}{|T_i^{n+1}|} |v_j^n-\kappa| \right] \int_{T_i^{n+1}} \phi(x,t^{n+1})\mathrm{d}x \\
&\quad + \sum_{n\in\mathbb{N}} \left[\sum_{i\in I^{n+1}} \sum_{j\in J^{n,n+1}(i)} \frac{|T_j^{n,n+1}|}{|T_i^{n+1}|} |v_j^n-\kappa| \int_{T_i^{n+1}} \phi(x,t^{n+1})\mathrm{d}x \right. \\
&\qquad \left. - \sum_{i\in I^{n+1}} \sum_{j\in J^{n,n+1}(i)} |v_j^n-\kappa| \int_{T_j^{n,n+1}} \phi(x,t^{n+1})\mathrm{d}x \right] \\
&=: T_{13} + T_{14}.
\end{aligned}
$$

For the term T_{14} we use a symmetrization trick.

$$
\begin{aligned}
T_{14} &= \sum_{n\in\mathbb{N}} \left[\sum_{i\in I^{n+1}} \sum_{j\in J^{n,n+1}(i)} \frac{|T_j^{n,n+1}|}{|T_i^{n+1}|} |v_j^n-\kappa| \int_{T_i^{n+1}} \phi(x,t^{n+1})\mathrm{d}x \right. \\
&\qquad \left. - \sum_{i\in I^n} \sum_{j\in J^{n,n+1}(i)} |v_j^n-\kappa| \int_{T_j^{n,n+1}} \phi(x,t^{n+1})\mathrm{d}x \right] \\
&= \sum_{n\in\mathbb{N}} \left[\sum_{i\in I^{n+1}} \sum_{j\in J^{n,n+1}(i)} \sum_{l\in J^{n,n+1}(i)} \frac{|T_j^{n,n+1}|}{|T_i^{n+1}|} |v_j^n-\kappa| \int_{T_l^{n,n+1}} \phi(x,t^{n+1})\mathrm{d}x \right. \\
&\qquad \left. - \sum_{i\in I^n} \sum_{j\in J^{n,n+1}(i)} \sum_{l\in J^{n,n+1}(i)} \frac{|T_l^{n,n+1}|}{|T_i^{n+1}|} |v_j^n-\kappa| \int_{T_j^{n,n+1}} \phi(x,t^{n+1})\mathrm{d}x \right] \\
&= \frac{1}{2} \sum_{n\in\mathbb{N}} \sum_{i\in I^{n+1}} \sum_{j,l\in J^{n,n+1}(i)} \left\{ |v_j^n-\kappa| \left[\frac{|T_j^{n,n+1}|}{|T_i^{n+1}|} \int_{T_l^{n,n+1}} \phi(x,t^{n+1})\mathrm{d}x \right.\right. \\
&\qquad \left. - \frac{|T_l^{n,n+1}|}{|T_i^{n+1}|} \int_{T_j^{n,n+1}} \phi(x,t^{n+1})\mathrm{d}x \right] \\
&\quad \left. - |v_l^n-\kappa| \left[\frac{|T_j^{n,n+1}|}{|T_i^{n+1}|} \int_{T_l^{n,n+1}} \phi(x,t^{n+1})\mathrm{d}x - \frac{|T_l^{n,n+1}|}{|T_i^{n+1}|} \int_{T_j^{n,n+1}} \phi(x,t^{n+1})\mathrm{d}x \right] \right\} \\
&\le \frac{1}{2} \sum_{n\in\mathbb{N}} \sum_{i\in I^{n+1}} \sum_{j,l\in J^{n,n+1}(i)} |v_j^n - v_l^n| \\
&\quad \times \left| \frac{|T_j^{n,n+1}|}{|T_i^{n+1}|} \int_{T_l^{n,n+1}} \phi(x,t^{n+1})\mathrm{d}x - \frac{|T_l^{n,n+1}|}{|T_i^{n+1}|} \int_{T_j^{n,n+1}} \phi(x,t^{n+1})\mathrm{d}x \right| \\
&\le \frac{1}{2} \sum_{n\in\mathbb{N}} \sum_{i\in I^{n+1}} \sum_{j,l\in \tilde{K}^{n,n+1}(i)} |u_j^n - u_l^n| \\
&\quad \times \frac{h_i^{n+1}}{|T_i^{n+1}|} \int_{T_l^n\cap T_i^{n+1}} \int_{T_j^n\cap T_i^{n+1}} \int_0^1 |\nabla\phi(x+\theta(y-x),t^{n+1})| \mathrm{d}\theta \mathrm{d}y \mathrm{d}x \\
&= \langle \mu_{h1}, |\nabla\phi| \rangle,
\end{aligned}
$$

where we have used

$$\sum_{j,l\in J^{n,n+1}(i)} |v_j^n - v_l^n| = \sum_{j,l\in \tilde{K}^{n,n+1}(i)} |u_j^n - u_l^n|.$$

It remains to consider the terms T_{13} and T_{20}. In order to do this we insert the discrete entropy inequality (5.3). Define

$$T_1 := \sum_{n\in\mathbb{N}} \sum_{i\in I^{n+1}} \left[|u_i^{n+1} - \kappa| - \sum_{j\in J^{n,n+1}(i)} \frac{|T_j^{n,n+1}|}{|T_i^{n+1}|} |v_j^n - \kappa| \right] \frac{1}{k^n} \int_{t^n}^{t^{n+1}} \int_{T_i^{n+1}} \phi(x,t)\mathrm{d}x\,\mathrm{d}t,$$

$$T_2 := \sum_{n\in\mathbb{N}} k^n \sum_{i\in I^{n+1}} \sum_{j\in J^{n,n+1}(i)} \sum_{l\in\mathcal{N}^{n,n+1}(j)} \left[G_{jl}^{n,n+1}(v_j^n, v_l^n, \kappa) - G_{jl}^{n,n+1}(v_j^n, v_j^n, \kappa) \right]$$
$$\times \frac{1}{k^n} \int_{t^n}^{t^{n+1}} \frac{1}{|T_i^{n+1}|} \int_{T_i^{n+1}} \phi(x,t)\mathrm{d}x\,\mathrm{d}t.$$

Since $\phi \geq 0$ we have by Lemma 5.1

$$T_1 + T_2 \leq 0.$$

We now compare T_{13} with T_1 and T_{20} with T_2. This part is almost identical to the proof of Theorem 1 in [5]. Let us start with the first pair. We have

$$\begin{aligned}
T_{13} - T_1 &= \sum_{n\in\mathbb{N}} \sum_{i\in I^{n+1}} \left[|u_i^{n+1} - \kappa| - \sum_{j\in J^{n,n+1}(i)} \frac{|T_j^{n,n+1}|}{|T_i^{n+1}|} |v_j^n - \kappa| \right] \\
&\quad \times \left[\frac{1}{k^n} \int_{t^n}^{t^{n+1}} \int_{T_i^{n+1}} \phi(x,t^{n+1})\mathrm{d}x\,\mathrm{d}t - \frac{1}{k^n} \int_{t^n}^{t^{n+1}} \int_{T_i^{n+1}} \phi(x,t)\mathrm{d}x\,\mathrm{d}t \right] \\
&\leq \sum_{n\in\mathbb{N}} \sum_{i\in I^{n+1}} \left| u_i^{n+1} - \sum_{j\in\tilde{K}^{n,n+1}(i)} \frac{|T_j^n \cap T_i^{n+1}|}{|T_i^{n+1}|} u_j^n \right| \\
&\quad \times \int_{t^n}^{t^{n+1}} \int_{T_i^{n+1}} \int_0^1 |\phi_t(x, t + \theta(t^{n+1} - t))|\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}t \\
&= \langle \mu_{h2}, |\phi_t| \rangle.
\end{aligned}$$

It remains to compare T_{20} and T_2. We first simplify T_{20} using the definition of u_h (see (2.18) and (2.15)) and that $\operatorname{div}_x F = 0$ (see Assumption 2.1).

$$\begin{aligned}
T_{20} &= -\int_{\mathbb{R}^+} \int_{\mathbb{R}^d} \left[F(x,t,u_h(x,t)\top\kappa) - F(x,t,u_h(x,t)\perp\kappa) \right] \nabla\phi(x,t)\mathrm{d}x\mathrm{d}t \\
&= \sum_{n\in\mathbb{N}} \sum_{i\in I^n} \sum_{j\in K^{n,n+1}(i)} \int_{t^n}^{t^{n+1}} \int_{T_j^{n,n+1}} \\
&\qquad\qquad \left[F(x,t,u_h(x,t)\top\kappa) - F(x,t,u_h(x,t)\perp\kappa) \right] \nabla\phi(x,t)\mathrm{d}x\mathrm{d}t \\
&= \sum_{n\in\mathbb{N}} \sum_{i\in I^n} \sum_{j\in K^{n,n+1}(i)} \sum_{l\in\mathcal{N}^{n,n+1}(j)} \int_{t^n}^{t^{n+1}} \int_{S_{jl}^{n,n+1}} \\
&\qquad\qquad \left[F(\gamma,t,v_j^n\top\kappa) - F(\gamma,t,v_j^n\perp\kappa) \right] n_{jl}^{n,n+1} \phi(\gamma,t)\mathrm{d}\gamma\mathrm{d}t
\end{aligned}$$

$$= \sum_{n\in\mathbb{N}} \sum_{(j,l)\in\mathcal{E}^{n,n+1}} \int_{t^n}^{t^{n+1}} \int_{S_{jl}^{n,n+1}} \Big\{ \big[F(\gamma,t,v_l^n \top \kappa) - F(\gamma,t,v_l^n \perp \kappa)\big] - \big[F(\gamma,t,v_j^n \top \kappa) - F(\gamma,t,v_j^n \perp \kappa)\big] \Big\} n_{jl}^{n,n+1} \phi(\gamma,t)\mathrm{d}\gamma\mathrm{d}t.$$

Due to the consistency of the numerical fluxes (see Assumption 2.5) we insert some terms such that

$$T_{20} = T_{200} + T_{201},$$

where

$$T_{200} := -\sum_{n\in\mathbb{N}} k^n \sum_{(j,l)\in\mathcal{E}^{n,n+1}} \big[G_{jl}^{n,n+1}(v_j^n, v_j^n) - G_{jl}^{n,n+1}(v_j^n, v_l^n) + G_{jl}^{n,n+1}(v_j^n, v_l^n) - G_{jl}^{n,n+1}(v_l^n, v_l^n)\big] \frac{1}{k^n\,|S_{jl}^{n,n+1}|} \int_{t^n}^{t^{n+1}} \int_{S_{jl}^{n,n+1}} \phi(\gamma,t)\mathrm{d}\gamma\mathrm{d}t,$$

$$T_{201} := \sum_{n\in\mathbb{N}} \sum_{(j,l)\in\mathcal{E}^{n,n+1}} \int_{t^n}^{t^{n+1}} \int_{S_{jl}^{n,n+1}} \Bigg\{ \Bigg(\big[F(\gamma,t,v_l^n \top \kappa) - F(\gamma,t,v_l^n \perp \kappa)\big] n_{jl}^{n,n+1} - \frac{1}{|S_{jl}^{n,n+1}|} G_{jl}^{n,n+1}(v_l^n, v_l^n, \kappa) \Bigg) - \Bigg(\big[F(\gamma,t,v_j^n \top \kappa) - F(\gamma,t,v_j^n \perp \kappa)\big] n_{jl}^{n,n+1} - \frac{1}{|S_{jl}^{n,n+1}|} G_{jl}^{n,n+1}(v_j^n, v_j^n, \kappa) \Bigg) \Bigg\} \phi(\gamma,t)\mathrm{d}\gamma\mathrm{d}t.$$

First, we estimate the consistency error T_{201} of the numerical flux. Due to the definition of $G_{jl}^{n,n+1}$ (see (5.1)) we have

$$\begin{aligned} T_{201} = \sum_{n\in\mathbb{N}} \sum_{(j,l)\in\mathcal{E}^{n,n+1}} \frac{1}{k^n |S_{jl}^{n,n+1}|} \int_{t^n}^{t^{n+1}} \int_{t^n}^{t^{n+1}} \int_{S_{jl}^{n,n+1}} \int_{S_{jl}^{n,n+1}} & \Bigg\{ \Bigg[\Bigg(F(\gamma,t,v_l^n \top \kappa) n_{jl}^{n,n+1} - \frac{1}{|S_{jl}^{n,n+1}|} g_{jl}^{n,n+1}(v_l^n \top \kappa, v_l^n \top \kappa) \Bigg) \\ & - \Bigg(F(\gamma,t,v_l^n \perp \kappa) n_{jl}^{n,n+1} - \frac{1}{|S_{jl}^{n,n+1}|} g_{jl}^{n,n+1}(v_l^n \perp \kappa, v_l^n \perp \kappa) \Bigg) \Bigg] \\ & - \Bigg[\Bigg(F(\gamma,t,v_j^n \top \kappa) n_{jl}^{n,n+1} - \frac{1}{|S_{jl}^{n,n+1}|} g_{jl}^{n,n+1}(v_j^n \top \kappa, v_j^n \top \kappa) \Bigg) \\ & - \Bigg(F(\gamma,t,v_j^n \perp \kappa) n_{jl}^{n,n+1} - \frac{1}{|S_{jl}^{n,n+1}|} g_{jl}^{n,n+1}(v_j^n \perp \kappa, v_j^n \perp \kappa) \Bigg) \Bigg] \Bigg\} \\ & \times \big[\phi(\gamma,t) - \phi(\xi,s)\big] \mathrm{d}\xi\mathrm{d}\gamma\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Note that the term involving $\phi(\xi,s)$ equals zero due to the consistency of the numerical fluxes; see Assumption 2.5. Since $F \in C^1$ (see Assumption 2.1) there exists a constant C_{F,ϕ,u_0} such that for all $(\gamma,s,v) \in (\mathrm{supp}(\phi) \cap (\mathbb{R}^d \times [t^n, t^{n+1}[)) \times [U_m, U_M]$

$$\left| F(\gamma,s,v) n_{jl}^{n,n+1} - \frac{1}{|S_{jl}^{n,n+1}|} g_{jl}^{n,n+1}(v,v) \right| \le C_{F,\phi,u_0}(\,\mathrm{diam}\,(S_{jl}^{n,n+1}) + k^n). \tag{5.12}$$

Moreover, we have for $(\gamma,\xi,t,s) \in (S_{jl}^{n,n+1})^2 \times [t^n, t^{n+1}[^2$

$$\big|\phi(\gamma,t)-\phi(\xi,s)\big| \leq (\,\mathrm{diam}\,(S_{jl}^{n,n+1})+k^n)\int_0^1 (|\nabla\phi|+|\phi_t|)(\xi+\theta(\gamma-\xi),t+\theta(s-t))\mathrm{d}\theta.$$

Hence, we can estimate

$$|T_{201}| \leq 4C_{F,\phi,u_0}\sum_{n\in\mathbb{N}}\sum_{(j,l)\in\mathcal{E}^{n,n+1}}\langle\lambda_{jl}^{n,n+1},|\nabla\phi|+|\phi_t|\rangle.$$

Now, let us consider T_2.

$$\begin{aligned}T_2 \,=\, &\sum_{n\in\mathbb{N}}k^n\sum_{i\in I^{n+1}}\sum_{j\in J^{n,n+1}(i)}\sum_{l\in\mathcal{N}^{n,n+1}(j)}\big[G_{jl}^{n,n+1}(v_j^n,v_l^n,\kappa)-G_{jl}^{n,n+1}(v_j^n,v_j^n,\kappa)\big]\\ &\times\left[\frac{1}{k^n}\int_{t^n}^{t^{n+1}}\frac{1}{|T_i^{n+1}|}\int_{T_i^{n+1}}\phi(x,t)\mathrm{d}x\,\mathrm{d}t-\frac{1}{k^n}\int_{t^n}^{t^{n+1}}\frac{1}{|T_j^{n,n+1}|}\int_{T_j^{n,n+1}}\phi(x,t)\mathrm{d}x\,\mathrm{d}t\right]\\ &+\sum_{n\in\mathbb{N}}k^n\sum_{i\in I^{n+1}}\sum_{j\in J^{n,n+1}(i)}\sum_{l\in\mathcal{N}^{n,n+1}(j)}\big[G_{jl}^{n,n+1}(v_j^n,v_l^n,\kappa)-G_{jl}^{n,n+1}(v_j^n,v_j^n,\kappa)\big]\\ &\times\frac{1}{k^n}\int_{t^n}^{t^{n+1}}\frac{1}{|T_j^{n,n+1}|}\int_{T_j^{n,n+1}}\phi(x,t)\mathrm{d}x\,\mathrm{d}t\\ =:&\, T_3+T_4.\end{aligned}$$

Note that due to the monotony of the numerical fluxes $g_{jl}^{n,n+1}$ (see Assumption 2.5) and due to the definition of $G_{jl}^{n,n+1}$ (see (5.1)) we have for all $\kappa\in\mathbb{R}$

$$\begin{aligned}&|G_{jl}^{n,n+1}(v_j^n,v_l^n,\kappa)-G_{jl}^{n,n+1}(v_j^n,v_j^n,\kappa)|\\ &\leq 2|g_{jl}^{n,n+1}(v_j^n,v_l^n)-g_{jl}^{n,n+1}(v_j^n,v_j^n)|.\end{aligned}\tag{5.13}$$

Using this we get for T_3

$$\begin{aligned}T_3 =& \sum_{n\in\mathbb{N}}k^n\sum_{i\in I^{n+1}}\sum_{j\in J^{n,n+1}(i)}\sum_{l\in\mathcal{N}^{n,n+1}(j)}\big[G_{jl}^{n,n+1}(v_j^n,v_l^n,\kappa)-G_{jl}^{n,n+1}(v_j^n,v_j^n,\kappa)\big]\\ &\times\left[\frac{1}{k^n|T_i^{n+1}|}\int_{t^n}^{t^{n+1}}\int_{T_i^{n+1}}\phi(x,t)\mathrm{d}x\,\mathrm{d}t-\frac{1}{k^n|T_j^{n,n+1}|}\int_{t^n}^{t^{n+1}}\int_{T_j^{n,n+1}}\phi(x,t)\mathrm{d}x\,\mathrm{d}t\right]\\ \leq& \,2\sum_{n\in\mathbb{N}}k^n\sum_{i\in I^{n+1}}\sum_{j\in J^{n,n+1}(i)}\sum_{l\in\mathcal{N}^{n,n+1}(j)}\big|g_{jl}^{n,n+1}(v_j^n,v_l^n)-g_{jl}^{n,n+1}(v_j^n,v_j^n)\big|\\ &\times\frac{h_i^{n+1}}{k^n\,|T_i^{n+1}|\,|T_j^{n,n+1}|}\int_{t^n}^{t^{n+1}}\int_{T_i^{n+1}}\int_{T_j^{n,n+1}}\int_0^1|\nabla\phi(x+\theta(y-x),t)|\mathrm{d}\theta\mathrm{d}x\mathrm{d}y\mathrm{d}t\\ =& \,\langle\mu_{h3},|\nabla\phi|\rangle.\end{aligned}$$

Now, we write the term T_4 as a sum over edges.

$$\begin{aligned}T_4 =& \sum_{n\in\mathbb{N}}k^n\sum_{i\in I^{n+1}}\sum_{j\in J^{n,n+1}(i)}\sum_{l\in\mathcal{N}^{n,n+1}(j)}\big[G_{jl}^{n,n+1}(v_j^n,v_l^n,\kappa)-G_{jl}^{n,n+1}(v_j^n,v_j^n,\kappa)\big]\\ &\times\frac{1}{k^n|T_j^{n,n+1}|}\int_{t^n}^{t^{n+1}}\int_{T_j^{n,n+1}}\phi(x,t)\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

$$= \sum_{n\in\mathbb{N}} k^n \sum_{(j,l)\in\mathcal{E}^{n,n+1}} \Big[G_{jl}^{n,n+1}(v_j^n, v_l^n)\phi_j^{n,n+1} - G_{jl}^{n,n+1}(v_j^n, v_j^n)\phi_j^{n,n+1} + G_{jl}^{n,n+1}(v_l^n, v_l^n)\phi_l^{n,n+1} - G_{jl}^{n,n+1}(v_j^n, v_l^n)\phi_l^{n,n+1}\Big],$$

where

$$\phi_j^{n,n+1} := \frac{1}{k^n|T_j^{n,n+1}|}\int_{t^n}^{t^{n+1}}\int_{T_j^{n,n+1}} \phi(x,t)\mathrm{d}x\,\mathrm{d}t$$

for $j \in I^{n,n+1}$.

Now, we can compare T_{200} with T_4.

$$T_{200} - T_4 = \sum_{n\in\mathbb{N}} k^n \sum_{(j,l)\in\mathcal{E}^{n,n+1}} \Big\{\big[G_{jl}^{n,n+1}(v_j^n, v_l^n) - G_{jl}^{n,n+1}(v_j^n, v_j^n)\big][\phi_{jl}^{n,n+1} - \phi_j^{n,n+1}] - \big[G_{jl}^{n,n+1}(v_j^n, v_l^n) - G_{jl}^{n,n+1}(v_l^n, v_l^n)\big][\phi_{jl}^{n,n+1} - \phi_l^{n,n+1}]\Big\},$$

where we have used the abbreviation

$$\phi_{jl}^{n,n+1} := \frac{1}{k^n}\int_{t^n}^{t^{n+1}} \frac{1}{|S_{jl}^{n,n+1}|}\int_{S_{jl}^{n,n+1}} \phi(\gamma,t)\mathrm{d}\gamma\mathrm{d}t.$$

Using the mean value theorem we get

$$\begin{aligned} |\phi_{jl}^{n,n+1} - \phi_j^{n,n+1}| &\le \frac{h_j^{n,n+1} + k^n}{(k^n)^2|T_j^{n,n+1}|\,|S_{jl}^{n,n+1}|}\int_{t^n}^{t^{n+1}}\int_{t^n}^{t^{n+1}}\int_{T_j^{n,n+1}}\int_{S_{jl}^{n,n+1}}\int_0^1 \\ &\quad |\nabla\phi(x+\theta(\gamma-x), t+\theta(s-t))| \\ &\quad + |\phi_t(x+\theta(\gamma-x), t+\theta(s-t))|\mathrm{d}\theta\mathrm{d}\gamma\mathrm{d}x\mathrm{d}s\mathrm{d}t \\ &= \langle \mu_{jl}^{n,n+1}, |\nabla\phi| + |\phi_t|\rangle. \end{aligned}$$

Using this and (5.13) we get that

$$T_{20} - T_4 \le \langle \mu_{h4}, |\nabla\phi| + |\phi_t|\rangle,$$

which concludes the proof. □

6. Proof of the a posteriori error estimate. It is well known that the Kruzkov technique for proving error estimates leads to a posteriori error estimates; cf. [6]. The main idea for proving Theorem 3.3 consists of estimating the constant in Approximation Lemma 3.6, to take care of the domain of dependence of the error, and to estimate the measures appropriately. In case of a fixed grid, this has been done in [15] where an adaptive strategy is introduced as well.

A more refined version of Approximation Lemma 3.6 contains test functions. These test functions are chosen specifically and can be interpreted as a solution of a linearized dual problem to (2.1). This is the content of the next lemma.

LEMMA 6.1 (see [15, Lem. 2.10]). *Let the assumptions in Approximation Lemma 3.6 hold. Let* $\omega, R, T \in \mathbb{R}^+$ *be given and let* $\rho \in C_0^1(\mathbb{R}^+; [0,1])$ *be such that* $\rho' \le 0$ *and*

$$\begin{aligned} \rho &= 1 \quad \text{on } [0,R], \\ \rho &= 0 \quad \text{on } [R+1,\infty[. \end{aligned}$$

Set

$$\psi(x,t) := \frac{T-t}{T}\rho(|x-x_0|+\omega t) \quad on \quad \mathbb{R}^d \times [0,T],$$
$$\psi(x,t) := 0 \quad on\ \mathbb{R}^d \times [T,\infty[,$$

where $x_0 \in \mathbb{R}^d$ *is given.*

Then we have

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^+}\Big\{|\tilde{u}(x,t)-u(x,t)|\psi_t(x,t) + \big[F(x,t\tilde{u}(x,t)\top u(x,t)) - F(x,t,\tilde{u}(x,t)\bot u(x,t))\big]\nabla\psi(x,t)\Big\}\mathrm{d}t\mathrm{d}x$$
$$\geq -\mu_0(\{\psi(.,0)\neq 0\}) - 2\sqrt{bc}[\mu(\{\psi\neq 0\})]^{1/2} - a\mu(\{\psi\neq 0\}), \tag{6.1}$$

with

$$a := 2\omega + \frac{1}{T} + 2,$$
$$b := 4 + 2^{d+2},$$
$$c := \|u\|_{\mathrm{BV}}\left[2\left(2\omega+\frac{1}{T}\right) + V_{[U_m,U_M]}(8+2^{d+5})\right] + \|u_0\|_{\mathrm{BV}}\big[2^{d+4}V_{[U_m,U_M]}+1\big] + 2V_{[U_m,U_M]}\max\{U_m,U_M\}[|B_{R+1}(0)|-|B_R(0)|]\,T.$$

REMARK 6.2. *Note that due to Theorem* 3 *in* [5] *we have under Assumption* 2.1

$$\|u\|_{BV} := \|u\|_{BV(D_{R+1}(x_0))} \leq C(F,u_0,R,T).$$

In the special case where $F(x,t,s)=f(s)$, *one has the estimate* [10, *Thm.* 2.3.1]

$$\|u(.,t)\|_{BV(\mathbb{R}^d)} \leq \|u_0\|_{BV(\mathbb{R}^d)} \quad \forall t>0.$$

In the next few lemmas we estimate the measures which are defined in Theorem 5.2. Putting all these results together then gives the assertion of our a posteriori error estimate in Theorem 3.3.

LEMMA 6.3. *Let* μ_{h1} *be defined in* (5.6). *Then*

$$\mu_{h1}(D_{R+1}(x_0)) \leq \frac{1}{2}\sum_{n=0}^{N_0}\sum_{i\in I_D^{n+1}} h_i^{n+1}|T_i^{n+1}| \sum_{j,l\in\tilde{K}^{n,n+1}(i)} \tilde{r}_{ij}^{n,n+1}\tilde{r}_{il}^{n,n+1}|u_j^n - u_l^n|.$$

Proof. Let $\chi_{D_{R+1}(x_0)}$ denote the characteristic function of the set $D_{R+1}(x_0)$. By definition of μ_{h1} we have

$$\mu_{h1}(D_{R+1}(x_0)) = \frac{1}{2}\sum_{n=0}^{N_0}\sum_{i\in I_D^{n+1}}\sum_{j,l\in\tilde{K}^{n,n+1}(i)}|u_j^n-u_l^n| \times \frac{h_i^{n+1}}{|T_i^{n+1}|}\int_{T_l^n\cap T_i^{n+1}}\int_{T_j^n\cap T_i^{n+1}}\int_0^1 \chi_{D_{R+1}(x_0)}(x+\theta(y-x),t^{n+1})\mathrm{d}\theta\mathrm{d}y\mathrm{d}x.$$

For $n \leq N_0$ and $i \in I_D^{n+1}$ we have that $T_l^n \cap T_i^{n+1}, T_j^n \cap T_i^{n+1} \subset T_i^{n+1}$ for $j,l \in \tilde{K}^{n,n+1}(i)$. Therefore, we estimate $\chi_{D_{R+1}}(x+\theta(y-x),t^{n+1}) \leq 1$ which gives the assertion. ∎

LEMMA 6.4. *Let μ_{h2} be defined in* (5.7). *Then*

$$\mu_{h2}(D_{R+1}(x_0)) \le \sum_{n=0}^{N_0-1} k^n \sum_{i\in I_D^{n+1}} |T_i^{n+1}| \left| u_i^{n+1} - \sum_{j\in\tilde{K}^{n,n+1}(i)} \tilde{r}_{ij}^{n,n+1} u_j^n \right|.$$

The proof is obvious and therefore omitted.

LEMMA 6.5. *Let μ_{h3} be defined in* (5.8). *Then*

$$\mu_{h3}(D_{R+1}(x_0)) \le 2V_{[U_m,U_M]} \sum_{n=0}^{N_0} k^n \sum_{i\in I_D^{n+1}} h_i^{n+1} \sum_{(j,l)\in\tilde{\mathcal{E}}^{n,n+1}(i)} |\tilde{S}_{jl}^{n,n+1}(i)|\, |u_j^n - u_l^n|.$$

Proof. Note that analogously to the proof of Lemma 6.3 we have

$$\frac{h_i^{n+1}}{k^n\, |T_i^{n+1}|\, |T_j^{n,n+1}|} \int_{t^n}^{t^{n+1}} \int_{T_i^{n+1}} \int_{T_j^{n,n+1}} \int_0^1 \chi_{D_{R+1}(x_0)}(x+\theta(y-x),t)\mathrm{d}\theta\mathrm{d}x\mathrm{d}y\mathrm{d}t \le h_i^{n+1}.$$

If indices $j,l \in \mathcal{E}^{n,n+1}$ correspond to an edge which lies on ∂T_i^{n+1}, then by construction (see (2.15))

$$v_j^n = v_l^n,$$

and hence the flux difference vanishes. Therefore, only flux differences across interior edges of elements T_i^{n+1} remain which are estimated using the Lipschitz property of the numerical fluxes; see Assumption 2.5 and Remark 4.6. Hence, we have

$$\mu_{h3}(D_{R+1}(x_0)) \le 2V_{[U_m,U_M]} \sum_{n=0}^{N_0} k^n \sum_{i\in I_D^{n+1}} h_i^{n+1} \sum_{(j,l)\in\mathcal{E}^{0,n,n+1}(i)} |S_{jl}^{n,n+1}|\, |v_j^n - v_l^n|,$$

where

$$\mathcal{E}^{0,n,n+1}(i) := \{(j,l)\in\mathcal{E}^{n,n+1} |\, T_j^{n,n+1} \subset T_i^{n+1} \quad \text{and} \quad T_l^{n,n+1} \subset T_i^{n+1}\}$$

denotes the set of edges in $\mathcal{T}_h^{n,n+1}$ which lie in the interior of T_i^{n+1}. Using the identification of indices due to the prolongation step (see (2.15)) and Notation 3.2 we get the assertion. ∎

Using similar arguments as before one can prove the following two lemmas.

LEMMA 6.6. *Let $\mu_{jl}^{n,n+1}$ and $\lambda_{jl}^{n,n+1}$ be defined in* (5.10) *and* (5.11), *respectively. Then*

$$\mu_{jl}^{n,n+1}(D_{R+1}(x_0)) \le h_j^{n,n+1} + k^n,$$
$$\lambda_{jl}^{n,n+1}(D_{R+1}(x_0)) \le k^n |S_{jl}^{n,n+1}| \left[\mathrm{diam}(S_{jl}^{n,n+1}) + k^n\right]^2.$$

LEMMA 6.7. *Let μ_{h4} be defined in* (5.9). *Then*

$$\begin{aligned}
&\mu_{h4}(D_{R+1}(x_0)) \\
&\le 4V_{[U_m,U_M]} \sum_{n=0}^{N_0} k^n \sum_{i\in I_D^{n+1}} (h_i^{n+1} + k^n) \sum_{(j,l)\in\tilde{\mathcal{E}}^{n,n+1}(i)} |\tilde{S}_{jl}^{n,n+1}(i)|\, |u_j^n - u_l^n| \\
&\qquad + C_{F,R,T,u_0} \sum_{n=0}^{N_0} \sum_{(j,l)\in\mathcal{E}_D^{n,n+1}} k^n |S_{jl}^{n,n+1}| \left[\mathrm{diam}(S_{jl}^{n,n+1}) + k^n\right]^2,
\end{aligned}$$

where C_{F,R,T,u_0} is defined in (5.12).

7. Proof of the a priori error estimate. The proof of the a priori error estimate in Theorem 3.5 follows from the a posteriori error estimate in Theorem 3.3 by estimating the term Q. In order to show this we manipulate the various parts in such a way that we can apply the BV-regularity results given in section 4; see Propositions 4.4 and 4.5.

Proof. We have to estimate $Q = Q_1 + Q_2 + Q_3 + Q_4$, where

$$Q_1 = \frac{1}{2} \sum_{n=0}^{N_0} \sum_{i \in I_D^{n+1}} h_i^{n+1} |T_i^{n+1}| \sum_{j,l \in \tilde{K}^{n,n+1}(i)} \tilde{r}_{ij}^{n,n+1} \tilde{r}_{il}^{n,n+1} |u_j^n - u_l^n|,$$

$$Q_2 = \sum_{n=0}^{N_0-1} k^n \sum_{i \in I_D^{n+1}} |T_i^{n+1}| \left| u_i^{n+1} - \sum_{j \in \tilde{K}^{n,n+1}(i)} \tilde{r}_{ij}^{n,n+1} u_j^n \right|,$$

$$Q_3 = 6V_{[U_m,U_M]} \sum_{n=0}^{N_0} k^n \sum_{i \in I_D^{n+1}} (h_i^{n+1} + k^n) \sum_{(j,l) \in \tilde{\mathcal{E}}^{n,n+1}(i)} |\tilde{S}_{jl}^{n,n+1}(i)| \, |u_j^n - u_l^n|,$$

$$Q_4 = C_{F,R,T,u_0} \sum_{n=0}^{N_0} \sum_{(j,l) \in \mathcal{E}_D^{n,n+1}} k^n |S_{jl}^{n,n+1}| \left[\operatorname{diam}(S_{jl}^{n,n+1}) + k^n \right]^2.$$

Now, we need geometrical estimates to apply the stability estimates of section 4.

Using the mesh regularity (Assumptions 3.1 and 2.5) and the overlap condition (Assumption 3.1) we get

$$h_i^{n+1} \le (C_{\mathrm{ov}} \alpha)^{-1/d} h_j^{n,n+1} \quad \forall i \in I^{n+1} \;\; \forall j \in J^{n,n+1}(i),$$
$$|S_{jl}^{n,n+1}| \le |\partial T_j^{n,n+1}| \le \frac{1}{\alpha} (h_j^{n,n+1})^{d-1} \quad \forall (j,l) \in \mathcal{E}_D^{n,n+1}.$$

Using these estimates together with the CFL-condition and the inverse CFL-condition (see Assumption 3.1) we get the following estimates:

$$Q_1 \le (C_{\mathrm{ov}} \alpha)^{-1/d} \frac{V_{[U_m,U_M]}}{2\alpha^2 \eta} \sum_{n=0}^{N_0} k^n \sum_{i \in I_D^{n+1}} |T_i^{n+1}| \sum_{j,l \in \tilde{K}^{n,n+1}(i)} |u_j^n - u_l^n|,$$

$$Q_3 \le 6V_{[U_m,U_M]} \frac{1}{\alpha^2} \left[(C_{\mathrm{ov}} \alpha)^{-1/d} + \frac{1}{2}(1-\xi)\alpha^2 \frac{1}{V_{[U_m,U_M]}} \right]$$
$$\times \sum_{n=0}^{N_0} k^n \sum_{i \in I_D^{n+1}} |T_i^{n+1}| \sum_{j,l \in \tilde{K}^{n,n+1}(i)} |u_j^n - u_l^n|,$$

$$Q_4 \le C_{F,R,T,u_0} \left[1 + \frac{1}{2}(1-\xi) \frac{\alpha^2}{V_{[U_m,U_M]}} \right]^2 \frac{1}{\alpha^2} |D_{R+1}(x_0)| \, h.$$

Further estimating Q_1 and Q_3 by Proposition 4.4 and Q_2 by Proposition 4.5 and applying Theorem 3.3 concludes the proof. ∎

Note that the constant C_e in Theorem 3.5 can be specified explicitly.

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