

# Non-oscillatory Lax-Friedrichs Type Central Finite Volume Methods for 3-D Flows on Unstructured Tetrahedral Grids

P. Arminjon, A. Madrane, and A. St-Cyr

**Abstract.** We present a 3D finite volume generalization of the 1-dimensional Lax-Friedrichs and Nessyahu-Tadmor schemes for hyperbolic equations on unstructured tetrahedral grids. The non-oscillatory central difference scheme of Nessyahu and Tadmor, in which the resolution of the Riemann problem at the cell interfaces is by-passed thanks to the use of the staggered Lax-Friedrichs scheme, is extended here to a two-step, three-dimensional non-oscillatory centered scheme in finite volume formulation. Piecewise linear cell interpolants using Venkatakrishnan's limiter combined with diverse techniques to estimate the gradients lead to a non-oscillatory spatial resolution of order 2. The fact that the expected second order resolution is not fully attained in 3D for nonlinear systems might be caused by the absence of grid adaptation in our calculations. Numerical results for a linear advection problem with continuous initial conditions in 3D show the accuracy and stability of the method. We also include results for the 3D Euler system (shock tube problem)

## 1. Introduction

In earlier papers [5, 6] we presented a two-dimensional finite volume method generalizing the one-dimensional Lax-Friedrichs [12] and Nessyahu-Tadmor [16] difference schemes for hyperbolic conservation laws to unstructured triangular grids, while in [4, 2] we have also constructed a corresponding extension for 2-dimensional Cartesian grids. Other recent contributions in the case of Cartesian grids can be found in [10, 11, 13, 14].

All these extensions to 2-dimensional problems lead to non-oscillatory, essentially second order accurate finite volume methods.

In [2, 3] we presented an extension to staggered structured three-dimensional Cartesian grids of our two-dimensional finite volume method for Cartesian grids. We proved that the first order three-dimensional finite volume extension of the Lax-Friedrichs scheme is monotone under an appropriate CFL condition. Some difficulties encountered in the numerical experiments led us to consider a special

type of limiting function proposed by Venkatakrisnan. The method still displayed too much dissipation and requires some adjustments.

In this paper, we first present a finite volume extension of the first order accurate Lax-Friedrichs scheme to three-dimensional tetrahedral grids. We then construct the extension of our two-dimensional finite volume method to a quasi-second order accurate non-oscillatory finite volume method of "central type", for staggered unstructured tetrahedral grids, which can be viewed as a finite volume generalization of the 1-dimensional Nessyahu-Tadmor difference scheme to 3-dimensional unstructured tetrahedral grids.

## 2. Mathematical modelling

### 2.1. Governing equations

Let  $\Omega \subset R^3$  be the domain of interest of the flow and  $\Gamma$  be its boundary, we write  $\Gamma = \Gamma_B \cup \Gamma_\infty \cup \Gamma_E$ , where  $\Gamma_B$  denotes that part of the body boundary which is relevant for the computational domain.  $\Gamma_\infty$  is the (upwind) farfield boundary, and  $\Gamma_E$  is the (downwind) exit part of the boundary. The equations describing 3-dimensional compressible inviscid flows are the Euler equations, written here in their conservation form, given by

$$\frac{\partial U}{\partial t} + \vec{\nabla} \cdot \vec{\mathcal{F}}(U) = 0$$

where  $U = (\rho, \rho u, \rho v, \rho w, E)^T$ ,  $\vec{\mathcal{F}}(U) = (F(U), G(U), H(U))^T$ , (1)

$\vec{\mathcal{F}}(U)$  denotes the convective flux,  $\rho$  is the density,  $\vec{V} = (u, v, w)$  is the velocity vector,  $E = \rho e = \rho e + \frac{1}{2}\rho(u^2 + v^2 + w^2)$  is the total energy per unit volume;  $p$  is the pressure of the fluid. Let  $A, B$  and  $C$  denote the Jacobian matrices  $\partial F(U)/\partial U$ ,  $\partial G(U)/\partial U$  and  $\partial H(U)/\partial U$ , respectively. Eq.(1) can be written in the nonconservative form:

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} + B(U)\frac{\partial U}{\partial y} + C(U)\frac{\partial U}{\partial z} = 0 \quad (2)$$

## 3. Discretization with respect to space and time

### 3.1. Definitions, description of the first-order method

We assume that  $\Omega$  is a polyhedral bounded domain of  $R^3$ , and start from an arbitrary FEM tetrahedral grid  $\mathcal{T}_h$ , where  $h$  is the maximal length of the edges in  $\mathcal{T}_h$ . We need the following notations:  $T_{ij}$  denotes the set of all tetrahedra which share edge  $[i, j]$  as a common edge. Let  $M$  denote the set of indices of all sub-tetrahedra of the original tetrahedral grid, defined by the median planes such as:  $iM_1GG_1, iM_1GG_3, jM_1GG_1$  and  $jM_1GG_3$  on fig. 1. Let  $m(i, j, \tau, l)$  be one such index contained in  $M$ , where  $\tau \in T_{ij}$  and  $l$  ( $1 \leq l \leq 4$ ) is the index of one of the four

sub-tetrahedra sharing edge  $[i, j]$  and contained in tetrahedron  $\tau$ . If  $l = 1, 3$  then the index denotes sub-tetrahedra having node  $a_i$  in common, and, if  $l = 2, 4$  they share node  $a_j$ . Any such sub-tetrahedron will be noted  $l_k$  where  $k \in M$ .  $K(i)$  is the set of nodes (vertices) which are neighbours of node  $a_i$ . Let  $nv$  be the number of vertices of  $\tau \in \mathcal{T}_h$ . For every vertex  $a_i$ ,  $1 \leq i \leq nv$  (written  $i$  for simplicity on fig. 1) of an arbitrary tetrahedron  $\tau \in \mathcal{T}_h$ , we define the (vertex centered) barycentric cell  $C_i$  as follows (this approach was introduced in [1]): every tetrahedron of  $\mathcal{T}_h$  is subdivided in 24 sub-tetrahedra  $l_k$  by the median planes. The cell  $C_i$  is the union of the resulting sub-tetrahedra sharing  $a_i$  as a vertex  $C_i = \bigcup_{j \in K(i)} \bigcup_{\tau \in \mathcal{T}_{ij}} l_{m(i,j,\tau,\{1,3\})}$ . The faces of interest associated with each sub-tetrahedron  $l_k$  ( $k \in M$ ) are, first, those which do not share any edge midpoint (such as  $M_1$  in fig. 1) noted  $\partial l_k^1$  (such as  $jGG_1$  or  $jGG_3$  on fig. 1), and secondly, the faces which do not contain vertex  $a_i$  in their composition, noted  $\partial l_k^2$ . The outward normals to those faces will be noted  $n_k$  for  $\partial l_k^1$  and  $\nu_k$  for  $\partial l_k^2$  (such as, on fig. 1,  $M_1GG_1$  or  $M_1GG_3$ ). Note that we use normal vectors such that  $\|n_k\| = \text{Area}(\partial l_k^1)$  and  $\|\nu_k\| = \text{Area}(\partial l_k^2)$ . The boundary  $\partial C_i$  of  $C_i$  can be constructed with  $\partial l_k^2$  ( $\partial C_i = \bigcup_{j \in K(i)} \bigcup_{\tau \in \mathcal{T}_{ij}} \partial l_{m(i,j,\tau,\{1,3\})}^2$ ). As was the case in our 2-dimensional extensions, our 3-D extension also uses a

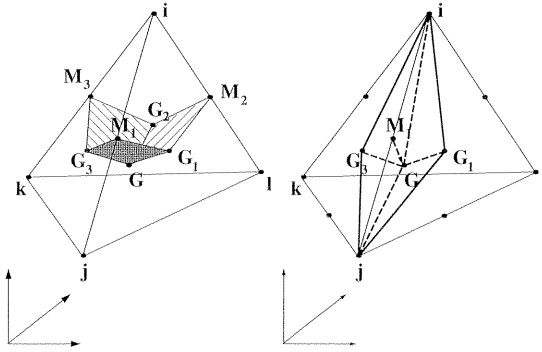


FIGURE 1. Barycentric cell  $C_i$  and diamond cell  $L_{ij}$

dual grid, with dual cells  $L_{ij}$  associated with the edges of  $\mathcal{T}_h$ . The dual ("diamond") cell  $L_{ij}$  is composed by the sub-tetrahedra (defined above) sharing edge  $[i, j]$  ( $L_{ij} = \bigcup_{\tau \in \mathcal{T}_{ij}} l_{m(i,j,\tau,\{1,2,3,4\})}$ ). Owing to the fact that  $\partial l_k^1$  denotes one of the facets of  $\partial L_{ij}$ , we can formulate the dependence of  $\partial L_{ij}$  on the selected faces (i.e. with superscript index 1) of the sub-tetrahedra which compose  $L_{ij}$  :

$$\partial L_{ij} = \bigcup_{\tau \in \mathcal{T}_{ij}} \partial l_{m(i,j,\tau,\{1,2,3,4\})}^1 = \bigcup_{\tau \in \mathcal{T}_{ij}} \partial l_{m(i,j,\tau,\{1,3\})}^1 \cup \partial l_{m(i,j,\tau,\{2,4\})}^1 \quad (3)$$

Let  $i, j, k, l$  be the four nodes defining a tetrahedron  $\tau$  sharing node  $a_i$ , then the cell boundary element of cell  $C_i$ , associated with  $\tau$  and edge  $[i, j]$  is

$$\partial C_i \cap \tau \cap L_{ij} = \text{Triangle}(M_1GG_1) \cup \text{Triangle}(M_1GG_3) = \partial l_{m(i,j,\tau,\{1,3\})}^2$$

$$= \partial l_{m(i,j,\tau,1)}^2 \cup \partial l_{m(i,j,\tau,3)}^2 \quad (4)$$

Let  $m_{ij}$  denote the midpoint of edge  $[i, j]$ , also written as  $M_1$  above, and let  $U_i^n \cong U(a_i, t^n)$  and  $U_{ij}^{n+1} \cong U(m_{ij}, t^{n+1})$  denote the nodal (resp. cell average) values in the first and second grid at time  $t = t^n$  and  $t = t^{n+1}$ , respectively ( $n$  even).

The union of all the barycentric cells constitutes a partition of the computational domain  $\Omega_h$  and the same holds for diamond cells:

$$\Omega_h = \bigcup_{i=1}^{nv} C_i, \quad \Omega_h = \bigcup_{k=1}^{ne} L_k$$

where  $nv$  and  $ne$  are the number of vertices and number of edges, respectively, of the original finite element triangulation  $\mathcal{T}_h$ . We now define the two steps of our first order accurate (staggered, Lax-Friedrichs type) finite volume method:

**First time step:**

In the first (and further odd) time step of our 3-dimensional finite volume scheme, we start from initial (or previously obtained) cell average values  $\vec{U}_i$  for the barycentric cells of the first grid, and compute cell average values  $\vec{U}_{ij}$  for the (staggered) dual cells of the second grid. This is done by integrating (1) on an extended control volume  $L_{ij} \times [t^n, t^{n+1}]$ , where  $L_{ij}$  is a (diamond) cell of the dual grid, and by assuming that  $\vec{U}_{L_{ij}}(t^n + \Delta t)$  is piecewise constant on the cells of the second grid. Using (1) and applying the divergence theorem to the flux term, we obtain

$$\int_{t^n}^{t^{n+1}} \int_{L_{ij}} U_i d\vec{x} dt + \int_{t^n}^{t^{n+1}} \int_{\partial L_{ij}} \vec{\mathcal{F}}(U) \cdot \vec{n} dAdt = 0 \quad (5)$$

Let  $m(1, 3)$  be equal to  $m(i, j, \tau, \{1, 3\})$  and  $m(2, 4) = m(i, j, \tau, \{2, 4\})$  for simplicity, (where  $m(1, 3)$  is in fact dependent on  $i, j, \tau$ ); we obtain

$$\begin{aligned} U_{ij}^{n+1} &= \frac{1}{2}(U_i^n + U_j^n) - \frac{\Delta t}{Vol(L_{ij})} \sum_{\tau \in \mathcal{T}_{ij}} \{ \vec{\mathcal{F}}(U_i^n) \cdot \vec{n}_{m(1,3)} + \vec{\mathcal{F}}(U_j^n) \cdot \vec{n}_{m(2,4)} \} \\ &\quad - \frac{\Delta t}{Vol(L_{ij})} \int_{\partial L_{ij} \cap \Gamma_B} \vec{\mathcal{F}}(U(\vec{x}^n)) \cdot \vec{n} dA - \frac{\Delta t}{Vol(L_{ij})} \int_{\partial L_{ij} \cap \Gamma_\infty} \vec{\mathcal{F}}(U(\vec{x}^n)) \cdot \vec{n} dA \end{aligned} \quad (6)$$

**Second time step:** Starting from the known diamond cell value  $U_{ij}^{n+1}$ , we compute the barycentric cell value  $U_i^{n+2}$  ( $1 \leq i \leq nv$ ) by integrating (1) on an extended control volume  $C_i \times [t^{n+1}, t^{n+2}]$ , applying the divergence theorem to the flux term, and observing that

$$\partial C_i = \bigcup_{j \in K(i)} \{ \partial C_i \cap \partial C_j \} \cup \{ \partial C_i \cap \Gamma_B \} \cup \{ \partial C_i \cap \Gamma_\infty \}.$$

We obtain

$$\begin{aligned}
 U_i^{n+2} = & \sum_{j \in K(i)} \frac{Vol(C_i \cap L_{ij})}{Vol(C_i)} U_{ij}^{n+1} - \frac{\Delta t}{Vol(C_i)} \sum_{j \in K(i)} \sum_{\tau \in T_{ij}} \vec{\mathcal{F}}(U_i^{n+1}) \cdot \vec{\nu}_{m(1,3)} \\
 & - \frac{\Delta t}{Vol(C_i)} \int_{\partial C_i \cap \Gamma_B} \vec{\mathcal{F}}(U^{n+1}) \cdot \vec{\nu} dA - \frac{\Delta t}{Vol(C_i)} \int_{\partial C_i \cap \Gamma_\infty} \vec{\mathcal{F}}(U^{n+1}) \cdot \vec{\nu} dA \quad (7)
 \end{aligned}$$

### 3.2. High-order accurate approximations

To obtain second-order accuracy, we introduce cellwise piecewise linear interpolation (MUSCL,[17]), the derivatives of the  $P_1$  (Galerkin) interpolation [8], Least Squares [6], and the Gauss-Green formula [7]. **First step:** We integrate (1) on an extended control volume  $L_{ij} \times [t^n, t^{n+1}]$ , assuming we have obtained from the cell average values  $U_i^n$  piecewise linear reconstructions given by

$$\begin{aligned}
 U_h(x, y, z, t^n)|_{C_i} = & \mathcal{L}_i(x, y, z, t^n) \\
 = & U_i^n + (x - x_i)P_i^n + (y - y_i)Q_i^n + (z - z_i)R_i^n \quad (x, y, z) \in C_i \quad (8)
 \end{aligned}$$

For the integration with respect to time, in order to ensure "nearly" second-order accuracy, we adopt a "quasi-midpoint formula" time discretization, where the convective flux is computed at the intermediate time  $t^{n+\frac{1}{2}}$  ( thus requiring the computation of **predicted** values  $U_h(x, y, t^{n+\frac{1}{2}})$  along  $\partial L_{ij}$  ), in the integration with respect to time.

#### Predictor (First step):

On each face of the cell  $L_{ij}$ , using the Euler equations, we define a predicted vector

$$U_{a_i, G_1, G}^{n+1/2} = U_{a_i, G_1, G}^n - \frac{\Delta t}{2} \{ F'(U_{a_i, G_1, G}^n) P_i^n + G'(U_{a_i, G_1, G}^n) Q_i^n + H'(U_{a_i, G_1, G}^n) R_i^n \} \quad (9)$$

where, using (8), the value of  $U_h^n$  along the face  $a_i, G_1, G$  of diamond cell  $L_{ij}$  is taken equal to

$$\begin{aligned}
 U_h(x, y, z, t^n) \cong & U_i^n + \frac{1}{2}(x_{a_i, G_1, G} - x_i)P_i^n + \frac{1}{2}(y_{a_i, G_1, G} - y_i)Q_i^n \\
 & + \frac{1}{2}(z_{a_i, G_1, G} - z_i)R_i^n \equiv U_{a_i, G_1, G}^n \quad (10)
 \end{aligned}$$

The corrector can now be written, using (9), as

#### Corrector (First step)

$$\begin{aligned}
 Vol(L_{ij})U_{ij}^{n+1} = & \left\{ \int_{L_{ij} \cap C_i} \mathcal{L}(x, y, z, t^n) d\vec{x} + \int_{L_{ij} \cap C_j} \mathcal{L}(x, y, z, t^n) d\vec{x} \right\} \\
 & + \Delta t \sum_{\tau \in T_{ij}} \left\{ \int_{\partial l_{m(i,j,\tau,\{1,3\})}^1} + \int_{\partial l_{m(i,j,\tau,\{2,4\})}^1} \right\} \vec{\mathcal{F}}(U_h^{n+\frac{1}{2}}) \cdot \vec{n} dA
 \end{aligned}$$

$$+ \Delta t \int_{\partial L_{ij} \cap \Gamma_B} \vec{\mathcal{F}}(U_h^n) \cdot \vec{n}_{ij} dA + \Delta t \int_{\partial L_{ij} \cap \Gamma_\infty} \vec{\mathcal{F}}(U_h^n) \cdot \vec{n}_{ij} dA = 0 \quad \text{where} \quad (11)$$

$$\int_{L_{ij} \cap C_i} \mathcal{L}(x, y, z, t^n) d\vec{x} = \sum_{\tau \in T_{ij}} \text{Vol}(l_{m(1,3)}) \mathcal{L}(x_{\mathcal{G}_{m(1,3)}}, y_{\mathcal{G}_{m(1,3)}}, z_{\mathcal{G}_{m(1,3)}}, t^n) \quad (12)$$

where  $\mathcal{G}_{m(1,3)}$  is the centroid of sub-tetrahedron  $l_{m(1,3)}$ . Note that from the coding viewpoint there is a simplification since the two sub-tetrahedra (1, 3) also form a convex polyhedron the barycenter of which can be calculated, and the last integral can be simplified.

### Second step

To obtain the second step of the time discretization, we integrate (1) on the cell  $C_i \times [t^{n+1}, t^{n+2}]$ , assuming we have obtained, from the diamond cell average values  $U_{ij}^{n+1}$  computed in the first time step, piecewise linear reconstructions, given by

$$\begin{aligned} U_h(x, y, z, t^{n+1})|_{L_{ij}} &= \mathcal{L}_{ij}(x, y, z, t^{n+1}) \\ &= U_{ij}^{n+1} + (x - x_{ij})P_{ij}^{n+1} + (y - y_{ij})Q_{ij}^{n+1} + (z - z_{ij})R_{ij}^{n+1} \end{aligned} \quad (13)$$

Proceeding as for the first step, we obtain

### Predictor (second step)

$$\begin{aligned} U_{G, G_1, M_1, G_3}^{n+\frac{3}{2}} &= U_{G, G_1, M_1, G_3}^{n+1} \\ &- \frac{\Delta t}{2} \{ F'(U_{G, G_1, M_1, G_3}^{n+1}) P_{ij}^{n+1} + G'(U_{G, G_1, M_1, G_3}^{n+1}) Q_{ij}^{n+1} + H'(U_{G, G_1, M_1, G_3}^{n+1}) R_{ij}^{n+1} \} \end{aligned} \quad (14)$$

where

$$\begin{aligned} U_h(x, y, z, t^{n+1}) &\cong U_{ij}^{n+1} \\ &+ \frac{1}{2}(x_G - x_{M_1})P_{ij}^{n+1} + \frac{1}{2}(y_G - y_{M_1})Q_{ij}^{n+1} + \frac{1}{2}(z_G - z_{M_1})R_{ij}^{n+1} \equiv U_{G, G_1, M_1, G_3}^{n+1} \end{aligned} \quad (15)$$

defines an approximation of the value of  $U$  on the boundary element  $[G, G_1, M_1, G_3]$  of cell  $C_i$ .

### Corrector (second step)

$$\begin{aligned} &\text{Vol}(C_i)U_i^{n+2} - \sum_{j \in K(i)} \int_{C_i \cap L_{ij}} \mathcal{L}_{ij}(x, y, z, t^{n+1}) d\vec{x} \\ &+ \Delta t \sum_{j \in K(i)} \sum_{\tau \in T_{ij}} \int_{\partial l_{m(i,j,\tau),(1,3)}^2} \vec{\mathcal{F}}(U(x, y, z, t^{n+3/2})) \cdot \vec{\nu} dA \\ &+ \Delta t \int_{\partial C_i \cap \Gamma_B} \vec{\mathcal{F}}(U_h^{n+1}) \cdot \vec{\nu} dA + \Delta t \int_{\partial C_i \cap \Gamma_\infty} \vec{\mathcal{F}}(U_h^{n+1}) \cdot \vec{\nu} dA = 0 \end{aligned} \quad (16)$$

where

$$\begin{aligned} \int_{C_i \cap L_{ij}} \mathcal{L}_{ij}(x, y, z, t^{n+1}) d\vec{x} = \\ \sum_{j \in K(i)} \sum_{\tau \in T_{ij}} \{Vol(l_{m(i,j,\tau,1,3)}) \mathcal{L}(x_{\mathcal{G}_m(i,j,\tau,1,3)}, y_{\mathcal{G}_m(i,j,\tau,1,3)}, z_{\mathcal{G}_m(i,j,\tau,1,3)}, t^{n+1})\} \end{aligned} \quad (17)$$

is computed according to 12 with the respective barycenters of the sub-tetrahedra (again, a simplification is possible).

**3.2.1. APPROXIMATION OF THE SLOPES AND LIMITATION** In order to compute the gradient  $(P_i^n, Q_i^n, R_i^n)$  of the piecewise linear interpolant  $L(x, y, z, t^n)$  for the cell  $C_i$ , we use the  $P_1$  (Galerkin) interpolation [8], Least Squares [6] and Gauss-Green [7]. For the limitation we use several procedures, see [17]. Numerical experiments have led us to choose the Gauss-Green formula for the gradients used in the reconstruction for the cells  $C_i$  and a least squares weighted procedure for the cells  $L_{ij}$ .

**3.2.2. STABILITY** We refer to [15] for a stability study of linear multidimensional advection models that is also valid in the 3-D case. For the second time step:

$$\begin{aligned} \Delta t_i = \frac{Vol(C_i)}{\lambda_{max}^i \int_{\partial C_i} dA} \quad \text{where} \quad \lambda_{max}^i = \max(\lambda_i, \max_{j \text{ neighbour of } i} \lambda_j) \\ \text{and} \quad \lambda_i = \|\vec{V}_i\| + c_i \end{aligned}$$

$\vec{V}_i, c_i$  refer to the values in the cell  $C_i$  of the velocity vector and sound speed, respectively. We then choose the minimum  $\Delta t_i$  for all indices  $i$  ( $1 \leq i \leq nv$ )  $\Delta t = \min_{1 \leq i \leq nv} \{\Delta t_i\}$  with a similar time step definition for the first (odd) step ( $t^n \rightarrow t^{n+1}$ ).

## 4. Numerical experiments

### 4.1. Linear advection problem

First, to confirm the spatial accuracy of the scheme, a simple advection test is performed on a smooth initial distribution. The distribution is advected near the boundary of the cube  $[0, 1]^3$  and compared with the exact solution of the following problem:

$$u_t + u_x + u_y + u_z = 0 \quad (18)$$

$$u(x, y, z, t = 0) = f(r^2) = \begin{cases} \cos(4\pi r^2) & \text{if } r^2 \leq 1/8 \\ 0 & \text{otherwise} \end{cases}$$

where  $r^2 = (x - 1/2)^2 + (y - 1/2)^2 + (z - 1/2)^2$ . Table 1 shows the  $L_1$ -error and ac-

Diameter	Nodes	$L^1$ -error	Order
0.134057	4095	0.0487568	
0.123977	5733	0.0416913	2.0027

TABLE 1.  $L^1$  - error and accuracy order, continuous problem

curacy order; the limiting function used was the one proposed by Venkatakrishnan [18].

#### 4.2. Shock tube problem

To illustrate the accuracy of our scheme, we present numerical results for a 3-D extension of the shock tube problem introduced by Sod . In this problem, an initial discontinuity in the thermodynamical state of the gas breaks into a shock wave followed by a contact discontinuity and a rarefaction wave. Finite element meshes with 909,1809 and 3609 nodes are used and the initial conditions at  $t = 0$  are specified by the data

$$\begin{cases} \rho = 1; u = 0; v = 0; w = 0; p = 1.0 & x \in [0, 1/2] \\ \rho = 0.125; u = 0; v = 0; w = 0; p = 0.1 & x \in [1/2, 1] \end{cases} \quad (19)$$

The profiles of density, velocity, pressure and temperature are compared with the analytical solution at  $t = 0.16$ . The numerical solution of the 3-dimensional shock

h	Nodes	$L^2 \rho$	Order	$L^2 V_z$	Order	$L^2 P$	Order
0.01	909	2.898e-05		9.982e-05		2.778e-05	
0.005	1809	1.555e-05	0.898	6.399e-05	0.641	1.196e-05	1.216

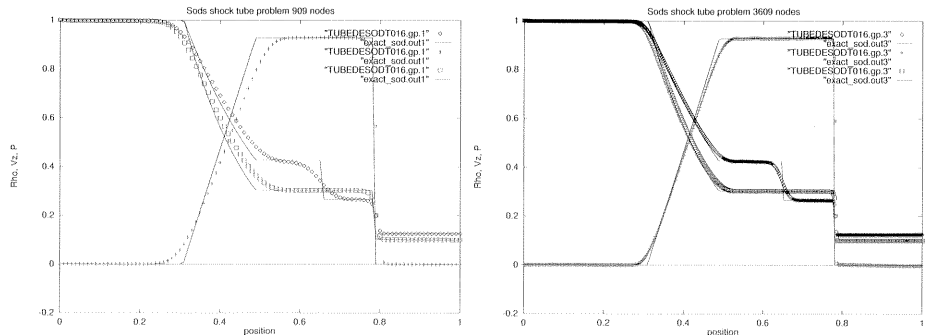
TABLE 2.  $L^2$  - error and accuracy order, Sod's problem

FIGURE 2. Sod's shock tube problem with 909 and 3609 gridpoints

tube problem is clearly much more likely to suffer from excessive numerical dissipation than its 1-dimensional analogue see e. g. [9]. In our case, see table 2 and Fig.



2, this dissipation might be due to the fact that the tetrahedra are fairly stretched (the aspect ratio is more than 1 : 25). This creates an additional difficulty for the capture of the shock and particularly of the contact discontinuity. Mesh adaptation, which was not available to us, would certainly bring a decisive improvement, as was observed in our 2-dimensional experiments [6].

## 5. Conclusion

In this paper we have presented a new construction of three-dimensional finite volume methods for inviscid flows using staggered unstructured tetrahedral grids. The first and second-order methods can be considered as finite volume generalizations of the Lax-Friedrichs and Nessyahu-Tadmor 1-dimensional difference schemes, respectively. Actual calculations have been restricted to the higher-order method, as the first-order Lax-Friedrichs-type method, in three dimensions, proved to be even more dissipative than in one dimension. Nevertheless the quality of the results does not fully meet original expectations, based on the very high quality obtained with our method for 2-dimensional flows [6], where we were using grid adaptation. We could probably attribute the excessive dissipation in part to the grids, which have a relatively small number of nodes, a high aspect ratio, and have not benefited from the improvements brought about by grid adaptation. We are currently working on this problem.

## References

- [1] ANGRAND F., BOULARD V., DERVIEUX A., PERIAUX J., VIJAYASUNDARAM G., (1984), *Triangular finite element methods for the Euler equations*, 6th Int. Conf. on Comp. Meth. in Appl. Sciences and Engineering, Glowinski R., Lions J.L., Eds., North Holland .
- [2] P. ARMINJON, A. MADRANE AND A. ST-CYR (1999), *New Lax-Friedrichs-type finite volume schemes on 2 and 3D Cartesian staggered grids*, 7th annual conference of the CFD Society of Canada, May 30 to June 1, 1999, Halifax, J. Militzer, editor, pp.(3-3)-(3-10).
- [3] P. ARMINJON, A. MADRANE, A. ST-CYR, *New 2 and 3-dimensional non-oscillatory central finite volume methods for staggered Cartesian grids*, to appear in Applied Numer. Math.
- [4] P. ARMINJON, D. STANESCU AND M. C. VIALON, (1995), *A two-dimensional finite volume extension of the Lax-Friedrichs and Nessyahu-Tadmor schemes for compressible flows*, Proc.of the 6th. Int. Symp. on Comp. Fluid Dynamics, Lake Tahoe (Nevada) September 4-8, 1995, M.Hafez and K. Oshima, editors, Vol. IV, pp. 7-14.
- [5] P. ARMINJON AND M. C. VIALON, (1995), *Généralisation du schéma de Nessayahu-Tadmor pour une équation hyperbolique à deux dimensions d'espace*, C.R. Acad. Sci. Paris, t.320, série I, January 1995, pp. 85-88.
- [6] P. ARMINJON, M.C. VIALON AND A. MADRANE, (1994), *A Finite Volume Extension of the Lax-Friedrichs and Nessyahu-Tadmor Schemes for Conservation Laws on*

- Unstructured Grids*, revised version with numerical applications, Int. J. of Comp. Fluid Dynamics (1997), Vol. 9, No. 1, 1-22.
- [7] T. J. BARTH AND D. C. JESPERSEN, (1989), *The design and application of upwind schemes on unstructured meshes*, AIAA Paper No. 89-0366, 27th Aerospace sciences meeting, January 9-12, 1989, Reno, Nevada.
- [8] FEZOU, L., LANTERI, S., LARROUTUROU, B. AND OLIVIER, C., (1989), *Résolution Numérique des Équations de Navier-Stokes pour un Fluide Compressible en Maillage triangulaire*, INRIA Res. Rep. No. 1033, Inst. National de Rech. en Informatique et Automatique, Rocquencourt, 78153 Le Chesnay, France.
- [9] FEZOU, L., STEVE, H. AND SELMIN, V. (1988) *Simulation numérique d'écoulements compressibles 3-D par un schéma décentré en maillage non-structuré*, INRIA Res. Rep. No. 0825, Institut National de Recherche en Informatique et Automatique, Rocquencourt, 78153 Le Chesnay, France.
- [10] G. JIANG AND E. TADMOR, (1998), *Non-oscillatory Central Schemes for Multi-dimensional Hyperbolic Conservation Laws*, SIAM J. on Scientific Computing, 19, pp.1892-1917.
- [11] T. KATSAOUNIS AND D. LEVY, (1999), *A modified structured central schme for 2D hyperbolic conservation laws*, Appl. Math. Lett. 12, pp. 89-96.
- [12] P. D. LAX, (1954), *Weak solutions of nonlinear hyperbolic equations and their numerical computation*, Comm. Pure and Applied Math. 7, pp. 159-193.
- [13] D. LEVY, G. PUPPO AND G. RUSSO, (2000), *A third order central WENO scheme for 2D hyperbolic conservations laws*, Appl. Numer. Math. 33, pp. 415-421.
- [14] D. LEVY, G. PUPPO AND G. RUSSO, (2000), *Compact central WENO scheme for multidimensional conservations laws*, SIAM J. Sci. Comput. 22, pp. 656-672.
- [15] A. MADRANE, (1998), *Constructions of new finite volume/finite element methods for transonic/supersonic compressible flows*, Ph. D. dissertation, Université de Montréal, Département de Mathématiques et Statistique.
- [16] H. NESSYAHU AND E. TADMOR, (1990), *Non-oscillatory central differencing for hyperbolic conservation laws*, J. Comp. Phys., 87, No. 2, pp. 408-463.
- [17] B. VAN LEER, (1979), *Towards the ultimate conservative difference scheme V. A Second-Order Sequel to Godunov's Method*, J. Comp. Physics, 32, pp. 101-136.
- [18] V. VENKATAKRISHNAN, (1995), *Convergence to Steady State Solutions of the Euler Equations on Unstructured Grids with Limiters*, J. Comp. Phys. 118, pp. 120-130.

P. Arminjon, A. St-Cyr

Département de mathématiques et de statistique Université de Montréal,  
C.P. 6128, Succ. centre-ville Montréal, Québec, H3C 3J7, Canada  
arminjon@crm.umontreal.ca  
stcyr@dms.umontreal.ca

A. Madrane

German Aerospace Center DLR Inst. of Design Aerodynamics,  
Lilienthalplatz 7, 38108 Braunschweig, Germany  
aziz.madrane@dlr.de

