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CENTRAL DIFFERENCING BASED NUMERICAL SCHEMES FOR HYPERBOLIC CONSERVATION LAWS WITH RELAXATION TERMS*

LORENZO PARESCHI†

Abstract. Many applications involve hyperbolic systems of conservation laws with source terms. The numerical solution of such systems may be challenging, especially when the source terms are stiff. Uniform accuracy with respect to the stiffness parameter is a highly desirable property but it is, in general, very difficult to achieve using underresolved discretizations. For such problems we develop different second order uniformly accurate high-resolution nonoscillatory central schemes. The schemes retain the simplicity of central schemes for hyperbolic conservation laws and avoid the use of Riemann solvers. In particular, we show that these schemes possess a discrete analogue of the continuous asymptotic limit and are able to capture the correct behavior even if the initial layer and the small relaxation time are not numerically resolved.

Key words. high-resolution central schemes, hyperbolic conservation laws, stiff terms, Runge–Kutta methods

AMS subject classifications. 65M99, 65L06, 82D05

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1. Introduction.

Several physical phenomena important in applications are described by hyperbolic systems with relaxation terms, for example, discrete kinetic theory of rarefied gases [9], hydrodynamical models for semiconductors [19], viscoelasticity [28], linear and nonlinear waves [32], multiphase and phase transitions [30], and radiation hydrodynamics [20].

The development of efficient numerical schemes for such problems is challenging, since in many applications the relaxation time varies from values of order 1 to very small values if compared to the time scale determined by the characteristic speed of the system. For small relaxation times the hyperbolic system with relaxation is said to be stiff and, typically, its solutions are well approximated by solutions of a suitably reduced set of conservation laws known as an equilibrium system [7].

Usually it is extremely difficult, if not impossible, to split the problem into separate regimes and to use different solvers in the stiff and nonstiff regions. Thus one has to use the original relaxation system in the entire computational domain. The construction of a scheme that works for all ranges of the relaxation time, using coarse grids that do not resolve the small relaxation time, has been studied mainly in the context of upwind methods using a method of lines approach combined with suitable operator splitting techniques [6, 13, 26, 27]. Alternatively, approaches based on the method of characteristics also have been considered [1, 24]. Most of these schemes are based on the solution to the Riemann problem and, except for the scheme proposed in [6], do not provide uniform second order accuracy with respect to the relaxation parameter without resolving the small scales or the initial layer. Numerical schemes

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†Department of Mathematics, University of Ferrara, Via Machiavelli 35, I-44100 Ferrara, Italy (pareschi@dm.unife.it, http://www.unife.it/∼prl).

for hyperbolic systems with more general stiff relaxation terms have been studied in [14, 22].

Recently, Riemann solver free numerical methods for hyperbolic conservation laws were proposed (see [23, 15, 11, 29]). These methods have attracted much attention, mainly because of their simplicity and robustness. The extension of these methods to the nonhomogeneous case can be done in a straightforward way for Jin–Xin relaxation schemes (see [22] for a more general treatment), and it is more delicate for Nessyahu–Tadmor (NT) central schemes or, more generally, for staggered schemes (see [8, 18]). We remark that other central schemes for treating stiff source terms was studied in [4].

In this paper we present different uniformly second order accurate, central difference approximations to nonlinear systems of hyperbolic conservation laws with relaxation.

Following [6, 13, 18] we call our schemes *robust* in the sense described above. They have the following properties:

- They possess a stability constraint independent of the small relaxation time. The CFL condition should be determined only by the convection part.
- They are uniformly second order accurate, in conserved as well as in nonconserved variables, with respect to the relaxation parameter, and provide the correct zero relaxation limit behavior using coarse grids that do not resolve the small relaxation time or the initial layer.
- They are high-resolution shock-capturing schemes that can properly handle the discontinuous features of the solution, yielding correct shock location and speed without numerical oscillations.
- They do not require the solution, exact or approximate, to the Riemann problem or the knowledge of the characteristic feature of the Jacobian matrix.

In addition to the aforementioned properties, although our schemes contain implicit terms in many circumstances—thanks to the special structure of the source term—one can avoid solving nonlinear algebraic equations. In particular, at variance with the schemes presented in [18], the schemes use only two evaluations of the relaxation term, which in some cases may represent the most expensive part of the computations. Both these aspects are of great importance in practical applications.

The schemes are built on a staggered grid and are based on the NT reconstruction strategy. Nonstaggered versions of the schemes can easily be obtained if one adapts the projection technique developed in [12]. In particular we will focus not only on conserved variables as in previous works [6, 13, 18] but also on nonconserved variables of the hyperbolic system with relaxation. We will show that in the small relaxation limit a reduction of accuracy may occur also on these variables independently on the initial layer, and we show how to fix this problem.

Finally, we remark that the time discretizations derived here can be effectively used also in the context of stiff ODEs of the type studied in [2, 3] and in combination with other space discretizations.

The rest of the paper is organized as follows. In section 2 we recall the bases of the mathematical models that characterize the hyperbolic systems with relaxation. In section 3 we introduce our discretization and derive different schemes with the desired second order accuracy. A detailed analysis of the stability properties of the schemes together with their small relaxation limit is presented in section 4. Finally, in section 5 we present several numerical examples which confirm the efficiency and robustness of the present approach.

2. Hyperbolic systems with relaxation. We consider one-dimensional hyperbolic systems with relaxation of the form

$$\partial_t U + \partial_x F(U) = \frac{1}{\varepsilon} R(U), \quad x \in \mathbb{R}, \tag{1}$$

where $U = U(x,t) \in \mathbb{R}^N$, $F : \mathbb{R}^N \to \mathbb{R}^N$, the Jacobian matrix $F'(U)$ has real eigenvalues, and $\varepsilon > 0$ is the relaxation time.

The operator $R : \mathbb{R}^N \to \mathbb{R}^N$ is said to be a relaxation operator, and consequently (1) defines a relaxation system in the sense of Whitham [32] and Liu [17] if there exists a constant $n \times N$ matrix Q with $\operatorname{rank}(Q) = n < N$ such that

$$QR(U) = 0 \quad \forall\, U \in \mathbb{R}^N. \tag{2}$$

This gives n independent conserved quantities $v = QU$. Moreover, such conserved quantities uniquely determine a local equilibrium value

$$U = \mathcal{E}(v) \text{ such that } R(\mathcal{E}(v)) = 0. \tag{3}$$

The image of $\mathcal{E}$ represents the manifold of local equilibria of the relaxation operator R.

Using (2) in (1) we obtain a system of n conservation laws which is satisfied by every solution of (1),

$$\partial_t(QU) + \partial_x(QF(U)) = 0. \tag{4}$$

For small values of the relaxation parameter ε from (1) we get $R(U) = 0$ which by (3) implies $U = \mathcal{E}(v)$. In this case system (1) is well approximated by the reduced system

$$\partial_t v + \partial_x G(v) = 0, \tag{5}$$

where $G(v) = QF(\mathcal{E}(v))$.

We refer to [32, 17, 7] for a theoretical study of these relaxation problems.

A simple prototype of a relaxation system in the case $N = 2$ is given by the p-system

$$\begin{aligned} \partial_t u + \partial_x v &= 0, \\ \partial_t v + \partial_x p(u) &= -\frac{1}{\varepsilon}(v - f(u)), \end{aligned} \tag{6}$$

which corresponds to taking $U = (u, v)$, $F(U) = (v, p(u))$, and $R(U) = (0, f(u) - v)$.

For small values of ε from the second equation in (6) we get the local equilibrium

$$v = f(u) \tag{7}$$

and, under Liu's subcharacteristic condition $p'(u) > f'(u)^2$, solutions to (6) converge to the solution of the scalar conservation law

$$\partial_t u + \partial_x f(u) = 0. \tag{8}$$

3. Central schemes.

3.1. Derivation. For simplicity we derive the schemes in the case of the single scalar equation

$$u_t + f(u)_x = \frac{1}{\varepsilon} g(u). \tag{9}$$

For the construction of our schemes we will use a staggered grid as in the NT approach. To this aim we introduce the grid points

$$x_j = j\Delta x, \quad x_{j+1/2} = x_j + \frac{1}{2}\Delta x, \quad j = \dots, -2, -1, 0, 1, 2, \dots \tag{10}$$

and use the standard notation

$$u_{j+1/2}^{n+1} = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} u(x, t^{n+1})\, dx. \tag{11}$$

Now if we integrate (9) over the cell $[x_j, x_{j+1}] \times [t^n, t^{n+1}]$ we obtain

$$\begin{aligned} u_{j+1/2}^{n+1} = {} & \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} u(x, t^n)\, dx + \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} f(u(x_j, t)) - f(u(x_{j+1}, t))\, dt \\ & + \frac{1}{\varepsilon \Delta x} \int_{t^n}^{t^{n+1}} \int_{x_j}^{x_{j+1}} g(u(x, t))\, dx\, dt. \end{aligned}$$

Following [23] at each time level $t^n = n\Delta t$ we reconstruct a piecewise linear approximation of $u(x, t)$ of the form

$$L_j(x, t) = u_j(t) + (x - x_j)\frac{u_j'}{\Delta x}, \quad x_{j-1/2} < x < x_{j+1/2}, \tag{12}$$

where u_j' is a numerical derivative such that

$$\frac{u_j'}{\Delta x} = u_x(x, t)|_{x=x_j} + O(\Delta x). \tag{13}$$

This gives

$$\begin{aligned} u_{j+1/2}^{n+1} = {} & \frac{1}{\Delta x} \left(\int_{x_j}^{x_{j+1/2}} L_j(x, t^n)\, dx + \int_{x_{j+1/2}}^{x_{j+1}} L_{j+1}(x, t^n)\, dx \right) \\ & + \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} f(u(x_j, t)) - f(u(x_{j+1}, t))\, dt + \frac{1}{\varepsilon \Delta x} \int_{t^n}^{t^{n+1}} \int_{x_j}^{x_{j+1}} g(u(x, t))\, dx\, dt \\ = {} & \frac{1}{2}(u_j^n + u_{j+1}^n) + \frac{1}{8}(u_j' - u_{j+1}') + \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} f(u(x_j, t)) - f(u(x_{j+1}, t))\, dt \\ & + \frac{1}{\varepsilon \Delta x} \int_{t^n}^{t^{n+1}} \int_{x_j}^{x_{j+1}} g(u(x, t))\, dx\, dt. \end{aligned}$$

The semidiscrete approximation can be rewritten in the form

$$u_{j+1/2}^{n+1} = \frac{1}{2}(u_j^n + u_{j+1}^n) - \lambda(f_{j+1} - f_j) + \frac{\Delta t}{\varepsilon} g_{j+1/2}, \tag{14}$$

where $\lambda = \Delta t/\Delta x$ and

$$f_j = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(u(x_j,t))\,dt + \frac{1}{8\lambda}u_j', \tag{15}$$

$$g_{j+1/2} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} g(u(x,t))\,dx\,dt. \tag{16}$$

Clearly, the main issue is the time integration of (16) which should guarantee both stability and second order accuracy.

As we will see, however, this time discretization has to be considered together with the time discretization of the flux term (15) in order to obtain an efficient and uniformly second order accurate scheme.

Remark 3.1. We point out that, to achieve second order accuracy in space for smooth solutions and to be consistent with the staggered grid, for the space discretization of (16) we can use a standard midpoint rule on the terms at $t = t^{n+1}$,

$$\frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} g(u^{n+1}(x))\,dx \approx g(u_{j+1/2}^{n+1}), \tag{17}$$

and a trapezoidal rule on all other terms with $t \in [t^n, t^{n+1}[$,

$$\frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} g(u(x,t))\,dx \approx \frac{1}{2}\left[g(u_{j+1}(t)) + g(u_j(t))\right]. \tag{18}$$

Clearly, suitable predictors for the values of $u_j(t)$ with $t \in]t^n, t^{n+1}[$ may be required by both time discretizations of (16) and (15).

To conclude this section we report some simple examples of nonoscillatory TVD numerical derivatives [23],

$$\begin{aligned} u_j' &= \mathrm{MM}(\Delta u_{j+1/2}, \Delta u_{j-1/2}),\\ u_j' &= \mathrm{MM}\left(\gamma(\Delta u_{j+1/2}), \frac{1}{2}(u_{j+1}-u_{j-1}), \gamma(\Delta u_{j-1/2})\right), \quad 1 \le \gamma \le 2, \end{aligned} \tag{19}$$

where $\Delta u_{j+1/2} = u_{j+1} - u_j$, $\Delta u_{j-1/2} = u_j - u_{j-1}$, and MM is the multivariable MinMod function given by

$$\mathrm{MM}(x_1, x_2, \ldots) = \begin{cases} \min_j\{x_j\} & \text{if } x_j > 0\ \forall j,\\ \max_j\{x_j\} & \text{if } x_j < 0\ \forall j,\\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

Another possible choice is the UNO [10] numerical derivative

$$u_j' = \mathrm{MM}\left(\Delta u_{j-1/2} + \frac{1}{2}\mathrm{MM}(\Delta^2 u_{j-1}, \Delta^2 u_j), \Delta u_{j+1/2} - \frac{1}{2}\mathrm{MM}(\Delta^2 u_j, \Delta^2 u_{j+1})\right)$$

with $\Delta^2 u_j = u_{j+1} + u_{j-1} - 2u_j$.

3.1.1. Predictor-corrector (PC) time discretizations. First we recall that for $g \equiv 0$ a fully second order scheme was obtained in [23] applying the midpoint rule to the flux

$$f_j \approx f(u_j^{n+1/2}) + \frac{1}{8\lambda}u_j' \tag{21}$$

and predicting the value $u_j^{n+1/2}$ by Taylor expansion

$$u_j^{n+1/2} = u_j^n - \frac{1}{2}\lambda f_j', \tag{22}$$

where $f_j'/\Delta x$ is at least a first order approximation of $f(u(x,t^n))_x|_{x=x_j}$, computed, for example, using the same MinMod limiter.

This is equivalent to using a standard second order PC scheme for the ODE

$$y' = f(y), \tag{23}$$

where the predictor is the explicit Euler method and the corrector the midpoint method

$$\begin{aligned} y_p^{n+1/2} &= y^n + \frac{1}{2}\Delta t f(y^n), \\ y^{n+1} &= y^n + \Delta t f(y_p^{n+1/2}). \end{aligned} \tag{24}$$

Of course this is not the unique possible choice to obtain second order accuracy in time.

For example, another second order accurate scheme is obtained using the trapezoidal rule on the fluxes

$$f_j \approx \frac{\Delta t}{2}\left(f(u_j^{n+1}) + f(u_j^n)\right) + \frac{1}{8\lambda}u_j' \tag{25}$$

and predicting the value u_j^{n+1} by

$$u_j^{n+1} = u_j^n - \lambda f_j' \tag{26}$$

corresponding to the PC scheme

$$\begin{aligned} y_p^{n+1} &= y^n + \Delta t f(y^n), \\ y^{n+1} &= y^n + \frac{1}{2}\Delta t(f(y_p^{n+1}) + f(y^n)). \end{aligned} \tag{27}$$

More generally, any other two-level explicit Runge–Kutta method for (23) applies provided that the first level is used as a predictor and the second as the time integrator in the corrector. We mention here other Runge–Kutta methods proposed in the context of central schemes in [5] and a more general treatment recently developed in [25].

3.2. Implicit-explicit central schemes. Since (9) contains stiff terms for small values of ε it is natural to use an implicit time discretization for the stiff relaxation term (16) combined with an explicit time discretization for the fluxes (15). In particular the combination of both time integrators should result in a scheme for which the overall accuracy is second order and, moreover, the accuracy should not deteriorate in the stiff limit as ε goes to zero.

In order to derive such time discretizations we consider the simple ODE

$$y' = f(y) + \frac{1}{\varepsilon}g(y). \tag{28}$$

Several Runge–Kutta schemes for this kind of equation, based on implicit discretizations of g and explicit discretizations of f, were derived recently for advection-diffusion problems in [2, 3].

3.2.1. Previous methods. First attempts to develop uniformly accurate second order methods in the context of central schemes were previously made in [8, 18]. Both methods were based on using directly the NT scheme, i.e., fixing the midpoint rule for the flux integration, and considering only the problem of the integration of the stiff source term.

In [8] a θ-method was constructed of the form

$$\begin{aligned} y_p^{n+1/2} &= y^n + \frac{1}{2}\Delta t f(y^n) + \frac{1}{2\varepsilon}\Delta t\, g(y_p^{n+1/2}), \\ y^{n+1} &= y^n + \frac{1}{2}\Delta t f(y_p^{n+1/2}) + \frac{1}{\varepsilon}\Delta t((1-\theta)g(y^n) + \theta g(y^{n+1})), \end{aligned} \tag{29}$$

where $\theta = \theta(\varepsilon) \in [0,1]$ is s.t. $\theta(\varepsilon = O(1)) = 1/2$ and $\theta(\varepsilon \ll 1) \approx 1$. The scheme is quite efficient since only two evaluations of f and, except for the first time step, of g are used. However, the derivation of the parameter θ that gives uniform second order accuracy is problem dependent and quite difficult in general.

At variance the method developed in [18] results in the three-level scheme

$$\begin{aligned} y_p^{n+1/2} &= y^n + \frac{1}{2}\Delta t f(y^n) + \frac{1}{2\varepsilon}\Delta t\, g(y_p^{n+1/2}), \\ y_p^{n+1/3} &= y^n + \frac{1}{3}\Delta t f(y^n) + \frac{1}{3\varepsilon}\Delta t\, g(y_p^{n+1/3}), \\ y^{n+1} &= y^n + \Delta t f(y_p^{n+1/2}) + \frac{1}{4\varepsilon}\Delta t(3g(y_p^{n+1/3}) + g(y^{n+1})), \end{aligned} \tag{30}$$

which gives uniform second order accuracy but is not efficient since it requires three evaluations of g at every time step. Moreover, when $g \equiv 0$ or $f \equiv 0$, the result of one of the two predictors is not used by the corrector.

3.2.2. Second order PC schemes. Here we are aiming to find a more general two-level PC scheme of the form

$$\begin{aligned} y_p^{n+\alpha} &= y^n + \alpha\Delta t f(y^n) + \alpha\frac{1}{\varepsilon}\Delta t\, g(y_p^{n+\alpha}), \\ y^{n+1} &= y^n + \Delta t(\mu f(y^n) + \nu f(y_p^{n+\alpha})) + \frac{1}{\varepsilon}\Delta t(\xi g(y_p^{n+\alpha}) + \eta g(y^{n+1})). \end{aligned} \tag{31}$$

Second order accuracy conditions for the scheme can be derived by considering the exact solution

$$Y(\Delta t) = \exp((A+B)\Delta t)Y(0) \tag{32}$$

in the time interval $[0, \Delta t]$ to the linear system

$$\partial_t Y = AY + BY, \tag{33}$$

where $Y \in \mathbb{R}^N$ and A, B are two constant matrices.

Applying (31) to (33) and equating the Taylor expansions up to order 2 of the resulting numerical solution and of (32), we obtain the following set of nonlinear algebraic equations for the weights:

$$\begin{aligned} \mu + \nu = 1, \qquad \xi + \eta = 1, \quad \alpha\,\nu = \frac{1}{2}, \\ \alpha\,\xi + \eta(\mu+\nu) = \frac{1}{2}, \quad \alpha\xi + \eta(\xi+\eta) = \frac{1}{2}. \end{aligned} \tag{34}$$

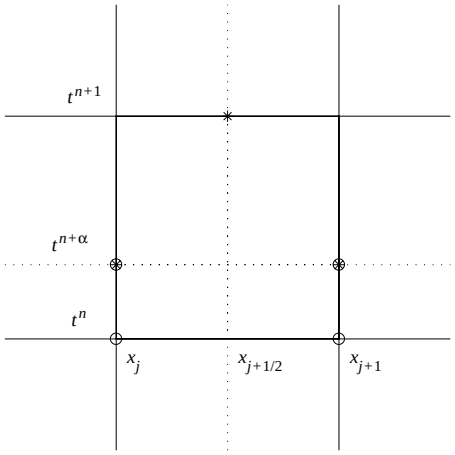

FIG. 1. *Quadrature points for the flux* ($\circ$) *and the source term* ($\times$) *in scheme PCC.*

The family of real solutions to (34) can be expressed in terms of the parameter $\alpha \neq 0, 1$ as

$$1-\mu=\nu=\frac{1}{2\alpha}, \quad 1-\eta=\xi=\frac{1}{2(1-\alpha)}. \tag{35}$$

For example, taking $\alpha = 1/3$ we get

$$\mu=-\frac{1}{2}, \quad \nu=\frac{3}{2}, \quad \xi=\frac{3}{4}, \quad \eta=\frac{1}{4}, \tag{36}$$

whereas for $\alpha = 2/3$ we have

$$\mu=\frac{1}{4}, \quad \nu=\frac{3}{4}, \quad \xi=\frac{3}{2}, \quad \eta=-\frac{1}{2}. \tag{37}$$

Note that for $\alpha > 1/2$ the discretization contains negative parameters in the implicit terms which may cause numerical breakdown in the intermediate regime $\Delta t = O(\varepsilon)$.

This creates the following family of second order PC central schemes (PCC) for (9) (see Figure 1):

$$u_j^{n+\alpha} = u_j^n - \lambda\alpha f_j' + \frac{\Delta t}{\varepsilon}\alpha g(u_j^{n+\alpha}), \tag{38}$$

$$\begin{aligned} u_{j+1/2}^{n+1} = {} & \frac{1}{2}(u_j^n + u_{j+1}^n) + \frac{1}{8}(u_j' - u_{j+1}') \\ & - \frac{\lambda}{2\alpha}\left[(2\alpha-1)(f(u_j^n) - f(u_{j+1}^n)) + (f(u_j^{n+\alpha}) - f(u_{j+1}^{n+\alpha}))\right] \\ & + \frac{\Delta t}{2(1-\alpha)\varepsilon}\left[\frac{1}{2}(g(u_j^{n+\alpha}) + g(u_{j+1}^{n+\alpha})) + (1-2\alpha)g(u_{j+1/2}^{n+1})\right]. \end{aligned} \tag{39}$$

We remark that a method based on the usual NT PC scheme (24) coupled with a midpoint quadrature rule on the stiff source term is obtained for $\alpha = 1/2$. Unfortunately, as we will see in section 4, L-stability of the stiff integrator will be guaranteed only if $0 < \alpha < 1/2$.

Remark 3.2. The second order conditions (34) are equivalent to the usual second order conditions of Runge–Kutta methods for the differential equation (28) (see [2]). This is not the case if one considers higher order methods.

Remark 3.3. We point out that schemes (38), (39) can be easily written in operator splitting form as

$$
\begin{aligned}
u_j^{(1)} &= u_j^n - \lambda\alpha f_j', \\
u_j^{n+\alpha} &= u_j^{(1)} + \frac{\Delta t}{\varepsilon}\alpha g(u_j^{n+\alpha}), \\
u_{j+1/2}^{(2)} &= \frac{1}{2}(u_j^n + u_{j+1}^n) + \frac{1}{8}(u_j' - u_{j+1}') \\
&\quad - \frac{\lambda}{2\alpha}\left[(2\alpha-1)(f(u_j^n) - f(u_{j+1}^n)) + (f(u_j^{n+\alpha}) - f(u_{j+1}^{n+\alpha}))\right], \\
u_{j+1/2}^{n+1} &= u_{j+1/2}^{(2)} + \frac{\Delta t}{2(1-\alpha)\varepsilon}\left[\frac{1}{2}(g(u_j^{n+\alpha}) + g(u_{j+1}^{n+\alpha})) + (1-2\alpha)g(u_{j+1/2}^{n+1})\right].
\end{aligned} \tag{40}
$$

This form is useful since it allows an efficient implementation of the methods and, as we will see in section 4, simplifies the analysis of the small relaxation limit.

3.2.3. Second order schemes with initial layer fix. From the split form (40) it is clear that PCC schemes do not possess the correct initial layer behavior since, except for $\alpha = 1/2$, the corrector contains the value $f(u_j^n)$, and u_j^n in the very first time step is not a projection over the local equilibrium as $\varepsilon \to 0$. This will introduce an initial error of $O(\Delta t)$ if the initial layer is not well resolved and the initial data are not in local equilibrium (see section 4.2.1). Hence, extra care must be taken to properly handle the initial layer. A possible way to overcome this problem is to use a Richardson extrapolation technique only in the first time step, as in [6]. More precisely, if we denote by $u_{j+1/2}^{\Delta t}$ the numerical solution obtained with a first iteration of the scheme, and by $u_j^{\Delta t/2}$ the numerical solution obtained with two half-steps iterations, taking care of the staggered grid with a suitable reconstruction technique, our first time step solution will be given by

$$
\begin{aligned}
u_{j+1/2}^{\Delta t/2} &= \frac{1}{2}(u_j^{\Delta t/2} + u_{j+1}^{\Delta t/2}) + \frac{1}{8}(u_j' - u_{j+1}'), \\
u_{j+1/2}^{1} &= 2\,u_{j+1/2}^{\Delta t/2} - u_{j+1/2}^{\Delta t},
\end{aligned} \tag{41}
$$

where $u_j'/\Delta x$ denotes a first order approximation of $u_x(x, \Delta t/2)|_{x=x_j}$.

We will denote by PCCF the PC central scheme (38), (39) with this simple initial layer fix.

However, since in order to have the correct behavior the numerical scheme should have a projection on the very first step, one may try to find a scheme of the form

$$
\begin{aligned}
y_p^* &= y^n + \beta\frac{1}{\varepsilon}\Delta t\, g(y_p^*), \\
y_p^{**} &= y^n + \alpha\Delta t f(y_p^*) + \frac{1}{\varepsilon}\Delta t\,(\xi g(y_p^*)) + \eta g(y_p^{**})), \\
y^{n+1} &= y^n + \Delta t(\mu f(y_p^*) + \nu f(y_p^{**})) + \frac{1}{\varepsilon}\Delta t(\gamma g(y_p^*) + \sigma g(y_p^{**})).
\end{aligned} \tag{42}
$$

Note that this Runge–Kutta scheme is still a two-level method since only two evaluations of f and g are used; however, it does not possess a natural PC interpretation since y_p^* solves only an equation in g.

Applying the same asymptotic technique to satisfy our second order accuracy requirement, we obtain the following set of nonlinear algebraic equations:

$$
\begin{aligned}
\mu + \nu &= 1, & \gamma + \sigma &= 1,\\
\alpha\,\nu &= \frac{1}{2}, & \alpha\,\sigma &= \frac{1}{2},\\
\beta\,\mu + \nu(\xi+\eta) &= \frac{1}{2}, & \beta\gamma + \sigma(\xi+\eta) &= \frac{1}{2}.
\end{aligned}
\tag{43}
$$

A family of solutions to (43) that satisfies the condition $\xi + \eta = 1$ can be expressed in terms of $\alpha \neq 0,\ 1/2$, and ξ as

$$
\beta = \frac{\alpha - 1}{2\alpha - 1}, \quad \mu = \gamma = \frac{2\alpha - 1}{2\alpha}, \quad \nu = \sigma = \frac{1}{2\alpha}, \quad \eta = 1 - \xi. \tag{44}
$$

Nonpositive parameters in the implicit terms are avoided if $0 < \alpha < 1/2$ or $\alpha > 1$.

Hence, we have the following family of second order Runge–Kutta central schemes (RKC) for (9):

$$
u_j^* = u_j^n + \frac{\Delta t}{(2\alpha - 1)\varepsilon}(\alpha - 1) g(u_j^*), \tag{45}
$$

$$
u_j^{**} = u_j^n - \lambda\alpha f'(u_j^*) + \frac{\Delta t}{\varepsilon}[\xi g(u_j^*) + (1-\xi) g(u_j^{**})], \tag{46}
$$

$$
\begin{aligned}
u_{j+1/2}^{n+1} = {} & \frac{1}{2}(u_j^n + u_{j+1}^n) + \frac{1}{8}(u_j' - u_{j+1}')\\
& - \frac{\lambda}{2\alpha}\left[(2\alpha - 1)(f(u_j^*) - f(u_{j+1}^*)) + (f(u_j^{**}) - f(u_{j+1}^{**}))\right]\\
& + \frac{\Delta t}{2\alpha\varepsilon}\left[\frac{(2\alpha - 1)}{2}(g(u_j^*) + g(u_{j+1}^*)) + \frac{1}{2}(g(u_j^{**}) + g(u_{j+1}^{**}))\right].
\end{aligned}
\tag{47}
$$

Remark 3.4. A splitting form of this scheme is given by

$$
\begin{aligned}
u_j^* &= u_j^n + \frac{\Delta t}{(2\alpha - 1)\varepsilon}(\alpha - 1) g(u_j^*),\\
u_j^{(1)} &= u_j^n - \lambda\alpha f'(u_j^*),\\
u_j^{**} &= u_j^{(1)} + \frac{\Delta t}{\varepsilon}[\xi g(u_j^*) + (1-\xi) g(u_j^{**})],\\
u_{j+1/2}^{(2)} &= \frac{1}{2}(u_j^n + u_{j+1}^n) + \frac{1}{8}(u_j' - u_{j+1}')\\
&\quad - \frac{\lambda}{2\alpha}\left[(2\alpha - 1)(f(u_j^*) - f(u_{j+1}^*)) + (f(u_j^{**}) - f(u_{j+1}^{**}))\right],\\
u_{j+1/2}^{n+1} &= u_{j+1/2}^{(2)} + \frac{\Delta t}{2\alpha\varepsilon}\left[\frac{(2\alpha - 1)}{2}(g(u_j^*) + g(u_{j+1}^*)) + \frac{1}{2}(g(u_j^{**}) + g(u_{j+1}^{**}))\right].
\end{aligned}
\tag{48}
$$

We point out that this splitting, as well as (40), can be effectively used also in the context of upwind schemes. Compared to the Runge–Kutta splittings proposed in [13, 6], the splitting (48) has the advantage of avoiding negative weights on the stiff terms and avoiding Richardson extrapolation in the first time step.

4. Analysis of the schemes.

4.1. Stability analysis. The stability of the previous schemes will be studied from the split forms (40) and (48) by considering separately the stability of the implicit steps, which involve the solution of a stiff ODE, and the stability of the explicit steps, which involve the solution of a conservation law.

4.1.1. Stability of PCC schemes. To study the stability of the implicit time integrator we will apply, in the case $f \equiv 0$, the schemes (31) with the weights given by (35) to the model problem

$$y' = \lambda y, \quad y(0) = 1. \tag{49}$$

This gives

$$y^{n+1} = R(z) y^n,$$

where R is the function of absolute stability

$$R(z) = \frac{1 + \alpha z}{(1 - z(1-\alpha))(1 - z(2\alpha - 1))}, \tag{50}$$

and $z = \Delta t \lambda$.

It is easy to show that these schemes are L-stable for $0 < \alpha < 1/2$. Thus the value $\alpha = 1/2$ (corresponding to the standard NT scheme for the fluxes) does not satisfy the L-stability requirements. In particular, the maximum damping of (50) is obtained for $\alpha = 1 - \sqrt{2}/2 \approx 0.3$.

Therefore, for $0 < \alpha < 1/2$ a sufficient CFL condition will be given only by the convective part since the stiff source terms are treated by L-stable integrators.

For the scalar problem (9) in the case $g \equiv 0$, using the same arguments as in [23], we can prove the following.

PROPOSITION 4.1. *Let the numerical derivatives be chosen such that*

$$0 \le u'_j \mathrm{sign}(\Delta u_{j\pm 1/2}) \le A |\mathrm{MM}(\Delta u_{j+1/2}, \Delta u_{j-1/2})|, \tag{51}$$

$$0 \le f'_j \mathrm{sign}(\Delta u_{j\pm 1/2}) \le C |\mathrm{MM}(\Delta u_{j+1/2}, \Delta u_{j-1/2})|, \tag{52}$$

where A and C are positive constants.

Then the family of PC schemes (38), (39) *with $g \equiv 0$ and $0 < \alpha < 1/2$ is TVD if the following CFL condition is satisfied:*

$$\lambda \max_j |a(u_j)| \le B, \tag{53}$$

$$B \equiv \lambda \frac{C}{A} \le \frac{1}{2\alpha A}(2(\alpha - 1) + \sqrt{\alpha^2(4A - A^2 + 4) - 8\alpha + 4}), \tag{54}$$

where $a(u_j) = \partial_u f(u = u_j)$.

Proof. The schemes are TVD if their modified numerical flux satisfies [23]

$$\lambda \left| \frac{f_{j+1} - f_j}{u_{j+1} - u_j} \right| = \lambda \left| \frac{\Delta f_{j+1/2}}{\Delta u_{j+1/2}} \right| \le \frac{1}{2},$$

where in our case

$$f_j = \frac{1}{2\alpha}((2\alpha - 1) f(u_j^n) + f(u_j^{n+\alpha})) + \frac{1}{8\lambda} u'_j, \quad 0 < \alpha < 1/2. \tag{55}$$

Thus

$$\lambda\left|\frac{\Delta f_{j+1/2}}{\Delta u_{j+1/2}}\right| \le \lambda\frac{(1-2\alpha)}{2\alpha}\left|\frac{f(u^n_{j+1})-f(u^n_j)}{\Delta u_{j+1/2}}\right| + \lambda\frac{1}{2\alpha}\left|\frac{f(u^{n+\alpha}_{j+1})-f(u^{n+\alpha}_j)}{\Delta u_{j+1/2}}\right| + \frac{1}{8}\left|\frac{\Delta u'_{j+1/2}}{\Delta u_{j+1/2}}\right|.$$

From the CFL condition we have

$$\lambda\left|\frac{f(u^n_{j+1})-f(u^n_j)}{\Delta u_{j+1/2}}\right| \le B$$

and

$$\lambda\left|\frac{f(u^{n+\alpha}_{j+1})-f(u^{n+\alpha}_j)}{\Delta u_{j+1/2}}\right| \le \lambda\left|\frac{f(u^{n+\alpha}_{j+1})-f(u^{n+\alpha}_j)}{u^{n+\alpha}_{j+1}-u^{n+\alpha}_j}\right|\left|\frac{u^{n+\alpha}_{j+1}-u^{n+\alpha}_j}{\Delta u_{j+1/2}}\right| \le B\left|\frac{u^{n+\alpha}_{j+1}-u^{n+\alpha}_j}{\Delta u_{j+1/2}}\right| \le B\left(1+\alpha\lambda\left|\frac{f'_{j+1}-f'_j}{\Delta u_{j+1/2}}\right|\right),$$

where the last term has been estimated using the predictor (38).

Now, as in [23], from our assumptions we have the bounds

$$\left|\frac{f'_{j+1}-f'_j}{\Delta u_{j+1/2}}\right| \le \frac{AB}{\lambda}, \qquad \left|\frac{\Delta u'_{j+1/2}}{\Delta u_{j+1/2}}\right| \le A.$$

Collecting all the previous estimates, we obtain that our TVD requirement results in the quadratic inequality

$$\frac{B}{2\alpha}\left(2-2\alpha+\alpha AB\right)+\frac{1}{8}A \le \frac{1}{2},$$

which gives the CFL limitation (54).

Remark 4.1. The right-hand side of (54) is an increasing function of α, and hence the final CFL limitation for the TVD property is stronger than that of the original NT scheme which corresponds to taking $\alpha = 1/2$. For example, in the case $A = 1$ we have for $\alpha = 1/3$ the limitation $B \le (\sqrt{19}-4)/2 \approx 0.18$, whereas for $\alpha = 2/5$ we have $B \le (\sqrt{12}-3)/2 \approx 0.23$. However, as already pointed out in [23], these constraints are sufficient, but not necessary, conditions for the TVD property, and in practice one may use higher values of B (typically up to 0.5).

4.1.2. Stability of RKC schemes. For schemes (42) with the weights given by (44) the function of absolute stability of the stiff integrator is given by

$$R(z) = -\frac{1}{2}\,\frac{(2\,\alpha\,\xi+1-2\,\alpha)\,z^2+(4\,\alpha\,\xi-2\,\alpha-2\,\xi+2)\,z-2+4\,\alpha}{(1+(\xi-1)\,z)\,((-1+\alpha)\,z+1-2\,\alpha)}. \tag{56}$$

Thus this class of schemes is L-stable only if

$$\lim_{z\to\infty} R(z) = -\frac{1}{2}\,\frac{2\,\alpha\,\xi+1-2\,\alpha}{(\xi-1)\,(-1+\alpha)} = 0. \tag{57}$$

Taking

$$\xi = \mu = \gamma = \frac{2\alpha - 1}{2\alpha}, \tag{58}$$

we have a class of L-stable schemes for $\alpha \neq 0, 1/2, 1$. The maximum damping of (56) is then obtained for $\alpha = 1 \pm \sqrt{2}/2$. However, we have fully positive parameters only for $\alpha > 1$; thus the value $\alpha = 1 + \sqrt{2}/2$ should be considered "optimal" in stiff regimes. For example, for $\alpha = 5/3$ we have

$$\beta = \frac{2}{7}, \quad \mu = \xi = \gamma = \frac{7}{10}, \quad \nu = \eta = \sigma = \frac{3}{10}, \tag{59}$$

whereas for $\alpha = 2$ we have

$$\beta = \frac{1}{3}, \quad \mu = \xi = \gamma = \frac{3}{4}, \quad \nu = \eta = \sigma = \frac{1}{4}. \tag{60}$$

In this case, since all the parameters of the schemes are positive, the CFL condition for TVD stability in the case $g \equiv 0$ is independent of α and is exactly the same as in NT schemes. This is stated by the following.

PROPOSITION 4.2. *Let the numerical derivatives be chosen such that relations* (51), (52) *hold. Then the family of PC schemes* (46), (47) *with* $g \equiv 0$ *and* $\alpha > 1$ *is TVD if the following CFL condition is satisfied:*

$$\lambda \max_j |a(u_j)| \leq B, \quad B = \lambda \frac{C}{A} \leq \frac{1}{2A}\left(\sqrt{4A - A^2 + 4} - 2\right). \tag{61}$$

Proof. It is enough to observe that the modified numerical flux is the same as in PC schemes but with $\alpha > 1$:

$$f_j = \frac{1}{2\alpha}((2\alpha - 1)f(u_j^n) + f(u_j^{n+\alpha})) + \frac{1}{8\lambda}u_j', \quad \alpha > 1. \tag{62}$$

Hence, we get the quadratic inequality

$$\frac{B}{2\alpha}(2\alpha - 1 + 1 + \alpha AB) + \frac{1}{8}A \leq \frac{1}{2},$$

which after simplification gives the usual CFL limitation (61).

Once more, we observe that these sufficient TVD constraints are more restrictive than the "practical" CFL limitation of $B \leq 0.5$.

Remark 4.2. Alternatively, as in [18], one also can study the stability of the full scheme applied to the model problem

$$y' = \lambda_1 y + \lambda_2 y, \quad y'(0) = 1. \tag{63}$$

This gives

$$y^{n+1} = R(z_1, z_2)y^n,$$

where R is the function of absolute stability, $R(0, z_2)$ is the function of absolute stability of the implicit scheme, and $z_1 = \Delta t \lambda_1$, $z_2 = \Delta t \lambda_2$.

Remark 4.3. More general TVD estimates can be proved for the full problem (9) if one assumes that the relaxation operator $g(u)$ is a nonincreasing function of u. However, the study of the simplified case $g \equiv 0$ is particularly important since in the zero relaxation limit, as we will see in the following sections, the schemes we obtain for the limiting equilibrium conservation laws have the same structure of the schemes we obtain for $g \equiv 0$.

4.2. Zero relaxation limit. The most important issue when developing under-resolved numerical schemes is the behavior of the method as $\varepsilon \to 0$. To this aim we will rewrite our schemes in splitting form for the hyperbolic system with relaxation (1).

4.2.1. PCC schemes. We have for $0 < \alpha < 1/2$

$$
\begin{aligned}
U_j^{(1)} &= U_j^n - \lambda\alpha F_j', \\
U_j^{n+\alpha} &= U_j^{(1)} + \frac{\Delta t}{\varepsilon}\alpha R_j(U^{n+\alpha}), \\
U_{j+1/2}^{(2)} &= \frac{1}{2}(U_j^n + U_{j+1}^n) + \frac{1}{8}(U_j' - U_{j+1}') \\
&\quad - \frac{\lambda}{2\alpha}\left[(2\alpha-1)(F_j(U^n) - F_{j+1}(U^n)) + (F_j(U^{n+\alpha}) - F_{j+1}(U^{n+\alpha}))\right], \\
U_{j+1/2}^{n+1} &= U_{j+1/2}^{(2)} + \frac{\Delta t}{2\alpha\varepsilon}\left[\frac{1}{2}(R_j(U^{n+\alpha}) + R_{j+1}(U^{n+\alpha})) + (2\alpha-1)R_{j+1/2}(U^{n+1})\right].
\end{aligned} \tag{64}
$$

As $\varepsilon \to 0$ from the second and the last equation in (64), we get

$$
R(U^{n+\alpha}) = 0, \quad R(U^{n+1}) = 0. \tag{65}
$$

Thus by (3) both steps are projections of the solution towards its local equilibrium

$$
U^{n+\alpha} = \mathcal{E}(v^{n+\alpha}), \quad U^{n+1} = \mathcal{E}(v^{n+1}). \tag{66}
$$

Multiplying system (64) by Q and using relations (66) in the first and second equation, respectively, we have the limiting scheme

$$
\begin{aligned}
v_j^{n+\alpha} &= v_j^n - \lambda\alpha Q F_j', \\
v_{j+1/2}^{n+1} &= \frac{1}{2}(v_j^n + v_{j+1}^n) + \frac{1}{8}(v_j' - v_{j+1}') \\
&\quad - \frac{\lambda}{2\alpha}\left[(2\alpha-1)(QF_j(u^n) - QF_{j+1}(u^n)) + (G_j(v^{n+\alpha}) - G_{j+1}(v^{n+\alpha}))\right],
\end{aligned} \tag{67}
$$

where $G(v) = QF(\mathcal{E}(U))$.

Thanks to (66), scheme (67), except for the very first time step, will be equivalent to applying directly the second order central schemes we get for $R \equiv 0$ to the reduced system of equations (5), and Proposition 4.1 applies. Hence, the method provides the correct equilibrium limit and it is asymptotic preserving. As already observed, a suitable initial layer fix technique (41) is required in the first step to avoid degradation of accuracy to first order if the initial data do not satisfy the equilibrium equation (3).

4.2.2. RKC schemes. In this case for $\alpha > 1$ we can write

$$
\begin{aligned}
U_j^* &= U_j^n + \frac{\Delta t}{(2\alpha-1)\varepsilon}(\alpha-1)R(U_j^*), \\
U_j^{(1)} &= U_j^n - \lambda\alpha F'(U_j^*), \\
U_j^{**} &= U_j^{(1)} + \frac{\Delta t}{2\alpha\varepsilon}[(2\alpha-1)R(U_j^*) + R(U_j^{**})], \\
U_{j+1/2}^{(2)} &= \frac{1}{2}(U_j^n + U_{j+1}^n) + \frac{1}{8}(U_j' - U_{j+1}') \\
&\quad - \frac{\lambda}{2\alpha}\left[(2\alpha-1)(F(U_j^*) - F(U_{j+1}^*)) + (F(U_j^{**}) - F(U_{j+1}^{**}))\right], \\
U_{j+1/2}^{n+1} &= U_{j+1/2}^{(2)} + \frac{1}{2}\left[(U_j^{**} + U_{j+1}^{**}) - (U_j^{(1)} + U_{j+1}^{(1)})\right].
\end{aligned} \tag{68}
$$

As $\varepsilon \to 0$ from the first and second equations in (68), we obtain

$$R(U^*) = 0, \qquad R(U^{**}) = 0. \tag{69}$$

Thus by (3) both steps are projections of the solution towards its local equilibrium

$$U^* = \mathcal{E}(v^n), \qquad U^{**} = \mathcal{E}(v^{n+\alpha}). \tag{70}$$

After multiplication of system (68) by Q, since $QU^{**} = QU^{(1)}$, using relations (70) we have the limiting scheme

$$\begin{aligned} v_j^{n+\alpha} &= v_j^n - \lambda\alpha G_j', \\ v_{j+1/2}^{n+1} &= \frac{1}{2}(v_j^n + v_{j+1}^n) + \frac{1}{8}(v_j' - v_{j+1}') \\ &\quad - \frac{\lambda}{2\alpha}\left[(2\alpha-1)(G_j(v^n) - G_{j+1}(v^n)) + (G_j(v^{n+\alpha}) - G_{j+1}(v^{n+\alpha}))\right], \end{aligned} \tag{71}$$

where $G(v) = QF(\mathcal{E}(U))$.

Clearly, and independently of the initial data, (71) is a second order high-resolution central scheme for (5) for which Proposition 4.2 holds.

Remark 4.4. From the split form (68) it is clear that the last step in the scheme is not a projection towards the local equilibrium. This problem is less important than the initial layer problem, since it will not affect the accuracy of the conserved variables (typically the quantities of interest) but only the accuracy on the remaining $N - n$ variables in the very last step. In fact, this "final layer" will reduce to first order the accuracy of the nonconserved variables, and hence their distance from equilibrium, for small values of ε. However, using Richardson extrapolation in the very last step, exactly as described in section 3.2.3, recovers the desired accuracy. This scheme will be denoted by RKCF in what follows.

5. Numerical examples.

5.1. Broadwell model. To compare the accuracy of our approach with the results presented in [6, 13, 18, 24, 1] we test our schemes for the Broadwell model of rarefied gas dynamics. The kinetic model is characterized by a hyperbolic system with relaxation of the form (1) for $N = 3$ with

$$U = (\rho,\, m,\, z), \qquad F(U) = (m,\, z,\, m), \qquad R(U) = \left(0,\, 0,\, \frac{1}{2}[\rho^2 + m^2 - 2\rho z]\right).$$

Here ε represents the mean-free path of particles. The only conserved quantities are the density ρ and the momentum m.

In the fluid-dynamic limit $\varepsilon \to 0$ we have

$$z = z_E \equiv \frac{\rho^2 + m^2}{2\rho}, \tag{72}$$

and the Broadwell system is well approximated by the reduced system (5) for $n = 2$ with

$$v = (\rho,\, \rho u), \qquad G(v) = \left(\rho u,\, \frac{1}{2}[\rho + \rho u^2]\right), \qquad u = \frac{m}{\rho},$$

which represents the corresponding Euler equations of fluid dynamics.

We have considered a periodic smooth solution with initial data as in [6, 18] given by

$$
\begin{aligned}
\rho(x,0) &= 1 + a_\rho \sin\frac{2\pi x}{L},\\
u(x,0) &= \frac{1}{2} + a_u \sin\frac{2\pi x}{L},\\
z(x,0) &= a_z \frac{\rho(x,0)^2 + m(x,0)^2}{2\rho(x,0)}.
\end{aligned} \tag{73}
$$

Clearly, if $a_z \neq 1$ the initial data is not in equilibrium and we have an initial layer. In our computations we used the parameters

$$
a_\rho = 0.3, \quad a_u = 0.1, \quad a_z = 0.2, \quad L = 20,
$$

and we integrate the equations for $t \in [0, 30]$.

The results for the relative L_∞ norm of the errors for the conserved quantity ρ and for the nonconserved quantity z are reported in Table 1 and the corresponding convergence rates are reported in Table 2. We have chosen $\alpha = 1/3$ for scheme PCC and $\alpha = 5/3$ for scheme RKC. However, other values of α in the L-stability range give similar results. The CFL condition is $\Delta t/\Delta x = 1/3$ and the UNO limiter has been used. The extrapolation in the first step for scheme PCCF and in the last step for scheme RKCF have been implemented for simplicity directly using the schemes themselves. As expected, a reduction of accuracy of schemes PCC and RKC in the nonconserved variables is observed. It is evident that both the PCCF and RKCF schemes provide the desired uniform second order accuracy in the conserved as well as in the nonconserved variables. Note that RKCF is more accurate than scheme PCCF which presents a slight degradation of accuracy in the intermediate regimes for the nonconserved quantity. The magnitude of the errors for the conserved variables is comparable to the results recently presented in [18] but with better computational efficiency.

Next, we test the shock-capturing properties of the schemes in the case of nonsmooth solutions characterized by the following two Riemann problems [6]:

$$
\begin{aligned}
&\rho_l = 2, \quad m_l = 1, \quad z_l = 1, \quad x < 0.2,\\
&\rho_r = 1, \quad m_r = 0.13962, \quad z_r = 1, \quad x > 0.2,
\end{aligned} \tag{74}
$$

$$
\begin{aligned}
&\rho_l = 1, \quad m_l = 0, \quad z_l = 1, \quad x < 0,\\
&\rho_r = 0.2, \quad m_r = 0, \quad z_r = 1, \quad x > 0.
\end{aligned} \tag{75}
$$

The results are shown in Figures 2 and 3 for a CFL condition of $\Delta t/\Delta x = 1/3$. Both schemes, as expected, give an accurate description of the solution in all different regimes also using coarse meshes that do not resolve the small scales. In particular the shock formation in the fluid limit is well captured without spurious oscillations. We refer to [6, 13, 18, 24, 1] for a comparison of the present results with previous ones. Next, in Figure 4 we emphasize the different behavior of the schemes in the case of initial layer and in evaluating the distance from equilibrium. The improvement of accuracy due to the initial and final extrapolations is evident.

TABLE 1
*Accuracy test, L_∞ norm errors (in units of 10^{-3}).**

ε	1.0	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	
Scheme				Relative errors for ρ				N
PCC	0.156088	0.340207	0.414980	0.426457	0.427747	0.427876	0.427889	100
	0.037920	0.115425	0.157950	0.162140	0.162666	0.162721	0.162726	200
	0.009359	0.040442	0.070713	0.072268	0.072519	0.072548	0.072551	400
RKC	0.183913	0.248758	0.268285	0.274410	0.275367	0.275465	0.275475	100
	0.043718	0.058525	0.062650	0.064783	0.065284	0.065336	0.065342	200
	0.010614	0.014084	0.015113	0.015601	0.015832	0.015859	0.015862	400
PCCF	0.177767	0.248930	0.278652	0.281968	0.281309	0.281451	0.281461	100
	0.042111	0.057579	0.071300	0.068225	0.066985	0.067095	0.067101	200
	0.010208	0.013673	0.017737	0.017268	0.016255	0.016312	0.016317	400
RKCF	0.187796	0.255287	0.275393	0.282256	0.283302	0.283409	0.283420	100
	0.044189	0.059333	0.063576	0.065751	0.066312	0.066371	0.066377	200
	0.010669	0.014188	0.015233	0.015711	0.015941	0.015969	0.015971	400
				Relative errors for z				
PCC	0.316649	0.876426	0.982754	0.996487	0.998066	0.998226	0.998242	100
	0.083132	0.341265	0.414507	0.417861	0.418347	0.418399	0.418404	200
	0.021503	0.125947	0.191382	0.192039	0.192215	0.192239	0.192242	400
RKC	0.233554	0.318965	0.438123	0.558334	0.569924	0.570346	0.570376	100
	0.055991	0.075030	0.112013	0.183210	0.196598	0.196977	0.196991	200
	0.013737	0.018175	0.028674	0.064705	0.079302	0.079850	0.079861	400
PCCF	0.225721	0.310813	0.390588	0.346528	0.340753	0.341205	0.341222	100
	0.053785	0.066234	0.128782	0.095851	0.079851	0.080403	0.080418	200
	0.013156	0.014668	0.037960	0.031686	0.019124	0.019545	0.019565	400
RKCF	0.239403	0.310855	0.295029	0.326014	0.340293	0.340815	0.340835	100
	0.056743	0.073544	0.059888	0.066784	0.077785	0.078566	0.078584	200
	0.013832	0.017957	0.014577	0.012936	0.017799	0.018933	0.018955	400

*Note: The underlined values indicate the smallest L_∞ norm errors for fixed ε and N.

TABLE 2
Accuracy test, convergence rates.

ε	1.0	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	
Scheme				Convergence rates for ρ				N
PCC	2.041345	1.559460	1.393572	1.395157	1.394844	1.394794	1.394789	100–200
	2.018553	1.513026	1.159427	1.165810	1.165483	1.165396	1.165387	200–400
RKC	2.072731	2.087625	2.098374	2.082638	2.076557	2.075912	2.075848	100–200
	2.042282	2.054974	2.051493	2.053945	2.043920	2.042585	2.042449	200–400
PCCF	2.077721	2.112119	1.966498	2.047159	2.070245	2.068608	2.068527	100–200
	2.044539	2.074181	2.007092	1.982191	2.042974	2.040301	2.040004	200–400
RKCF	2.087401	2.105205	2.114931	2.101917	2.095001	2.094265	2.094192	100–200
	2.050203	2.064147	2.061288	2.065251	2.056496	2.055314	2.055192	200–400
				Convergence rates for z				
PCC	1.929399	1.360740	1.245435	1.253828	1.254434	1.254488	1.254493	100–200
	1.950904	1.438073	1.114938	1.121625	1.121979	1.121974	1.121972	200–400
RKC	2.060491	2.087858	1.967665	1.607631	1.535524	1.533811	1.533782	100–200
	2.027090	2.045509	1.965834	1.501554	1.309825	1.302670	1.302573	200–400
PCCF	2.069265	2.230395	1.600709	1.854098	2.093343	2.085328	2.085126	100–200
	2.031510	2.174882	1.762384	1.596958	2.061907	2.040474	2.039201	200–400
RKCF	2.076925	2.079570	2.300517	2.287351	2.129206	2.117009	2.116772	100–200
	2.036402	2.034042	2.038568	2.368099	2.127720	2.052993	2.051640	200–400

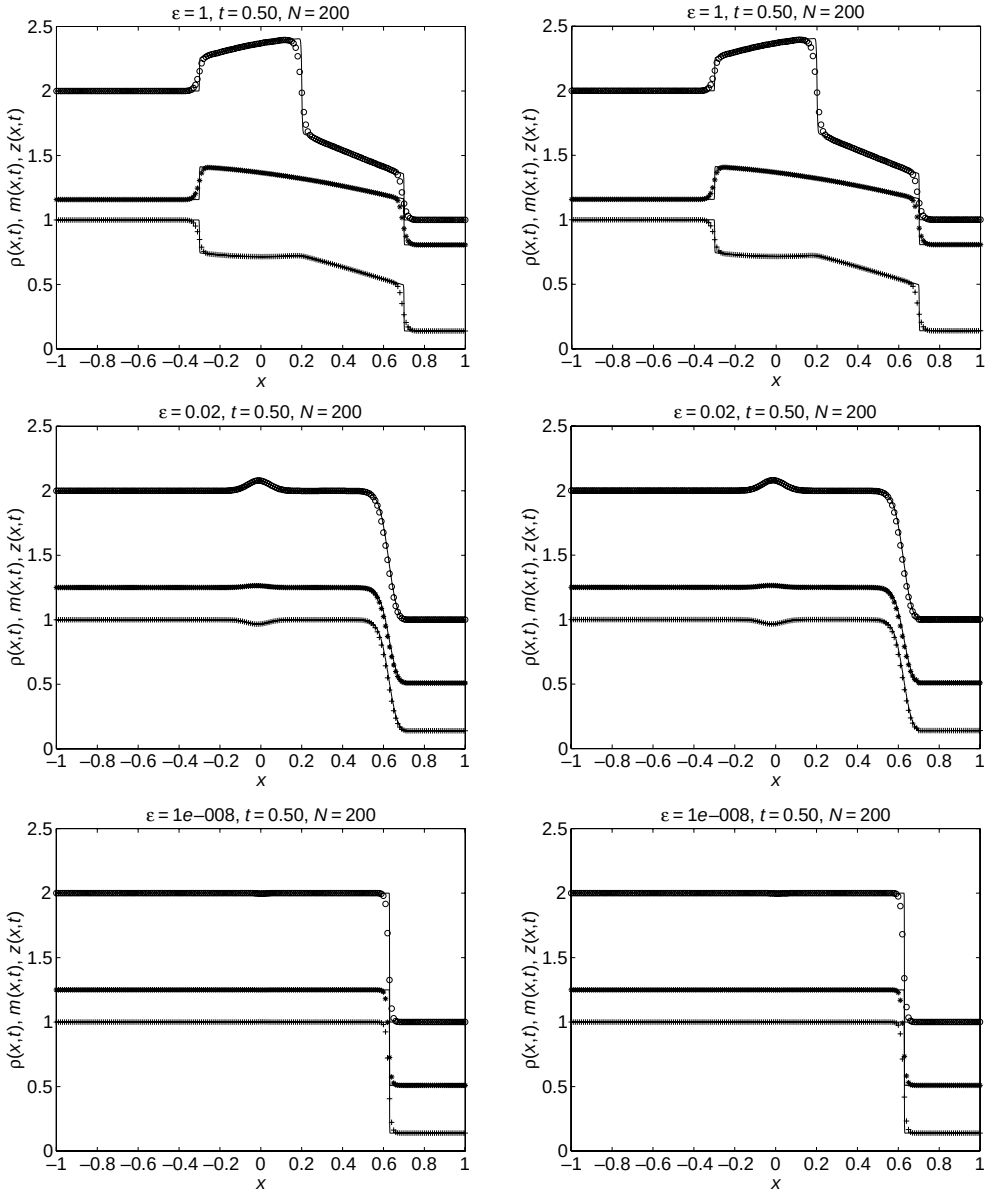

FIG. 2. *Numerical solution of the Broadwell equations with initial data* (74) *for* ρ ($\circ$), m ($+$), *and* z ($*$) *at time* $t = 0.5$. *Left column: PCCF scheme. Right column: RKCF scheme. From top to bottom:* $\varepsilon = 1.0,\ 0.02,\ 10^{-8}$.

5.2. Extended thermodynamics for a monoatomic gas. Finally, we apply our second order central schemes to a more general relaxation system for which an explicit analytical expression of the eigenvalues and eigenvectors is not known. We consider the one-dimensional model of extended thermodynamics for a monoatomic gas [21, 18]. In conservative form the equations can be written as a relaxation system of type (1) for $N = 5$ with

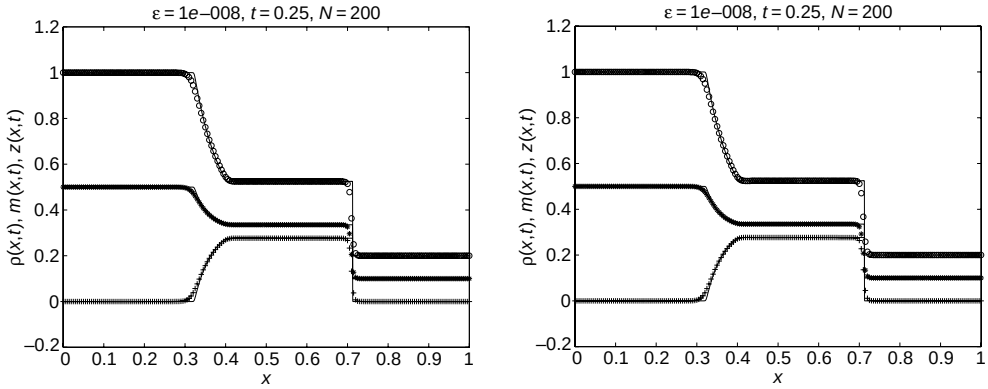

FIG. 3. *Numerical solution of the Broadwell equations with initial data* (75) *for* ρ ($\circ$), m ($+$), *and* z ($*$) *at time* $t = 0.25$ *for* $\varepsilon = 10^{-8}$. *Left: PCCF scheme. Right: RKCF scheme.*

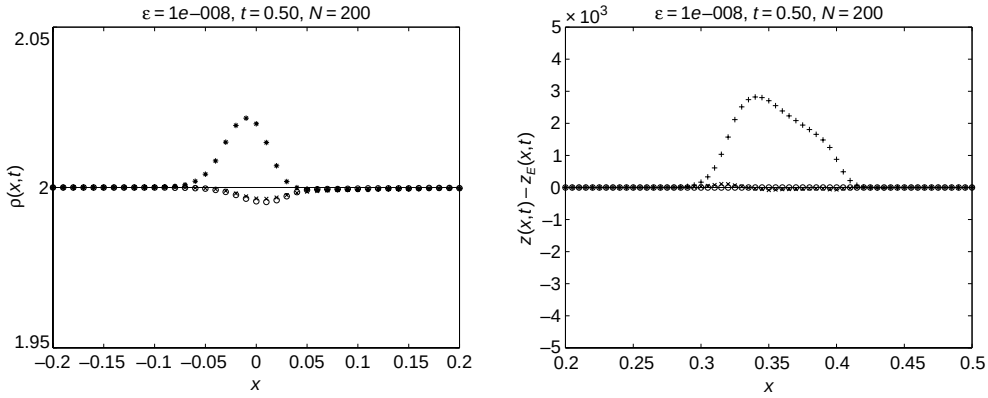

FIG. 4. *Initial layer for* ρ *in problem* (74) (*left*) *and departure from equilibrium* $z - z_E$ *in problem* (75) (*right*) *for* $\varepsilon = 10^{-8}$. *The x-scale has been magnified for clarity. Left problem: PCC* ($*$), *PCCF* ($\circ$), *RKCF* ($\times$). *Right problem: RKC* ($+$), *RKCF* ($\times$), *PCCF* ($\circ$).

$$
\begin{aligned}
U &= (\rho,\, m,\, z,\, w,\, h),\\
R(U) &= \left(0,\, 0,\, 0,\, \frac{2}{3}m^2 - w\rho,\, \frac{2}{3}\left[\frac{10}{3}mz - mw - h\rho\right]\right),\\
F(U) &= \left(m,\, \frac{2}{3}z + w,\, \frac{1}{2}h,\, \frac{1}{\rho^2}\left[\frac{9}{5}mw\rho + \frac{4}{15}h\rho^2 - \frac{4}{5}m^3\right],\right.\\
&\qquad \left.\frac{2}{\rho^3}\left\{\rho^2\left[\frac{7}{3}wz + \frac{10}{9}z^2 + \frac{8}{5}mh\right] + m^2\left[\frac{79}{45}m^2 - \frac{16}{3}\rho z - \frac{28}{15}w\rho\right]\right\}\right).
\end{aligned}
$$

For monoatomic gases ε is a constant proportional to the relaxation time of the system. The conserved quantities are the density ρ, the momentum m, and the energy z. Other quantities of physical interest are the pressure p, the heat flux q, and the stress σ given by

$$
p = \frac{2\rho z - m^2}{3\rho}, \qquad q = \frac{3\,h\rho^2 + 6\,m^3 - 10\,mz\rho - 6\,mw\rho}{6\rho^2}, \qquad \sigma = \frac{3\,w\rho - 2\,m^2}{3\rho}.
$$

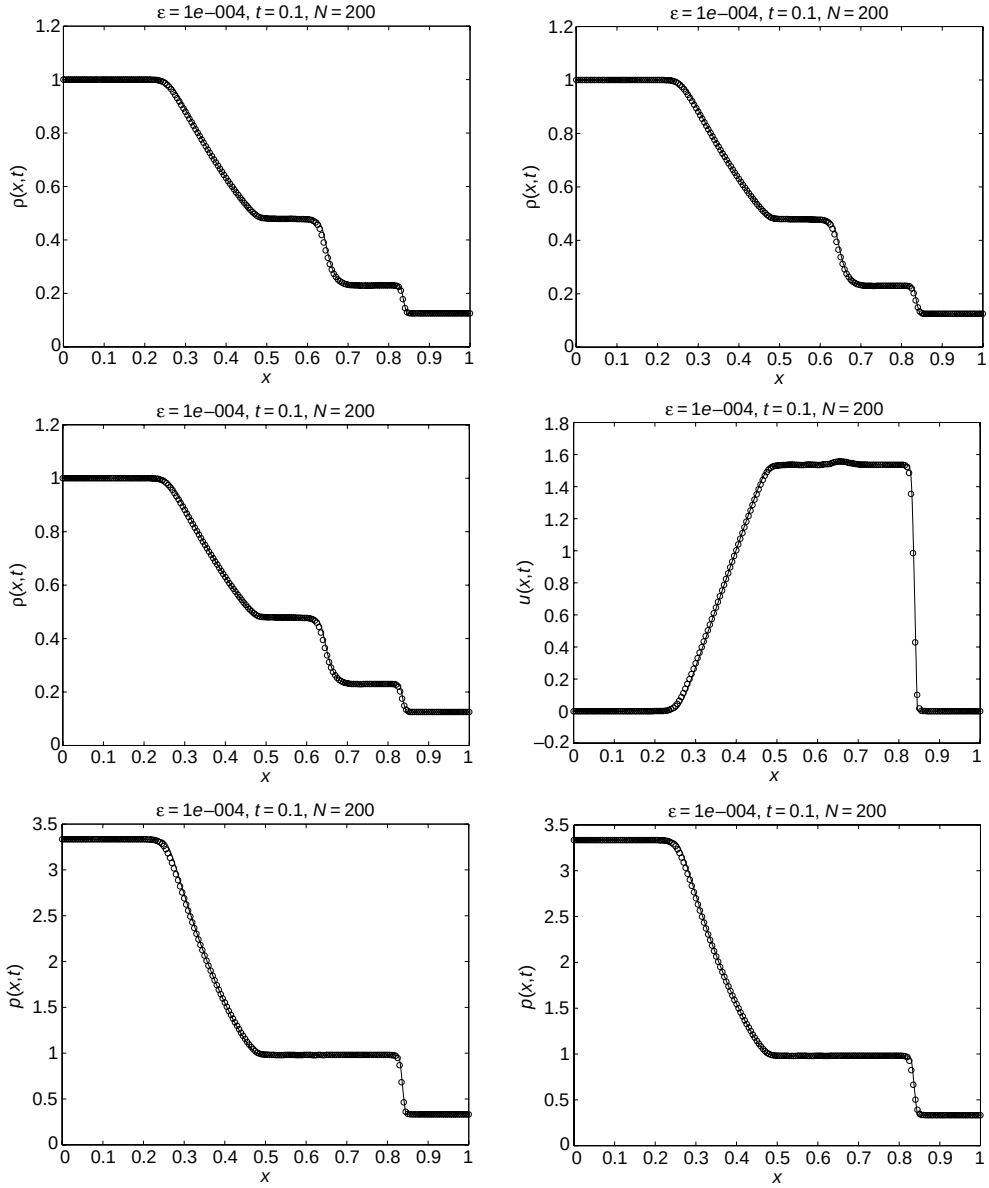

FIG. 5. *Numerical solution of the extended thermodynamics equations with initial data* (77) *for* $\varepsilon = 10^{-4}$ *at time* $t = 0.1$. *Left column: PCCF scheme. Right column: RKCF scheme. From top to bottom: density* ρ, *velocity* u, *and pressure* p.

As $\varepsilon \to 0$ we obtain the equilibrium relations

$$w = w_E \equiv \frac{2m^2}{3\rho}, \qquad h = h_E \equiv \frac{mw}{\rho} + \frac{10mz}{3\rho} \tag{76}$$

that correspond to $\sigma = q = 0$ and provide the usual Euler equations for a monoatomic

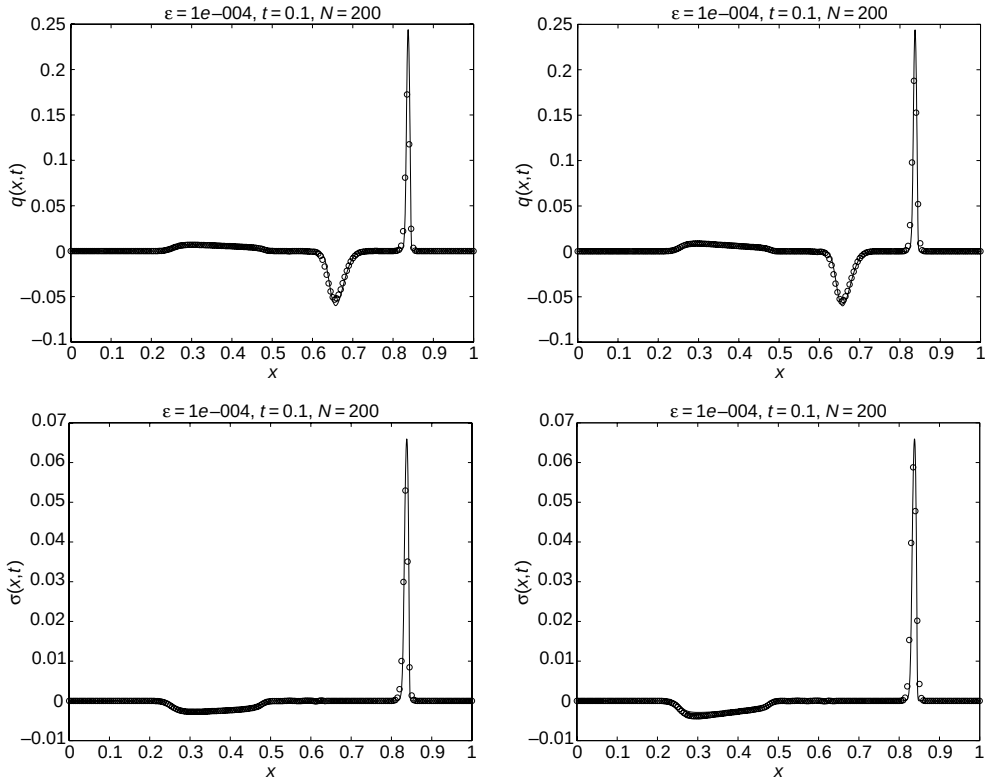

FIG. 6. *Numerical solution of the extended thermodynamics equations with initial data* (77) *for* $\varepsilon = 10^{-4}$ *at time* $t = 0.1$. *Left column: PCCF scheme. Right column: RKCF scheme. From top to bottom: heat flux* q *and stress* σ.

gas characterized by the reduced system (5) for $n = 3$ with

$$v = \left(\rho,\, \rho u,\, \frac{1}{2}\rho u^2 + \frac{3}{2}p\right), \qquad G(v) = \left(\rho u,\, \rho u^2 + p,\, \frac{1}{2}\rho u^3 + \frac{5}{2}up\right), \qquad u = \frac{m}{\rho}.$$

We test our schemes for a generalization of the classical Sod's problem [31]

$$\begin{aligned} &U = U_l = (1, 0, 5, 0, 0), && x < 0.5, \\ &U = U_r = (0.125, 0, 0.5, 0, 0), && x > 0.5 \end{aligned} \tag{77}$$

that corresponds to the test problem used in [18] (except for the initial value of the pressure).

The numerical results are presented in Figures 5 and 6 for a CFL condition of $\Delta t/\Delta x = 0.1$. It appears that both schemes give a very accurate description of the behavior of the system also using underresolved discretizations. Note that both schemes fully resolve the shear stress and the heat flux profiles using 200 grid points. We refer to [18] for a comparison of the present results.

6. Conclusions. We have presented a new class of efficient underresolved central schemes for the accurate solution of hyperbolic conservation laws with stiff source terms. The schemes use only two evaluations of the flux and the source terms and

provide second order accurate solutions on both conserved as well as nonconserved variables for a wide range of stiffness parameters even if the initial layer is not resolved. Moreover, in the zero relaxation limit they provide a consistent high-resolution scheme for the reduced equilibrium system of conservation laws. Although the schemes have been developed in the context of one-dimensional centrally based space discretizations, the results also apply to other spatial discretizations in more dimensions. For example, the extension of the present schemes to higher dimensions in a staggered central context can be done using the reconstruction strategy presented in [11, 12].

Finally, we remark that the strategy developed here applies to any kind of Runge–Kutta method in a staggered context. For example, this allows us to construct more efficient higher order schemes for systems of conservation laws with [16] or without [25] source terms.

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