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CRITICAL THRESHOLD FOR GLOBAL REGULARITY OF THE EULER–MONGE–AMPÈRE SYSTEM WITH RADIAL SYMMETRY*

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Abstract. We study the global well-posedness of the Euler–Monge–Ampère (EMA) system. We obtain a sharp, explicit critical threshold in the space of initial configurations which guarantees the global regularity of the EMA system with radially symmetric initial data. The result is obtained using two independent approaches—one using the *spectral dynamics* of Liu and Tadmor [*Comm. Math. Phys.*, 228 (2002), pp. 435–466] and the other based on the *geometric approach* of Brenier and Loeper [*Geom. Funct. Anal.*, 14 (2004), pp. 1182–1218]. The results are extended to 2D radial EMA with swirl.

Key words. Euler–Monge–Ampère system, critical threshold, radial symmetry

MSC codes. 35Q35, 35B30, 35K96, 76N10

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1. Introduction.

We are concerned with the global regularity of the pressureless Euler–Monge–Ampère (EMA) system

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1.1a}$$

$$\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = -\kappa \rho \nabla \phi, \tag{1.1b}$$

$$det(\mathbb{I} - D^2 \phi) = \rho, \tag{1.1c}$$

with density $\rho(\cdot, t) : \mathbb{R}^n \mapsto \mathbb{R}_+$, velocity $\mathbf{u}(\cdot, t) : \mathbb{R}^n \mapsto \mathbb{R}^n$, and potential $\phi(\cdot, t) : \mathbb{R}^n \mapsto \mathbb{R}$, subject to the corresponding initial conditions $(\rho_0(\cdot), \mathbf{u}_0(\cdot), \phi_0(\cdot))$ at $t = 0$. We set the constant $\kappa > 0$, which represents a repulsive force. Without loss of generality, we fix the potential under the assumption that $\phi(0) = 0$.

The EMA system (1.1) was introduced and studied by Loeper in [19] around its equilibrium state $(\rho, \mathbf{u}) = (1, \mathbf{0})$. It is closely related to the Euler–Poisson equations in plasma physics,

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1.2a}$$

$$\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = -\kappa \rho \, \nabla \phi, \tag{1.2b}$$

$$-\Delta \phi = \rho - 1. \tag{1.2c}$$

Indeed, these two systems are the same when $n = 1$. In higher dimensions, $n \geqslant 2$, one considers a perturbed solution around the equilibrium state $(1, \mathbf{0})$: rescaling $\phi = \epsilon \varphi$, then

$$\rho = det(\mathbb{I} - \epsilon D^2 \varphi) = 1 - \epsilon \Delta \varphi + \mathcal{O}(\epsilon^2),$$

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which yields (1.2c) modulo $\mathcal{O}(\epsilon^2)$ terms. Hence, we can view the EMA system (1.1) as a nonlinear counterpart of the Euler–Poisson equations (1.2) around the equilibrium state. Interestingly, if we scale $\kappa = \epsilon^{-2}$, both systems converges to the incompressible Euler equations as $\epsilon \to 0$; see, e.g., [3, 4].

The stability of Euler–Poisson equations near the equilibrium state $(\rho, \mathbf{u}) = (1, \mathbf{0})$ was analyzed in [6, 8, 10, 7]. The question of global regularity holds for a larger region in the space of initial configurations: *subcritical* initial data admit global strong solutions, while *supercritical* initial data lead to finite time singularity formations. This is known as the *critical threshold phenomenon.* Threshold conditions for Euler–Poisson equations were found in [5, 16, 22] for the one-dimensional cases and in [24, 11, 23] for the multidimensional cases with radial symmetry. The search for a multidimensional threshold beyond the radial case was addressed in related *restricted* models [17, 14, 13], but the general unrestricted case for (1.2) is still open.

In this paper we study the global regularity of the EMA system (1.1) with radial symmetry subject to subcritical initial data. We obtain a *sharp* and *explicit* critical threshold in the space of initial configurations, which distinguish between initial data admitting globally regular solutions and that admitting solutions with finite-time blowup. We state our main result.

THEOREM 1.1. *Consider the EMA system* (1.1) *with smooth radial initial data of the form*

$$\rho_0(\mathbf{x}) = \rho_0(r), \quad \mathbf{u}_0(\mathbf{x}) = \frac{\mathbf{x}}{r} u_0(r), \quad \phi_0(\mathbf{x}) = \phi_0(r), \quad r = |\mathbf{x}|. \tag{1.3}$$

Specifically, our smoothness assumption requires

$$\mathbf{U}_0(|\mathbf{x}|) := \left[u_0'(|\mathbf{x}|),\ \frac{u_0(|\mathbf{x}|)}{|\mathbf{x}|},\ \phi_0''(|\mathbf{x}|),\ \frac{\phi_0'(|\mathbf{x}|)}{|\mathbf{x}|} \right]^\top \in \left(H^s(\mathbb{R}^n)\right)^4, \qquad s > \frac{n}{2}.$$

Then, we have the following:

- *Subcritical threshold: if the initial condition satisfies*

$$|u_0'(r)| < \sqrt{\kappa(1 - 2\phi_0''(r))} \qquad \textit{for all } r > 0, \tag{1.4}$$

  then the system admits a global smooth solution

$$\mathbf{U} = \left[u'(|\mathbf{x}|, t),\ \frac{u(|\mathbf{x}|, t)}{|\mathbf{x}|},\ \phi''(|\mathbf{x}|, t),\ \frac{\phi'(|\mathbf{x}|, t)}{|\mathbf{x}|} \right]^\top \in C([0, T], (H^s(\mathbb{R}^n))^4)$$

  for any finite time T.
- *Supercritical threshold: if* (1.4) *fails to hold for some $r > 0$, then system* (1.1) *admits a solution which will generate singular shocks (and/or nonphysical shocks) in finite time; namely, there exist a finite critical time T_c and a location $r_c = r(T_c; r_0)$ such that the solution remains smooth in $[0, T_c)$, and*

$$\lim_{t \to T_c-} \partial_r u(r_c, t) = -\infty, \quad \lim_{t \to T_c-} \rho(r_c, t) = +\infty \ (\textit{or } 0). \tag{1.5}$$

Two remarks are in order.

Remark 1.2 (uniqueness). We note that the uniqueness of our radial solution, $U(|\mathbf{x}|, t)$, is dictated by its vanishing behavior at infinity. In particular, the H^s-boundedness of $\phi_{rr}(r, t)$ and $\phi_r(r, t)/r$ implies their vanishing behavior at infinity, and hence ϕ is dictated up to a constant by the Monge–Ampère equation (see its radial version in (3.2) below), which we fixed by setting, say, $\phi(0) = 0$. Furthermore, (3.2) then implies that $(\rho, \mathbf{u})$ approaches the equilibrium state $(1, \mathbf{0})$ at infinity.

Remark 1.3 (bounded away from vacuum). We observe (consult Remark 3.5) that for the n-dimensional EMA threshold (1.4) to hold, we need the lower bound $\rho_0(r) > 2^{-n}$. This is in agreement with the sharp 1D threshold $|u_0'(r)| < \sqrt{\kappa(2\rho_0(r) - 1)}$, which requires the lower bound $\rho_0 > 1/2$—one cannot expect a global smooth solution with initial density which is "far below" the equilibrium state $\rho_0 \equiv 1$. In particular, therefore, the presence of vacuum in the initial data will always lead to shock formations. On the other hand, if ρ_0 is not far below the constant equilibrium state $\rho_0 \equiv 1$, then one can find subcritical initial data with $|u_0'|$ small enough, such that the solutions exist globally in time.

The proof of Theorem 1.1 begins in section 2 with a general framework established in [23] on Eulerian dynamics with radial symmetry, followed by the spectral dynamics of radial EMA in section 3, and in section 4 we complete the proof of Theorem 1.1 by energy estimates.

When the dimension $n = 1$, the Monge–Ampère equation (1.1c) is simply $1 - \phi'' = \rho$. The subcritical global regularity condition (1.4) is reduced to $|u_0'(r)| < \sqrt{\kappa(2\rho_0(r) - 1)}$, and our result recovers the sharp threshold for 1D Euler–Poisson equations obtained in [5]. The interesting part of Theorem 1.1 comes in higher dimensions, $n \geqslant 2$, addressing the fully nonlinear Monge–Ampère part of the EMA system, which seems more difficult to treat when compared with the linear Poisson part in Euler–Poisson equations (1.2). Nevertheless, sharp threshold conditions for the radially symmetric Euler–Poisson equations (consult [24] for $u_0 > 0$ and [23] for general radial data) are stated implicitly and seem to depend on the dimension. This is in contrast to the explicit form of our threshold condition for the fully nonlinear EMA system, (1.4), which is independent of the dimension n.

Another perspective on having such a simple, elegant threshold condition is due to the geometric structure of the Monge–Ampère equation. In section 5, we pursue the geometric approach for the Monge–Ampère equation à la [2, 19], and we rederive the radial threshold condition (1.4).

In section 6, we discuss extension of these results beyond radial configurations. First, we further extend our result to the 2D radial EMA system with swirl.

THEOREM 1.4. *Consider the two-dimensional EMA system* (1.1) *with smooth radial initial data with swirl,*

$$\rho_0(\mathbf{x}) = \rho_0(r), \quad \mathbf{u}_0(\mathbf{x}) = \frac{\mathbf{x}}{r}u_0(r) + \frac{\mathbf{x}^{\perp}}{r}\Theta_0(r), \quad \phi_0(\mathbf{x}) = \phi_0(r), \quad r = |\mathbf{x}|. \tag{1.6}$$

Then, there exists a set $\Sigma \subset \mathbb{R}^6$, defined in (6.1)*, such that the following hold:*

- *Subcritical threshold: if the initial condition satisfies*

$$\left(u_0'(r), \frac{u_0(r)}{r}, \Theta_0'(r), \frac{\Theta_0(r)}{r}, \phi_0''(r), \frac{\phi_0'(r)}{r}\right) \in \Sigma \tag{1.7}$$

  for all $r > 0$, then the system admits a global smooth solution.
- *Supercritical threshold: if there exists an $r > 0$ such that* (1.7) *is violated, then the solution will become singular in finite time.*

Note that the set Σ is implicitly defined. It is not clear whether the threshold condition (1.7) can be expressed explicitly. This indicates that rotation adds another layer of intrinsic difficulty in extending our theory to general data.

Finally, we comment on the difficulties in both the approach based on spectral dynamics and the geometric approach in extending these results to the case of general data.

2. Preliminaries: Eulerian dynamics with radial symmetry.

The EMA system (1.1) falls into a general framework of pressureless Eulerian dynamics,

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0,$$
$$\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = \rho \mathbf{F},$$

where the force $\mathbf{F} = -\kappa \nabla \phi$ and the potential ϕ satisfies the Monge–Ampère equation (1.1c). The momentum equation can be equivalently written as the following dynamics of the velocity $\mathbf{u}$ in the nonvacuous region:

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{F}. \tag{2.1}$$

The equation is the usual Eulerian-type nonlinearity, and it is well known that the uniform boundedness of the $n \times n$ velocity gradient matrix, $\|\nabla \mathbf{u}(\cdot, t)\|_{L^\infty} < \infty$, is the key to global regularity. Taking the spatial gradient of (2.1) would yield

$$(\partial_t + \mathbf{u} \cdot \nabla)\nabla \mathbf{u} + (\nabla \mathbf{u})^2 = \nabla \mathbf{F}. \tag{2.2}$$

2.1. Spectral dynamics.

Let $\lambda_i = \lambda_i(\nabla \mathbf{u}), i = 1, 2, \ldots, n$, be the n eigenvalues of $\nabla \mathbf{u}$. Then, the spectral dynamics (2.2) can be written as

$$(\partial_t + \mathbf{u} \cdot \nabla)\lambda_i + \lambda_i^2 = \langle \mathbf{l}_i, (\nabla \mathbf{F})\mathbf{r}_i \rangle, \quad i = 1, \ldots, n, \tag{2.3}$$

where $(\mathbf{l}_i, \mathbf{r}_i)$ are the corresponding left and right eigenvectors of λ_i.

Spectral dynamics has been studied extensively in [16]. Although one can largely benefit from the explicit Ricatti structure, $(2.3)_\mathrm{i}$, it is a delicate step to control the term on the right, $\langle \mathbf{l}_i, (\nabla \mathbf{F})\mathbf{r}_i \rangle$, since in many cases $\nabla \mathbf{F}$ need not share the same eigensystem with $\nabla \mathbf{u}$; instead, one seeks an invariant expressed in terms of this eigensystem. As a typical example, one studies the dynamics of the *divergence,*

$$d := \nabla \cdot \mathbf{u} = \operatorname{trace}(\nabla \mathbf{u}) = \sum_{i=1}^{n} \lambda_i(\nabla \mathbf{u}).$$

Taking the trace of (2.3) would yield

$$(\partial_t + \mathbf{u} \cdot \nabla)d + \operatorname{trace}\big((\nabla \mathbf{u})^2\big) = \nabla \cdot \mathbf{F}.$$

While the forcing term $\nabla \cdot \mathbf{F}$ can be easier to control, one loses the explicit Ricatti structure encoded in the difference, $\operatorname{trace}\big((\nabla \mathbf{u})^2\big) - d^2 \neq 0$ for $n \geqslant 2$. The difference is related to the *spectral gap*, $\lambda_1(\nabla \mathbf{u}) - \lambda_2(\nabla \mathbf{u})$ (particularly in 2D). Examples are found in [17, 21, 9].

2.2. Radially symmetric solutions.

We focus on a special type of solution for the EMA system (1.1), with radial symmetry and without swirl,

$$\rho(\mathbf{x}, t) = \rho(r, t), \quad \mathbf{u}(\mathbf{x}, t) = \frac{\mathbf{x}}{r} u(r, t), \quad \phi(\mathbf{x}, t) = \phi(r, t). \tag{2.4}$$

Here, $r = |\mathbf{x}| \in \mathbb{R}_+$ is the radial variable, and ρ, u, and ϕ are scalar functions defined in $\mathbb{R}_+ \times \mathbb{R}_+$. To ensure regularity at the origin, we impose boundary conditions at $r = 0$,

$$\partial_r \rho(0, t) = 0, \quad u(0, t) = 0, \quad \partial_r \phi(0, t) = 0. \tag{2.5}$$

The persistence of such no-swirl solutions follows by noting that the velocity field is induced by radial potential,

$$\mathbf{u}(\mathbf{x},t) = \nabla U(\mathbf{x},t), \quad U(\mathbf{x},t) := \int_0^{|\mathbf{x}|} u(s,t)\mathrm{d}s,$$

in which case (2.1) with $\mathbf{F} = -\kappa\nabla\phi$ is encoded as an Eikonal equation,

$$\partial_t U + \frac{1}{2}|\nabla U|^2 = -\kappa\phi. \tag{2.6}$$

The gradient of (2.6) yields the momentum equation (1.1b). Take the Hessian of (2.6) to recover the velocity gradient equation (2.2). The next lemma, which is at the heart of matter, recalls that radial Hessians are rank-one modifications of a scalar matrix, and hence they all share the same eigenvectors; see, e.g., [20, eq. (1.3)].

Lemma 2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a radial scalar field $f(\mathbf{x}) = f(|\mathbf{x}|)$. Then, for any $\mathbf{x} \neq \mathbf{0}$ the eigensystem of its Hessian $D^2 f(\mathbf{x})$ is characterized by two distinct eigenvalues given by*
- $\lambda_1(D^2 f) = f'(|\mathbf{x}|)$ *associated with the eigenvector $\mathbf{v}_1 = \mathbf{x}$;*
- $\lambda_2(D^2 f) = \cdots = \lambda_n(D^2 f) = \frac{f(|\mathbf{x}|)}{|\mathbf{x}|}$ *associated with $\{\mathbf{v}_2,\ldots,\mathbf{v}_n\}$, which span $\{\mathbf{x}^\perp\}$.*

Indeed, the Hessian $D^2 f$ is given by

$$D^2 f(\mathbf{x}) = \frac{f(r)}{r}\,\mathbb{I} + \frac{1}{r^2}\left(f'(r) - \frac{f(r)}{r}\right)\mathbf{x}\,\mathbf{x}^\top, \qquad r := |\mathbf{x}|.$$

A straightforward computation yields

$$\big(\lambda_1\mathbb{I} - D^2 f(\mathbf{x})\big)\mathbf{v}_1 = \left(f'(r) - \frac{f(r)}{r}\right)\left(\mathbf{x} - \frac{1}{r^2}\mathbf{x}\,\langle\mathbf{x},\mathbf{x}\rangle\right) = 0,$$

and for any $\mathbf{v}$ such that $\langle\mathbf{x},\mathbf{v}\rangle = 0$,

$$\big(\lambda_2\mathbb{I} - \nabla f(\mathbf{x})\big)\mathbf{v} = -\frac{1}{r^2}\left(f'(r) - \frac{f(r)}{r}\right)\mathbf{x}\,\langle\mathbf{x},\mathbf{v}\rangle = 0.$$

It follows that the Hessians of all radial scalar fields share the same eigensystem. In particular, the velocity gradient $\nabla\mathbf{u}(\mathbf{x},t) = D^2U(r)$ and the forcing gradient $\nabla\mathbf{F}(\mathbf{x},t) = -\kappa D^2\phi(r)$ share the same eigenvectors. Hence, we can "diagonalize" the spectral dynamics (2.3) in terms of the distinct eigenvalues of $D^2U(r)$ and of $D^2\phi(r)$, independently of the corresponding eigenvectors

$$\begin{cases} p' = -p^2 - \kappa\mu, \\ q' = -q^2 - \kappa\nu. \end{cases} \tag{2.7}$$

Here, $' := \partial_t + u(r,t)\partial_r$ denotes differentiation along particle paths, (p,q) denote the two distinct eigenvalues of the velocity gradient $\nabla\mathbf{u}(r,t)$,

$$p(r,t) = \lambda_1(\nabla\mathbf{u}(\mathbf{x},t)) = \partial_r u(r,t), \quad q(r,t) = \lambda_2(\nabla\mathbf{u}(\mathbf{x},t)) = \frac{u(r,t)}{r}, \tag{2.8}$$

and (μ,ν) are the eigenvalues of potential gradient $D^2\phi(r,t)$,

$$\mu(r,t) = \lambda_1(D^2\phi(\mathbf{x},t)) = \partial_r^2\phi(r,t), \quad \nu(r,t) = \lambda_2(D^2\phi(\mathbf{x},t)) = \frac{\partial_r\phi(r,t)}{r}. \tag{2.9}$$

The following lemma shows that the boundedness of the radial derivative $p = \partial_r u(r,t)$ is sufficient for guaranteeing the boundedness of $\nabla \mathbf{u}$.

LEMMA 2.2. *Consider the radial velocity field* (2.4), (2.5), *where* $\mathbf{u}(\mathbf{x},t) = \frac{\mathbf{x}}{r} u(r,t)$ *and* $u(0,t) = 0$. *Then*

$$\|\nabla \mathbf{u}(\cdot,t)\|_{L^\infty} \leqslant \|\partial_r u(r,t)\|_{L^\infty}. \tag{2.10}$$

To verify (2.10) recall that $\nabla \mathbf{u}(\mathbf{x},t)$ is given by the radial Hessian

$$\nabla \mathbf{u}(\mathbf{x},t) = D^2 U(r) = q(r,t)\,\mathbb{I} + \big(p(r,t) - q(r,t)\big)\,\frac{\mathbf{x}\,\mathbf{x}^\top}{r^2}, \qquad \begin{cases} p(r,t) = \partial_r u(r,t), \\ q(r,t) = \dfrac{u(r,t)}{r}, \end{cases}$$

and hence for arbitrary unit vector $\mathbf{w}$,

$$\langle \nabla \mathbf{u}(\mathbf{x},t)\mathbf{w}, \mathbf{w}\rangle = \theta p + (1-\theta) q \leqslant \max\{p,q\}, \qquad \theta = \frac{|\langle \mathbf{x}, \mathbf{w}\rangle|^2}{r^2} \in [0,1].$$

Moreover, by (2.5), $u(0,t) = 0$ and hence

$$|q(r,t)| = \frac{1}{r}\left|\int_0^r \partial_s u(s,t)\,\mathrm{d}s\right| \leqslant \|p(\cdot,t)\|_{L^\infty},$$

and (2.10) follows from the last two inequalities.

3. Spectral dynamics for the radial Euler–Monge–Ampère system.

3.1. Thresholds for the Euler–Monge–Ampère system. In this section, we aim to study the spectral dynamics (2.7) of the radial EMA system. The goal is to obtain an L^∞ bound on p.

Let us start with expressing the Monge–Ampère equation (1.1c) as

$$\rho = det(\mathbb{I} - D^2\phi) = \prod_{i=1}^{n} (1 - \lambda_i(D^2\phi)) = (1-\mu)(1-\nu)^{n-1}. \tag{3.1}$$

From the definition (2.9), we observe the following relation between μ and ν:

$$\mu = \partial_r(r\nu) = r\partial_r \nu + \nu.$$

Hence, we have

$$(1-\mu)(1-\nu)^{n-1} = (1-\nu)^n - r\partial_r\nu\,(1-\nu)^{n-1} = \frac{1}{nr^{n-1}}\,\partial_r\big(r^n(1-\nu)^n\big).$$

Then, the Monge–Ampère equation (3.1) amounts to

$$\partial_r\big(r^n(1-\nu)^n\big) = nr^{n-1}\rho. \tag{3.2}$$

LEMMA 3.1. *Let* $\rho(\mathbf{x},t) = \rho(r,t)$ *be a radial solution of the continuity equation* (1.1a). *Define*

$$e(r,t) = \int_0^r s^{n-1}\rho(s,t)\,\mathrm{d}s. \tag{3.3}$$

Then, e *satisfies the transport equation*

$$e' = \partial_t e + u\partial_r e = 0.$$

Proof. Under radial symmetry (2.4), the continuity equation (1.1a) can be written as

$$\partial_t\rho + \partial_r(\rho u) = -\frac{(n-1)\rho u}{r}. \tag{3.4}$$

Then, we can compute

$$\begin{aligned}\partial_r(e') &= \partial_t\partial_r e + \partial_r(u\partial_r e) = \partial_t\big(r^{n-1}\rho\big) + \partial_r\big(r^{n-1}\rho u\big)\\ &= r^{n-1}\left(\partial_t\rho + \partial_r(\rho u) + \frac{(n-1)\rho u}{r}\right) = 0.\end{aligned}$$

This implies that $e'(\cdot,t)$ is a constant. By definition, $e'(0,t) = 0$. Therefore, we conclude that $e' = 0$. ∎

From (3.2), we get $r^n(1-\nu)^n = ne$. Applying Lemma 3.1, we obtain

$$\big(r(1-\nu)\big)' = 0.$$

Note that $r = r(t;r_0)$ is the characteristic path initiated at r_0, satisfying

$$r' = u(r,t), \quad r(0;r_0) = r_0.$$

This implies the dynamics of ν,

$$\nu' = \frac{r'}{r}(1-\nu) = \frac{u}{r}(1-\nu) = q(1-\nu). \tag{3.5}$$

The dynamics of (q,ν) form a closed ODE system along characteristic paths,

$$\begin{cases} q' = -q^2 - \kappa\nu,\\ \nu' = q(1-\nu). \end{cases} \tag{3.6}$$

PROPOSITION 3.2. *Consider the ODE system* (3.6) *with initial condition* (q_0,ν_0). *Then, the solution remains bounded in all time if and only if*

$$|q_0| < \sqrt{\kappa(1-2\nu_0)}. \tag{3.7}$$

Moreover, if (3.7) *is violated, there exists a finite time* T_c*, such that*

$$\lim_{t\to T_c^-} q(t) = -\infty, \quad \lim_{t\to T_c^-}\nu(t) = \begin{cases} \infty, & \nu_0 > 1,\\ 1, & \nu_0 = 1,\\ -\infty, & \nu_0 < 1. \end{cases}$$

Proof. Let us first consider the case when $\nu_0 \geqslant 1$. We claim that the solution must blow up in finite time. Suppose (q,ν) are bounded in any finite time. Then, we have

$$\nu(t) = 1 + (\nu_0 - 1)\exp\left[\int_0^t q(s)\,ds\right] \geqslant 1 \quad \text{for all } t \geqslant 0.$$

Then, we get

$$q' \leqslant -q^2 - \kappa,$$

which must blow up in finite time; namely, there exists a T_c such that

$$\lim_{t\to T_c^-} q(t) = -\infty.$$

This leads to a contradiction. Furthermore, if $\nu_0 > 1$, we have

$$\lim_{t\to T_c^-} \nu(t) = \infty.$$

If $\nu_0 = 1$, then $\nu(t) \equiv 1$. This corresponds to the case when $\rho(t) \equiv 0$.

Next, we consider the case $\nu_0 < 1$. Define

$$w = \frac{q}{1-\nu}, \quad v = \frac{1}{1-\nu}. \tag{3.8}$$

The dynamics of (w, v) forms a linear system

$$\begin{aligned} w' &= \frac{q'(1-\nu) + q\nu'}{(1-\nu)^2} = \frac{(-q^2 - \kappa\nu)(1-\nu) + q^2(1-\nu)}{(1-\nu)^2} = \frac{-\kappa\nu}{1-\nu} = \kappa(1-v), \\ v' &= \frac{\nu'}{(1-\nu)^2} = \frac{q}{1-\nu} = w. \end{aligned}$$

The trajectory is an ellipse in the (w, v) phase plane. Indeed, we have

$$\big(w^2 + \kappa(1-v)^2\big)' = 2w \cdot k(1-v) + 2\kappa(1-v)\cdot(-w) = 0.$$

The only possible blowup is when $v \to 0$. Clearly, v remains away from zero if and only if the initial condition satisfies

$$w_0^2 + \kappa(1-v_0)^2 < \kappa.$$

Expressing the condition in (q_0, ν_0), we end up with (3.7).

If (3.7) is violated, there exists a time T_c such that $v(T_c) = 0$. Then, we have

$$\begin{aligned} &\lim_{t\to T_c^-} \nu(t) = 1 - \lim_{t\to T_c^-} \frac{1}{v(t)} = -\infty, \\ &\lim_{t\to T_c^-} q(t) = \lim_{t\to T_c^-} \frac{\nu'(t)}{1-\nu(t)} = -\lim_{t\to T_c^-} \big(\log(1-\nu(t))\big)' = -\infty. \qquad \square \end{aligned}$$

Next, we discuss the dynamics of μ. From the Monge–Ampère equation (3.1), we have

$$\mu = 1 - \frac{\rho}{(1-\nu)^{n-1}}.$$

Recall the dynamic of ρ (3.4),

$$\rho' = -\rho\,\partial_r u - \frac{(n-1)\rho u}{r} = -\rho\big(p + (n-1)q\big).$$

This, together with (3.5), implies

$$\begin{aligned} \mu' &= -\frac{\rho'(1-\nu)^{n-1} + (n-1)\rho(1-\nu)^{n-2}\nu'}{(1-\nu)^{2n-2}} = -\frac{-\rho\big(p+(n-1)q\big) + (n-1)\rho q}{(1-\nu)^{n-1}} \\ &= \frac{\rho p}{(1-\nu)^{n-1}} = p(1-\mu). \end{aligned}$$

Therefore, the dynamics of (p, μ) also forms a closed ODE system along characteristic paths,

$$\begin{cases} p' = -p^2 - \kappa\mu, \\ \mu' = p(1-\mu). \end{cases} \tag{3.9}$$

Observe that it is the same as the dynamics of (q, ν) in (3.6). We obtain the same critical threshold condition.

PROPOSITION 3.3. *Consider the ODE system* (3.9) *with initial condition* (p_0, μ_0). *Then, the solution remains bounded in all time if and only if*

$$|p_0| < \sqrt{\kappa(1 - 2\mu_0)}. \tag{3.10}$$

Moreover, if (3.7) *is violated, there exists a finite time* T_c, *such that*

$$\lim_{t \to T_c^-} p(t) = -\infty, \quad \lim_{t \to T_c^-} \mu(t) = \begin{cases} \infty, & \mu_0 > 1, \\ 1, & \mu_0 = 1, \\ -\infty, & \mu_0 < 1. \end{cases} \tag{3.11}$$

We end up with the following sharp critical threshold result for the radial EMA system.

THEOREM 3.4. *Let* $(\rho, \mathbf{u}, \phi)$ *be a classical solution of the EMA system* (1.1) *with radial symmetry* (2.4).

- *If the initial condition satisfies*

$$|u_0'(r)| < \sqrt{\kappa(1 - 2\phi_0''(r))} \tag{3.12}$$

for all $r > 0$, *then* ρ *and* $\nabla \mathbf{u}$ *are uniformly bounded in all time.*

- *If there exists an* $r > 0$ *such that* (3.12) *is violated, then there exist a location* r_c *and a finite time* T_c, *such that*

$$\lim_{t \to T_c^-} u_r(r_c, t) = -\infty, \quad \lim_{t \to T_c^-} \rho(r_c, t) = \infty \text{ (or } 0). \tag{3.13}$$

Proof. For subcritical initial data satisfying (3.12), we can apply Proposition 3.3 along all characteristic paths and obtain boundedness of $\|p(\cdot, t)\|_{L^\infty}$ and $\|\mu(\cdot, t)\|_{L^\infty}$ in all time. Then, uniform boundedness on $\|\nabla \mathbf{u}(\cdot, t)\|_{L^\infty}$ follows directly from Lemma 2.2.

To obtain boundedness of ρ, we recall that $\rho = (1 - \mu)(1 - \nu)^{n-1}$. Therefore, it suffices to show boundedness of ν. Through an argument similar to Lemma 2.2, we have

$$|\partial_r \phi(r, t)| = \left| \partial_r \phi(0, t) + \int_0^r \partial_r^2 \phi(s, t) \, \mathrm{d}s \right| \leqslant r \|\mu(\cdot, t)\|_{L^\infty}.$$

Hence, $\|\nu(\cdot, t)\|_{L^\infty} \leqslant \|\mu(\cdot, t)\|_{L^\infty}$. Consequently, $\|\rho(\cdot, t)\|_{L^\infty} \leqslant \|\mu(\cdot, t)\|_{L^\infty}^n$ is bounded.

For supercritical initial data, suppose (3.12) is violated at $r = r_0 > 0$. Then, applying Proposition 3.3, the solution of the ODE system (3.9) with initial conditions $p(0) = u_0'(r_0)$ and $\mu(0) = \phi_0''(r_0)$ becomes unbounded in finite time T_c at the location $r_c = r(T_c; r_0)$. Moreover, if the solution is smooth in $[0, T_c)$, (3.11) directly implies (3.13). In particular, the case $\rho(r_c, T_c) = 0$ only happens if $\mu_0 = 1$ or, equivalently, $\rho_0(r_0) = 0$. ∎

Remark 3.5. According to (3.7) and (3.10), global solutions of their respective ODEs require that $\nu_0 > 1/2$ and, respectively, $\mu > 1/2$, and hence a global smooth solution of (3.1) requires

$$\rho_0 = det(\mathbb{I} - D^2\phi) = (1 - \mu_0)(1 - \nu_0)^{n-1} > \frac{p_0^2 + \kappa}{2\kappa} \left(\frac{q_0^2 + \kappa}{2\kappa} \right)^{n-1} \geqslant \frac{1}{2^n}.$$

This recovers the necessary lower bound, $\rho_0 > 1/2$, for global regularity in the case $n = 1$. Thus, $\rho_0(r_0) < 2^{-n}$ will necessarily lead to formation of shock discontinuities, and, in particular, a vacuous state of ρ_0 leads to formation of (nonphysical) shocks. On the other hand, if ρ_0 is not far away from the equilibrium state $\rho_0 = 1$, such that $\mu_0 > 1/2$ and $\nu_0 > 1/2$, we can always find p_0 and q_0 small enough, such that (3.7) and (3.10) hold.

3.2. A comparison with Euler–Poisson equations. In this section, we compare our critical threshold result for the EMA system (1.1) with the Euler–Poisson equations (1.2) under radial symmetry.

A sharp critical threshold was obtained in [23] for the radial Euler–Poisson equations following a similar procedure. We summarize the result here for the sake of self-consistency, using the same notation (p, q, μ, ν) as defined in (2.8) and (2.9).

The Poisson equation (1.2c) can be expressed as

$$-(\mu + (n-1)\nu) = \widetilde{\rho} - 1, \tag{3.14}$$

which implies

$$\partial_r\big(r^n(1-n\nu)\big) = nr^{n-1}\widetilde{\rho}.$$

Applying Lemma 3.1 with $e = r^n(\frac{1}{n} - \nu)$, we obtain

$$\big(r^n(1-n\nu)\big)' = ne' = 0.$$

It yields

$$\nu' = q(1-n\nu).$$

Hence, the dynamics of (q, ν) reads

$$\begin{cases} q' = -q^2 - \kappa\nu, \\ \nu' = q(1-n\nu). \end{cases} \tag{3.15}$$

In contrast to (3.6), the dynamics depends on the dimension n. The global behaviors are surprisingly different.

Proposition 3.6 (see [23, Theorem 3.15]). *Let $n \geqslant 2$. Consider the ODE system* (3.15) *with bounded initial data $(q_0, \nu_0 < \frac{1}{n})$. Then, the solution (q, ν) remains bounded in all time.*

Note that from the definition (3.3), $e_0(r) \geqslant 0$, where the inequality holds in the trivial case of $\widetilde{\rho}_0(s) = 0$ for $s \in [0, r]$. Therefore, $\nu_0 < \frac{1}{n}$ holds for generic initial data. This indicates different behaviors from the EMA system, where blowup can happen in the (q, ν) dynamics once (3.7) is violated.

The dynamics of (p, μ), however, is less understood for the Euler–Poisson equations. Indeed, we can calculate from (3.14) and (3.15) that

$$\begin{aligned} \mu' &= -\widetilde{\rho}\,' - (n-1)\nu' = \widetilde{\rho}\,(p + (n-1)q) - (n-1)q(1-n\nu) \\ &= p(1-\mu) + (n-1)\big[-p\nu - q(\mu - \nu)\big]. \end{aligned}$$

This does not yield a closed ODE system on (p, μ), except when $n = 1$, where the Poisson equation (1.2c) coincides with the Monge–Ampère equation (1.1c). Whether there is an explicit threshold condition that leads to a global bound for the (p, q, μ, ν) dynamics for the radial Euler–Poisson equations is open.

The explicit result in Theorem 3.4 indicates that the EMA system has some special structures compared with the Euler–Poisson equations, despite being fully nonlinear. It is related to the geometric structure of the Monge–Ampère equation, which will be discussed in section 5.

4. Global well-posedness. The local and global well-posedness theory for the EMA system (1.1) in Sobolev space H^s has been established by Loeper in [19] using energy estimates. The theory requires a smallness assumption on the potential ϕ to handle the nonlinearity from the Monge–Ampère equation.

We now establish a global well-posedness theory for the EMA system with radial symmetry. We make use of the critical threshold condition and do not require any smallness assumptions.

Let $\mathbf{U}$ be a vector-valued radial function defined as

$$\mathbf{U}(\mathbf{x},t) = \begin{bmatrix} p(|\mathbf{x}|,t) \\ q(|\mathbf{x}|,t) \\ \mu(|\mathbf{x}|,t) \\ \nu(|\mathbf{x}|,t) \end{bmatrix} = \begin{bmatrix} \partial_r u(|x|,t) \\ \frac{u(|\mathbf{x}|,t)}{|x|} \\ \partial_r^2 \phi(|\mathbf{x}|,t) \\ \frac{\partial_r \phi(|\mathbf{x}|,t)}{|x|} \end{bmatrix}. \tag{4.1}$$

From the dynamics (3.6) and (3.9), we know $\mathbf{U}$ satisfies

$$\partial_t \mathbf{U} + (\mathbf{u}\cdot\nabla)\mathbf{U} = \mathbf{F}(\mathbf{U}), \quad \mathbf{F}(\mathbf{U}) = \begin{bmatrix} -U_1^2 - \kappa U_3 \\ -U_2^2 - \kappa U_4 \\ U_1(1-U_3) \\ U_2(1-U_4). \end{bmatrix}.$$

Equivalently, we can write

$$\partial_t U_i + \nabla\cdot(U_i \mathbf{u}) = \widetilde{F}_i(\mathbf{U}) \quad \text{for all } i = 1,2,3,4, \tag{4.2}$$

with a nonlinear force $\widetilde{\mathbf{F}}$ which depends quadratically on $\mathbf{U}$,

$$\widetilde{\mathbf{F}}(\mathbf{U}) := \mathbf{F}(\mathbf{U}) + (\nabla\cdot\mathbf{u})\,\mathbf{U} = \begin{bmatrix} -U_1^2 - \kappa U_3 + U_1\big(U_1 + (n-1)U_2\big) \\ -U_2^2 - \kappa U_4 + U_2\big(U_1 + (n-1)U_2\big) \\ U_1(1-U_3) + U_3\big(U_1 + (n-1)U_2\big) \\ U_2(1-U_4) + U_4\big(U_1 + (n-1)U_2\big) \end{bmatrix}. \tag{4.3}$$

Here, we have used

$$\nabla\cdot\mathbf{u} = \sum_{i=1}^n \lambda_i(\nabla\mathbf{u}) = U_1 + (n-1)U_2. \tag{4.4}$$

Let us first state a local well-posedness theory as well as regularity criteria.

THEOREM 4.1. *Consider the EMA system* (1.1) *with radial symmetry* (2.4). $\mathbf{U}$ *is defined in* (4.1). *Assume the initial condition* $\mathbf{U}_0 \in H^s(\mathbb{R}^n)$ *for* $s > \frac{n}{2}$. *Then, there exists a time* $T > 0$ *such that the solution* $\mathbf{U}$ *satisfies*

$$\mathbf{U} \in C([0,T], H^s(\mathbb{R}^n))^4. \tag{4.5}$$

Moreover, the lifespan T *can be extended as long as*

$$\int_0^\top \|\mathbf{U}(\cdot,t)\|_{L^\infty}\,\mathrm{d}t < +\infty. \tag{4.6}$$

The local existence result follows from the standard energy method in which one obtains a closure of H^s estimates for $s > n/2$ so that $H^s(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$, as long as the Beale–Kato–Majda like condition $\int_0^T \|\nabla\cdot\mathbf{u}(\cdot,t)\|_{L^\infty}\mathrm{d}t < \infty$ holds; see, e.g., [15]. For completeness, we outline the details below.

Proof. Given any $s \geqslant 0$, denote $\Lambda^s = (-\Delta)^{s/2}$ as the fractional Laplacian operator. Define energy $Y_s(t)$ as

$$Y_s(t) = \frac{1}{2}\|\mathbf{U}\|^2_{H^s(\mathbb{R}^n)} = \frac{1}{2}\|\mathbf{U}\|^2_{L^2(\mathbb{R}^n)} + \frac{1}{2}\|\Lambda^s\mathbf{U}\|^2_{L^2(\mathbb{R}^n)}.$$

The L^2 energy can be estimated by

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\mathbf{U}\|^2_{L^2} &= -\int_{\mathbb{R}^n} U_i \cdot \partial_{x_j}(U_i u_j)\,\mathrm{d}x + \int_{\mathbb{R}^n} U_i \cdot \widetilde{F}_i(\mathbf{U})\,\mathrm{d}x \\
&\leqslant \int_{\mathbb{R}^n} \partial_{x_j}\left(\frac{U_i^2}{2}\right)\cdot u_j\,\mathrm{d}x + \|U_i\|_{L^2}\cdot\|\widetilde{F}_i(\mathbf{U})\|_{L^2} \\
&\leqslant -\int_{\mathbb{R}^n} \partial_{x_j}u_j \cdot \frac{1}{2}U_i^2\,\mathrm{d}x + C\|\mathbf{U}\|_{L^2}\cdot(1+\|\mathbf{U}\|_{L^\infty})\|\mathbf{U}\|_{L^2} \\
&\lesssim (1+\|\nabla\cdot\mathbf{u}\|_{L^\infty} + \|\mathbf{U}\|_{L^\infty})\|\mathbf{U}\|^2_{L^2} \lesssim (1+\|\mathbf{U}\|_{L^\infty})\|\mathbf{U}\|^2_{L^2}.
\end{aligned}$$

Here, we use the Einstein summation convention and drop the summation on i and j for simplicity. We also use the notation $\lesssim$, where $A \lesssim B$ means $A \leqslant CB$, with a constant C which might depend on parameters (like n, s, etc.). In the penultimate line, we make use of the quadratic dependence of $\widetilde{\mathbf{F}}$ on $\mathbf{U}$ in (4.3). Apply the Hölder inequality to get

$$\|\widetilde{\mathbf{F}}(\mathbf{U})\|_{L^2} \lesssim (1+\|\mathbf{U}\|_{L^\infty})\|\mathbf{U}\|_{L^2}.$$

The last inequality is due to (4.4).

Next, we estimate the $\dot{H}^s$ energy. Apply Λ^s to (4.2), multiply by $\Lambda^s\mathbf{U}$, and integrate in $\mathbb{R}^n$. We obtain

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s\mathbf{U}\|^2_{L^2} = -\int_{\mathbb{R}^n}\Lambda^s U_i\cdot\Lambda^s\partial_{x_j}(U_i u_j)\,\mathrm{d}x + \int_{\mathbb{R}^n}\Lambda^s U_i\cdot\Lambda^s(\widetilde{F}_i(\mathbf{U}))\,\mathrm{d}x = I + II.$$

To estimate I, we use a Kato–Ponce type commutator estimate [12] and get

$$\begin{aligned}
I &= -\int \Lambda^s U_i\cdot u_j\,\Lambda^s\partial_{x_j}U_i\,\mathrm{d}x - \int \Lambda^s U_i\cdot\big[\Lambda^s\partial_{x_j}, u_j\big]U_i\,\mathrm{d}x \\
&\leqslant \int \partial_{x_j}u_j\cdot\frac{1}{2}\,(\Lambda^s U_i)^2\,\mathrm{d}x + \|\Lambda^s U_i\|_{L^2}\,\big\|\big[\Lambda^s\partial_{x_j}, u_j\big]U_i\big\|_{L^2} \\
&\leqslant \frac{1}{2}\|\nabla\cdot\mathbf{u}\|_{L^\infty}\|\Lambda^s\mathbf{U}\|^2_{L^2} + C\|\Lambda^s U_i\|_{L^2}\big(\|\nabla u_j\|_{L^\infty}\|\Lambda^s U_i\|_{L^2} + \|\Lambda^{s+1}u_j\|_{L^2}\|U_i\|_{L^\infty}\big) \\
&\lesssim \|\nabla\mathbf{u}\|_{L^\infty}\|\Lambda^s\mathbf{U}\|^2_{L^2} + \|\Lambda^s\mathbf{U}\|_{L^2}\|\mathbf{U}\|_{L^\infty}\|\Lambda^s(\nabla\mathbf{u})\|_{L^2}.
\end{aligned}$$

Furthermore, using (4.4), we have

$$\|\Lambda^s(\nabla\mathbf{u})\|_{L^2} \lesssim \|\Lambda^s(\nabla\cdot\mathbf{u})\|_{L^2} \leqslant \|\Lambda^s U_1\|_{L^2} + (n-1)\|\Lambda^s U_2\|_{L^2} \lesssim \|\Lambda^s\mathbf{U}\|_{L^2}.$$

Applying the estimate above and (2.10), we obtain

$$I \lesssim \|\mathbf{U}\|_{L^\infty}\|\Lambda^s\mathbf{U}\|^2_{L^2}. \tag{4.7}$$

The II term can be estimated as

$$II \leqslant \|\Lambda^s\mathbf{U}\|_{L^2}\|\Lambda^s(\widetilde{\mathbf{F}}(\mathbf{U}))\|_{L^2} \lesssim (1+\|\mathbf{U}\|_{L^\infty})\|\Lambda^s\mathbf{U}\|^2_{L^2}. \tag{4.8}$$

Here, we have used the quadratic dependence of $\widetilde{\mathbf{F}}$ on $\mathbf{U}$ in (4.3) again, and we apply the fractional Leibniz rule to get

$$\|\Lambda^s(\widetilde{\mathbf{F}}(\mathbf{U}))\|_{L^2} \lesssim (1 + \|\mathbf{U}\|_{L^\infty})\|\Lambda^s \mathbf{U}\|_{L^2}.$$

Combining (4.7) and (4.8), we end up with

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s \mathbf{U}\|_{L^2}^2 \lesssim (1 + \|\mathbf{U}\|_{L^\infty})\|\Lambda^s \mathbf{U}\|_{L^2}.$$

From the L^2 and $\dot{H}^s$ energy estimates, we get

$$\frac{d}{dt}Y_s(t) \lesssim (1 + \|\mathbf{U}(\cdot,t)\|_{L^\infty})\, Y_s(t).$$

Local well-posedness follows from standard Sobolev embedding for any $s > \frac{n}{2}$. Moreover, we apply the Grönwall inequality

$$Y_s(t) \leqslant Y_s(0) \exp\left[C\int_0^t (1 + \|\mathbf{U}(\cdot,s)\|_{L^\infty})\, ds\right].$$

Hence, $Y_s(t)$ is bounded as long as (4.6) holds. ∎

Theorem 3.4 provides sufficient and necessary conditions to ensure the regularity criterion (4.6). Hence, our main Theorem 1.1 is a direct consequence of Theorems 3.4 and 4.1.

5. A geometric approach. In this section, we provide an alternative way to study the global well-posedness of the EMA system (1.1), taking advantage of the geometric structure of the system.

Let us start with the definition and notation for the pushforward mapping.

DEFINITION 5.1 (pushforward). *Let $\Omega \subset \mathbb{R}^n$. A measurable mapping $T : \Omega \to \mathbb{R}^n$ is called a pushforward from a measure μ in Ω to a measure ν in $T(\Omega)$ if for any measurable test function f,*

$$\int_\Omega f \circ T \; d\mu = \int_{T(\Omega)} f \; d\nu.$$

We use the notation $T_\sharp\, \mu = \nu$. Moreover, if $d\mu = \rho_1(\mathbf{x})d\mathbf{x}$ and $d\nu = \rho_2(\mathbf{x})d\mathbf{x}$, we denote $T_\sharp\, \rho_1 = \rho_2$.

The key ingredient is linking the solution of the Monge–Ampère equation to a pushforward mapping.

LEMMA 5.2 (Monge–Ampère equation represented as pushforward). *Let ψ_t be a solution of the Monge–Ampère equation*

$$det\big(D^2\psi_t(\mathbf{x})\big) = \rho(\mathbf{x},t).$$

Then, $\nabla\psi_t$ is a pushforward from $\rho(\mathbf{x},t)d\mathbf{x}$ to the Lebesgue measure $d\mathbf{x}$, namely,

$$(\nabla\psi_t)_\sharp\, \rho(\cdot,t) = 1. \tag{5.1}$$

The proof of the lemma can be shown by a simple change of variables formula. We notice that the representation (5.1) makes sense as long as $\rho(\cdot,t)$ is a measure. If we further assume that the density $\rho(\cdot,t)$ is bounded and away from vacuum,

$$0 < \rho_{\min}(t) \leqslant \rho(\cdot,t) \leqslant \rho_{\max}(t) < +\infty, \tag{5.2}$$

then $\nabla\psi_t$ is a diffeomorphism.

Once we find ψ_t that solves (5.1), the solution of the Monge–Ampère equation (1.1c) can be expressed as

$$\phi(\mathbf{x},t)=\frac{|\mathbf{x}|^2}{2}-\psi_t(\mathbf{x}). \tag{5.3}$$

In order to construct the pushforward mapping $\nabla\psi_t$ that satisfies (5.1), we make use of the characteristic flow $\mathbf{X}_t(\mathbf{x})$, defined as

$$\partial_t\mathbf{X}_t(\mathbf{x})=\mathbf{u}(\mathbf{X}_t(\mathbf{x}),t),\quad \mathbf{X}_0(\mathbf{x})=\mathbf{x}, \tag{5.4}$$

where $\mathbf{u}$ is the velocity field. $\mathbf{X}_t$ can be viewed as a pushforward mapping.

LEMMA 5.3 (characteristic flow represented as pushforward). *Suppose ρ satisfies* (1.1a)*, with a Lipschitz flow* $\mathbf{u}$*. Then,* $\mathbf{X}_t$ *is a diffeomorphism. It satisfies*

$$(\mathbf{X}_t)_\sharp\,\rho_0=\rho(\cdot,t). \tag{5.5}$$

The proof of the lemma is elementary if the flow $\mathbf{u}$ is Lipschitz. Moreover, $\mathbf{X}_t$ is a diffeomorphism. We denote its inverse mapping by $\mathbf{X}_t^{-1}$.

Let us define another mapping $\mathbf{\Gamma}$, which pushforward the Lebesgue measure $d\mathbf{x}$ to $\rho_0 d\mathbf{x}$, namely,

$$\mathbf{\Gamma}_\sharp\,1=\rho_0. \tag{5.6}$$

We further define

$$\tilde{\mathbf{X}}_t:=\mathbf{X}_t\circ\mathbf{\Gamma}.$$

From (5.5) and (5.6), we get $(\tilde{\mathbf{X}}_t)_\sharp\,1=\rho(\cdot,t)$. Then, if $\mathbf{\Gamma}$ is invertible, we have

$$\left(\tilde{\mathbf{X}}_t^{-1}\right)_\sharp\rho(\cdot,t)=\left(\mathbf{\Gamma}^{-1}\circ\mathbf{X}_t^{-1}\right)_\sharp\rho(\cdot,t)=1. \tag{5.7}$$

Hence, if $\widetilde{\mathbf{X}}_t^{-1}$ has a gradient form, the corresponding stream function is a solution of the Monge–Ampère equation (5.1).

The following lemma shows that $\widetilde{\mathbf{X}}_t^{-1}$ indeed has a gradient form under the radial symmetry.

LEMMA 5.4. *Let $(\rho,\mathbf{u},\phi)$ be a solution of* (1.1) *with radial symmetry* (2.4)*. Assume* $\mathbf{u}$ *is Lipschitz and that the initial density* ρ_0 *satisfies*

$$0<\rho_{\min}(0)\leqslant\rho_0(\cdot)\leqslant\rho_{\max}(0)<+\infty. \tag{5.8}$$

Then, there exists a pushforward mapping $\mathbf{\Gamma}$ *satisfying* (5.6)*. Also, there exists a radial function ψ_t, defined in* (5.11)*, such that*

$$\widetilde{\mathbf{X}}_t^{-1}(\mathbf{x})=\nabla\psi_t(\mathbf{x}). \tag{5.9}$$

Moreover, $\mathbf{\Gamma}$ *and* $\nabla\psi_t$ *are diffeomorphisms.*

Proof. First, we construct $\mathbf{\Gamma}$. From Lemma 5.2, we can write (5.6) equivalently as

$$det\big(\nabla\mathbf{\Gamma}^{-1}(\mathbf{x})\big)=\rho_0.$$

Under radial symmetry (2.4), the mapping $\mathbf{\Gamma}$ takes the form

$$\mathbf{\Gamma}(\mathbf{x}) = \frac{\mathbf{x}}{r}\,\Gamma(r), \quad \mathbf{\Gamma}^{-1}(\mathbf{x}) = \frac{\mathbf{x}}{r}\,\Gamma^{-1}(r), \quad r = |\mathbf{x}|.$$

Indeed, apply Lemma 2.1 with $f = \Gamma^{-1}$ to get

$$det\big(\nabla\mathbf{\Gamma}^{-1}(\mathbf{x})\big) = (\Gamma^{-1})'(r)\cdot\left(\frac{\Gamma^{-1}(r)}{r}\right)^{n-1} = \frac{\frac{d}{dr}\big(\Gamma^{-1}(r)^n\big)}{nr^{n-1}} = \rho_0.$$

The last equality holds if we define

$$\Gamma^{-1}(r) = \left[\int_0^r ns^{n-1}\rho_0(s)\,ds\right]^{\frac{1}{n}}.$$

This completes the construction of $\mathbf{\Gamma}^{-1}$. Moreover, as ρ_0 satisfies (5.8), $\mathbf{\Gamma}^{-1}$, as well as $\mathbf{\Gamma}$, is a diffeomorphism.

Next, we construct ψ_t. The dynamics of the mapping $\widetilde{\mathbf{X}}_t$ reads

$$\partial_t\widetilde{\mathbf{X}}_t(\mathbf{x}) = \mathbf{u}(\widetilde{\mathbf{X}}_t(\mathbf{x}), t), \quad \widetilde{\mathbf{X}}_0(\mathbf{x}) = \mathbf{\Gamma}(\mathbf{x}). \tag{5.10}$$

Under radial symmetry (2.4), $\widetilde{\mathbf{X}}_t$ takes the form

$$\widetilde{\mathbf{X}}_t(\mathbf{x}) = \frac{\mathbf{x}}{r}\,R_t(r), \quad \widetilde{\mathbf{X}}_t^{-1}(\mathbf{x}) = \frac{\mathbf{x}}{r}\,R_t^{-1}(r), \quad r = |\mathbf{x}|,$$

where R_t satisfies

$$\partial_t R_t(r) = u(R_t(r), t), \quad R_0(r) = \Gamma^{-1}(r).$$

We define a radial function ψ_t as

$$\psi_t(\mathbf{x}) = \psi_t(r) = \int_0^r R_t^{-1}(s)\,ds. \tag{5.11}$$

Then, we can verify that ψ_t satisfies (5.9),

$$\nabla\psi_t(\mathbf{x}) = \partial_r\psi_t(r)\,\frac{\mathbf{x}}{r} = \frac{\mathbf{x}}{r}\,R_t^{-1}(r) = \widetilde{\mathbf{X}}_t^{-1}(\mathbf{x}).$$

Moreover, as $\mathbf{u}$ is a Lipschitz flow, we have

$$\rho_{\max}(t) \leqslant \rho_{\max}(0)\exp\left(\int_0^t \|\nabla\cdot\mathbf{u}(\cdot,s)\|_{L^\infty}\,ds\right) < +\infty,$$
$$\rho_{\min}(t) \geqslant \rho_{\min}(0)\exp\left(-\int_0^t \|\nabla\cdot\mathbf{u}(\cdot,s)\|_{L^\infty}\,ds\right) > 0.$$

This verifies the condition (5.2). Hence, $\nabla\psi_t$ is a diffeomorphism. ∎

Combining (5.9) with (5.7), we find a solution of (5.1), defined in (5.11). This allows us to obtain an explicit expression of the characteristic path $\mathbf{X}_t$.

Proposition 5.5. *Under the same assumptions as Lemma* 5.4*, the characteristic flow* $\mathbf{X}_t$ *satisfies*

$$\mathbf{X}_t(\mathbf{x}) = \big(\mathbf{x} - \nabla\phi_0(\mathbf{x})\big) + \nabla\phi_0(\mathbf{x})\cos(\sqrt{\kappa}t) + \mathbf{u}_0(\mathbf{x})\frac{\sin(\sqrt{\kappa}t)}{\sqrt{\kappa}}. \tag{5.12}$$

Proof. Let us first calculate

$$\begin{aligned}\partial_t^2\mathbf{X}_t(\mathbf{x}) &= \partial_t\big(\mathbf{u}(\mathbf{X}_t(\mathbf{x}),t)\big) = \partial_t\mathbf{u}(\mathbf{X}_t(\mathbf{x}),t) + \nabla\mathbf{u}(\mathbf{X}_t(\mathbf{x}),t)\partial_t\mathbf{X}_t(\mathbf{x})\\ &= \partial_t\mathbf{u}(\mathbf{X}_t(\mathbf{x}),t) + \mathbf{u}(\mathbf{X}_t(\mathbf{x}),t)\cdot\nabla\mathbf{u}(\mathbf{X}_t(\mathbf{x}),t) = -\kappa\nabla\phi(\mathbf{X}_t(\mathbf{x}),t).\end{aligned}\tag{5.13}$$

Then we apply the relation (5.3) and get

$$\nabla\phi(\mathbf{X}_t(\mathbf{x}),t) = \mathbf{X}_t(\mathbf{x}) - \nabla\psi_t(\mathbf{X}_t(\mathbf{x})) = \mathbf{X}_t(\mathbf{x}) - \mathbf{\Gamma}^{-1}(\mathbf{x}).\tag{5.14}$$

Here, we have used (5.9), so that

$$\nabla\psi_t\circ\mathbf{X}_t = \widetilde{\mathbf{X}}_t^{-1}\circ\mathbf{X}_t = \mathbf{\Gamma}^{-1}\circ\mathbf{X}_t^{-1}\circ\mathbf{X}_t = \mathbf{\Gamma}^{-1}.$$

Therefore, $\mathbf{X}_t$ satisfies the second order equation

$$\partial_t^2\mathbf{X}_t(\mathbf{x}) = -\kappa\mathbf{X}_t(\mathbf{x}) + \kappa\mathbf{\Gamma}^{-1}(\mathbf{x}),\quad \mathbf{X}_0(\mathbf{x}) = \mathbf{x},\quad \partial_t\mathbf{X}_0(\mathbf{x}) = \mathbf{u}_0(\mathbf{x}).$$

It can be solved explicitly, resulting in

$$\mathbf{X}_t(\mathbf{x}) = \mathbf{\Gamma}^{-1}(\mathbf{x}) + (\mathbf{x} - \mathbf{\Gamma}^{-1}(\mathbf{x}))\cos(\sqrt{\kappa}t) + \mathbf{u}_0(\mathbf{x})\frac{\sin(\sqrt{\kappa}t)}{\sqrt{\kappa}}.$$

Moreover, we apply (5.14) with $t = 0$ and obtain $\mathbf{\Gamma}^{-1}(\mathbf{x}) = \mathbf{x} - \nabla\phi_0(\mathbf{x})$. This leads to the formula (5.12). ∎

COROLLARY 5.6 (energy conservation). *Given any bounded set $\Omega\subset\mathbb{R}^n$, define energy*

$$E(t) = \frac{1}{2}\int_{\mathbf{X}_t(\Omega)}\rho(\mathbf{x},t)\big(|\mathbf{u}(\mathbf{x},t)|^2 + \kappa|\nabla\phi(\mathbf{x},t)|^2\big)\,d\mathbf{x}.$$

Then, $E(t)$ is conserved in time.

Proof. First, we apply Lemma 5.3 and write

$$E(t) = \frac{1}{2}\int_\Omega\big(|\partial_t\mathbf{X}_t(\mathbf{x})|^2 + \kappa|\nabla\phi(\mathbf{X}_t(\mathbf{x}),t)|^2\big)\rho_0(\mathbf{x})\,d\mathbf{x}.\tag{5.15}$$

Then,

$$\begin{aligned}E'(t) &= \int_\Omega\Big[\partial_t\mathbf{X}_t(\mathbf{x})\cdot\partial_t^2\mathbf{X}_t(\mathbf{x}) + \kappa\nabla\phi(\mathbf{X}_t(\mathbf{x}),t)\cdot\partial_t\nabla\phi(\mathbf{X}_t(\mathbf{x}),t)\Big]\rho_0(\mathbf{x})\,d\mathbf{x}\\ &= \int_\Omega\partial_t\mathbf{X}_t(\mathbf{x})\cdot\Big[\partial_t^2\mathbf{X}_t(\mathbf{x}) + \kappa\nabla\phi(\mathbf{X}_t(\mathbf{x}),t)\Big]\rho_0(\mathbf{x})\,d\mathbf{x} = 0.\end{aligned}$$

Here, in the penultimate equality, we apply (5.14) and get $\partial_t\nabla\phi(\mathbf{X}_t(\mathbf{x}),t) = \partial_t\mathbf{X}_t(\mathbf{x})$. The last equality follows from (5.13). ∎

Taking the spatial gradient of (5.12) would yield

$$\nabla\mathbf{X}_t(\mathbf{x}) = \big(\mathbb{I} - D^2\phi_0(\mathbf{x})\big) + D^2\phi_0(\mathbf{x})\cos(\sqrt{\kappa}t) + \nabla\mathbf{u}_0(\mathbf{x})\frac{\sin(\sqrt{\kappa}t)}{\sqrt{\kappa}}.\tag{5.16}$$

We can recover the critical threshold condition that we obtained through the analysis of the spectral dynamics.

THEOREM 5.7. $\nabla\mathbf{X}_t(\mathbf{x})$ *remains positive definite in all time if and only if the initial condition satisfies* (3.12).

Proof. Since $\nabla\mathbf{X}_t(\mathbf{x})$ is symmetric, it is positive definite if and only if the eigenvalues $\lambda_i(\nabla\mathbf{X}_t(\mathbf{x})) > 0$.

From Lemma 2.1, $\nabla\mathbf{X}_t(\mathbf{x}), D^2\phi_0(\mathbf{x})$, and $\nabla\mathbf{u}_0(\mathbf{x})$ all share the same eigenvectors. Therefore, (5.16) implies

$$\lambda_i(\nabla\mathbf{X}_t(\mathbf{x})) = \big(1-\lambda_i(D^2\phi_0(\mathbf{x}))\big) + \lambda_i(D^2\phi_0(\mathbf{x}))\cos(\sqrt{\kappa}t) + \lambda_i(\nabla\mathbf{u}_0(\mathbf{x}))\frac{\sin(\sqrt{\kappa}t)}{\sqrt{\kappa}}.$$

Hence, $\lambda_i(\nabla\mathbf{X}_t(\mathbf{x})) > 0$ if and only if

$$\lambda_i(D^2\phi_0(\mathbf{x}))^2 + \frac{\lambda_i(\nabla\mathbf{u}_0(\mathbf{x}))^2}{\kappa} < \big(1-\lambda_i(D^2\phi_0(\mathbf{x}))\big)^2$$

or, equivalently,

$$\lambda_i(\nabla\mathbf{u}_0(\mathbf{x}))^2 < \kappa\left(1-2\lambda_i(D^2\phi_0(\mathbf{x}))\right).$$

This is precisely (3.10) and (3.7) for $i = 1, 2$, respectively.

Finally, the equivalency to (3.12) follows through the same argument as in Theorem 3.4. ∎

6. Beyond radial symmetry: The 2D system with swirl. We have established the global well-posedness theory for the EMA system with radial symmetry (2.4). A natural question is what happens when we do not impose radial symmetry. In this section, we briefly discuss potential extensions of our theory to more general data.

A major difficulty of implementing the spectral dynamics analysis to the general data is that $\nabla\mathbf{u}$ and $\nabla\mathbf{F}$ do not necessarily share the same eigenvectors, so the forcing term in (2.3) can be hard to control.

6.1. 2D radial EMA system with swirl. Let us consider the following type of solutions in 2D,

$$\rho(\mathbf{x},t) = \rho(r,t), \quad \mathbf{u}(\mathbf{x},t) = \mathbf{u}_1(\mathbf{x},t) + \mathbf{u}_2(\mathbf{x},t) = \frac{\mathbf{x}}{r}u(r,t) + \frac{\mathbf{x}^\perp}{r}\Theta(r,t), \tag{6.1}$$

where Θ characterizes the rotation. Under this setup, although $\nabla\mathbf{u}$ does not share the same eigenvectors as $\nabla\mathbf{F}$, the component $\nabla\mathbf{u}_1$ does. In fact, we can decompose $\nabla\mathbf{u}$ by the symmetric part $\nabla\mathbf{u}_1$ and antisymmetric part $\nabla\mathbf{u}_2$ and study their spectral dynamics separately. Elementary calculation yields the dynamics of (p, q, μ, ν), together with $(\Theta_r, \frac{\Theta}{r})$, as follows:

$$\begin{cases} q' = -q^2 - \kappa\nu + \left(\frac{\Theta}{r}\right)^2, \\ \nu' = q(1-\nu), \\ \left(\frac{\Theta}{r}\right)' = -2q\frac{\Theta}{r}, \end{cases} \quad \text{and} \quad \begin{cases} p' = -p^2 - \kappa\mu + 2\Theta_r\frac{\Theta}{r} - \left(\frac{\Theta}{r}\right)^2, \\ \mu' = p(1-\mu), \\ \Theta_r' = -(p+q)\Theta_r - (p-q)\frac{\Theta}{r}. \end{cases} \tag{6.2}$$

Global well-posedness follows from the solvability of the closed ODE system, with six variables.

DEFINITION 6.1. *Define a set* $\Sigma \subset \mathbb{R}^6$ *so that* $\sigma_0 = (p_0, q_0, (\Theta_r)_0, (\frac{\Theta}{r})_0, \mu_0, \nu_0) \in \Sigma$ *if and only if the ODE system* (6.2) *with initial condition* σ_0 *is bounded globally in time.*

Clearly, if the initial data are subcritical, satisfying (1.7), the boundedness of $\sigma(t)$ implies the boundedness of $\nabla\mathbf{u}$. Then the solution is globally regular. This finishes the proof of Theorem 1.4.

A natural question is whether we can find an explicit formulation of the subcritical region Σ similarly to what we did for the system without swirl. So, we shall examine the ODE system (6.2).

Note that the dynamics of $(q, \nu, \frac{\Theta}{r})$ form a closed system. Comparing with (3.6), we observe that the presence of $\frac{\Theta}{r}$ helps to avoid $q \to -\infty$. This phenomenon is known as *rotation prevents finite-time breakdown*, which has been studied in [18]. More precisely, we state the following result.

PROPOSITION 6.2. *Consider the dynamics $(q, \nu, \frac{\Theta}{r})$ with initial conditions $\nu(0) < 1$ and $\frac{\Theta}{r}(0) \neq 0$. Then, the solution $(q, \nu, \frac{\Theta}{r})$ remains bounded in all time.*

Proof. We follow Proposition 3.2 and define (w, v) as (3.8). The dynamics reads

$$\begin{cases} w' = \kappa(1-v) + \left(\frac{\Theta}{r}\right)^2 v, \\ v' = w. \end{cases}$$

To understand the influence of $\frac{\Theta}{r}$, we observe the conserved quantity

$$\left(\frac{\Theta}{r} v^2\right)' = -2q\frac{\Theta}{r} \cdot v^2 + \frac{\Theta}{r} \cdot 2vw = -\frac{\Theta}{r} \cdot 2v(qv - w) = 0.$$

This implies

$$\frac{\Theta}{r} = C_0 v^{-2}, \quad \text{where} \quad C_0 = \frac{\Theta}{r}(0) v(0)^2 \neq 0.$$

It leads to the closed system for (w, v),

$$\begin{cases} w' = \kappa(1-v) + C_0^2 v^{-3}, \\ v' = w. \end{cases}$$

We obtain an invariant quantity

$$\left(w^2 + \kappa(1-v)^2 + C_0^2 v^{-2}\right)' = 2w \cdot \left(\kappa(1-v) + C_0^2 v^{-3}\right) + \left(-2\kappa(1-v) - 2C_0^2 v^{-3}\right) \cdot w = 0.$$

So, we have

$$w^2 + \kappa(1-v)^2 + C_0^2 v^{-2} = C := w_0^2 + \kappa(1-v_0)^2 + C_0^2 v_0^{-2},$$

where the constant $C > 0$ is a finite number when $\nu(0) < 1$. Clearly, w is bounded with $|w| \leqslant \sqrt{C}$. Also, $\kappa(1-v)^2 + C_0^2 v^{-2} \leqslant C$ implies $v > 0$, and v is bounded. The boundedness of $(q, \nu, \frac{\Theta}{r})$ then follows as a direct consequence. ∎

Under radial symmetry (2.4), the dynamics of (p, μ) is the same as that of (q, ν). However, this is not the case with swirl. In fact, the dynamics of (p, ν, Θ_r) does not even form a closed system. It is unclear whether there is an explicit critical threshold condition on the initial data that leads to the boundedness of the six quantities.

6.2. General nonsymmetric data. Consider the EMA system (1.1) with general initial data. We found it difficult to trace its spectral dynamics since the eigenstructure of the gradient force is not accessible through any obvious time-invariant quantities. We shall briefly discuss the alternative geometric approach [2, 19].

Without radial symmetry, the flow map $\widetilde{\mathbf{X}}_t^{-1}$ might not have a gradient form. Lemma 5.4 no longer holds. To find a solution ψ_t for (5.1), we make use of the celebrated polar factorization by Brenier [1] and decompose

$$\widetilde{\mathbf{X}}_t = \nabla\Phi_t \circ \pi_t,$$

where π_t is a measure-preserving pushforward mapping, namely $(\pi_t)_\sharp 1 = 1$. We get

$$(\nabla\Phi_t)_\sharp 1 = (\widetilde{\mathbf{X}}_t)_\sharp 1 = \rho(\cdot,t) \quad \Rightarrow \quad \big((\nabla\Phi_t)^{-1}\big)_\sharp \rho(\cdot,t) = 1.$$

Take ψ_t to be the Legendre transformation of Φ_t so that $\nabla\psi_t = (\nabla\Phi_t)^{-1}$. Then, ψ_t is a solution of the Monge–Ampère equation (5.1).

PROPOSITION 6.3. *Let* $(\rho, \mathbf{u}, \phi)$ *be a solution of* (1.1). *Assume* ρ *satisfies* (5.2) *and* $\mathbf{u}$ *is Lipschitz. Then,* $\mathbf{X}_t$ *solves the differential equation*

$$\partial_t^2 \mathbf{X}_t(\mathbf{x}) = -\kappa \mathbf{X}_t(\mathbf{x}) + \kappa\, \pi_t \circ \mathbf{\Gamma}^{-1}(\mathbf{x}), \quad \mathbf{X}_0(\mathbf{x}) = \mathbf{x}, \quad \partial_t \mathbf{X}_0(\mathbf{x}) = \mathbf{u}_0(\mathbf{x}). \tag{6.3}$$

Proof. The polar factorization $\mathbf{X}_t \circ \mathbf{\Gamma} = \nabla\Phi_t \circ \pi_t$ implies $\psi_t \circ \mathbf{X}_t = \pi_t \circ \mathbf{\Gamma}^{-1}$. We apply (5.13) and calculate

$$\partial_t^2 \mathbf{X}_t(\mathbf{x}) = -\kappa\nabla\phi(\mathbf{X}_t(\mathbf{x}), t) = -\kappa\big(\mathbf{X}_t(\mathbf{x}) - \nabla\psi_t(\mathbf{X}_t(\mathbf{x}))\big) = -\kappa\big(\mathbf{X}_t(\mathbf{x}) - \pi_t(\mathbf{\Gamma}^{-1}(\mathbf{x}))\big). \qquad \square$$

Unlike the radially symmetric case where $\pi_t(\mathbf{x}) = \mathbf{x}$, the measure-preserving mapping π_t can vary at different times t. Therefore, we are not able to obtain an explicit solution of $\mathbf{X}_t$ from (6.3). Moreover, it is unclear whether π_t is a diffeomorphism, or whether it could lose differentiability at some finite time. This has a big impact on the regularity of the solution. We will leave the study of the regularity properties of π_t for future investigation.

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