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Partial differential equations

Global regularity of two-dimensional flocking hydrodynamics

Régularité globale des dynamiques d'alignement bidimensionnel à l'échelle hydrodynamique

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ABSTRACT

We study the systems of Euler equations that arise from agent-based dynamics driven by velocity *alignment*. It is known that smooth solutions to such systems must flock, namely the large-time behavior of the velocity field approaches a limiting "flocking" velocity. To address the question of global regularity, we derive sharp critical thresholds in the phase space of initial configuration that characterizes the global regularity and hence the flocking behavior of such *two-dimensional* systems. Specifically, we prove for that a large class of *sub-critical* initial conditions such that the initial divergence is "not too negative" and the initial spectral gap is "not too large", global regularity persists for all time.

RÉSUMÉ

Nous étudions les systémes des équations d'Euler qui résultent de dynamiques d'*alignement* entre agents. Il a été prouvé que, pour des solutions régulières de tels systémes, en temps grand, le champ de vitesse s'approche d'une vitesse limite uniforme. Nous identifions des seuils critiques dans l'espace de phase de la configuration initiale qui caractérisent la régularité globale et donc le comportement en temps grand de tels systèmes *bidimensionnels*. Plus précisément, nous prouvons que, pour une classe assez large de conditions initiales *sous-critiques* telles que la divergence initiale n'est « pas trop négative » et l'écart spectral initial n'est « pas trop grand », la régularité globale reste vraie en temps grand.

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1. Flocking hydrodynamics

We consider the system of Eulerian dynamics where the density $\rho(x,t)$ and velocity field $\mathbf{u}(x,t)=(u_1,\dots u_n):\mathbb{R}^n\times\mathbb{R}_+\mapsto\mathbb{R}^n$ are driven by nonlocal alignment forcing,

$$\left\{\begin{aligned}&\rho_t+\nabla\cdot(\rho\mathbf{u})=0,\\&\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}=\int a(x,y,t)(\mathbf{u}(y,t)-\mathbf{u}(x,t))\rho(y,t)\mathrm{d}y\end{aligned}\right\}\quad (x,t)\in\mathbb{R}^n\times\mathbb{R}_+. \tag{1.1}$$

A solution $(\rho,\mathbf{u})$ is sought subject to the compactly supported initial density $\rho(x,0)=\rho_0(x)\in L^1_+(\mathbb{R}^n)$ and uniformly bounded initial velocity $\mathbf{u}(x,0)=\mathbf{u}_0(x)\in W^{1,\infty}(\mathbb{R}^n)$. The alignment forcing on the right-hand side of (1.1) involves the non-negative interaction kernel $a(x,y,t)$.

Such systems arise as the macroscopic realization of some agent-based dynamics that describes the collective motion of N agents, each of which adjusts its velocity to a *weighted average* of velocities of its neighbors

$$\left\{\begin{aligned}&\dot{x}_i=\mathbf{v}_i\\&\dot{\mathbf{v}}_i=\frac{1}{\deg_i}\sum_{j=1}^{N}\phi(|x_i-x_j|)(\mathbf{v}_j-\mathbf{v}_i)\end{aligned}\right. \tag{1.2}$$

Here, the weighted average of the right-hand side of (1.2) is dictated by the influence function $\phi(\cdot)$, which is assumed to be decreasing, and $\deg_i$ is a weighting normalization factor. Different agent-based models employ different $\deg_i$'s, e.g., [3]. We focus here on two such models. The Cucker–Smale (CS) model [7] sets a uniform averaging $\deg_i\equiv N$ that leads to the *symmetric* interaction kernel $a(x,y)=\phi(|x-y|)$. The Motsch–Tadmor (MT) model [20] uses an *adaptive* normalization $\deg_i=\sum_j\phi(|x_i-x_j|)$, which leads to $a(x,y,t)=\dfrac{\phi(|x-y|)}{(\phi*\rho)(x,t)}$. The kernel is non-symmetric, but normalized such that $\int a(x,y,t)\rho(y,t)\,\mathrm{d}y=1$. The dynamics of (1.2) can be described in terms of the empirical distribution $f(x,\mathbf{v},t):=\frac{1}{N}\sum_j\delta_{x-x_j(t)}\otimes\delta_{\mathbf{v}-\mathbf{v}_j(t)}$. For large crowds of N agents, $N\gg1$, a limiting distribution of the approximate form $f(x,\mathbf{v},t)\approx\rho(x,t)\delta(\mathbf{v}-\mathbf{u}(x,t))$ is captured by the first two velocity moments, namely the density $\rho:=\langle f(x,\mathbf{v},t)\rangle$ and the momentum $\rho\mathbf{u}:=\langle\mathbf{v}f(x,\mathbf{v},t)\rangle$, which satisfy the *conservative* system [2,5,11,19]

$$\left\{\begin{aligned}&\rho_t+\nabla\cdot(\rho\mathbf{u})=0\\&(\rho\mathbf{u})_t+\nabla(\rho\mathbf{u}\otimes\mathbf{u})=\frac{\alpha(x,t)}{(\phi*\rho)(x,t)}\int\phi(|x-y|)(\mathbf{u}(y,t)-\mathbf{u}(x,t))\rho(x,t)\rho(y,t)\,\mathrm{d}y.\end{aligned}\right. \tag{1.3}$$

Here $\alpha(x,t)$ is the amplitude of alignment, $\alpha(x,t)=(\phi*\rho)(x,t)$ in the case of the CS model, and $\alpha(x,t)\equiv1$ in the MT model. When classical solutions to these equations are restricted to the support of $\rho(\cdot,t)$, one ends with the equivalent system (1.1) with $a(x,y,t)=\alpha(x,t)\phi(|x-y|)/(\phi*\rho)(x,t)$, namely

$$\left\{\begin{aligned}&\rho_t+\nabla\cdot(\rho\mathbf{u})=0,\\&\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}=\frac{\alpha(x,t)}{(\phi*\rho)(x,t)}\int\phi(|x-y|)(\mathbf{u}(y,t)-\mathbf{u}(x,t))\rho(y,t)\,\mathrm{d}y.\end{aligned}\right. \tag{1.4}$$

Since the alignment forcing on the right-hand side is non-local, dictated by the support of ϕ, it acts even within the vacuum region, where $\mathrm{dist}\{x,\mathrm{supp}\{\rho(\cdot,t)\}\}>0$, and (1.4) extends *throughout* $\mathbb{R}^n$. We elaborate on this issue in §1.3 below.

We note that the dynamics of both models can be interpreted in terms of the mean velocity $\overline{\mathbf{u}}(x,t)$

$$\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}=\alpha(x,t)\big(\overline{\mathbf{u}}(x,t)-\mathbf{u}(x,t)\big),\qquad \overline{\mathbf{u}}(x,t):=\frac{\phi*(\rho\mathbf{u})(x,t)}{(\phi*\rho)(x,t)}. \tag{1.5}$$

This formulation reveals that system (1.4) (and in its general form (1.1)) is dynamically aligned towards the mean $\overline{u}(x,t)$, and its large-time behavior is expected to approach a constant limiting velocity. This is the flocking hydrodynamics alluded to in the title, where a finite-size of non-vacuum state is approaching a limiting velocity as $t\to\infty$. Specifically, the dynamics can be characterized in terms of the diameters

$$D(t):=\sup_{x,y\in\mathrm{supp}\{\rho(\cdot,t)\}}|x-y|,\qquad V(t):=\sup_{x,y\in\mathrm{supp}\{\rho(\cdot,t)\}}|\mathbf{u}(x,t)-\mathbf{u}(y,t)|.$$

The system (1.1) converges to a flock if there exists a finite D such that

$$\sup_{t\geq0}D(t)\leq D_\infty\quad\text{and}\quad V(t)\stackrel{t\to\infty}{\longrightarrow}0. \tag{1.6}$$

This corresponds to the flocking behavior at the level of the agent-based description [11], [20, definition 1.1], where a cohesive flock of a finite diameter $\max_{i,j}|x_i(t)-x_j(t)|\leq D_\infty<\infty$, is approaching a limiting velocity, $\max_{i,j}|\mathbf{v}_i(t)-\mathbf{v}_j(t)|\to0$ as $t\to\infty$.

1.1. Strong solutions must flock

In this work, we focus on the case where ϕ is *global*. Since the agent-based model (1.2) exhibits flocking behavior in this case, [21], it is natural to expect a similar result for its macroscopic description (1.4). This is the content of the following theorem.

Theorem 1.1 *(Strong solutions must flock [27]). Let $(\rho(\cdot,t),\mathbf{u}(\cdot,t)) \in (L^\infty \cap L^1) \times W^{1,\infty}$ be a global strong solution of the system (1.4) subject to a compactly supported initial density $\rho_0 = \rho(\cdot,0) \geq 0$ and a bounded initial velocity $\mathbf{u}_0 = \mathbf{u}(\cdot,0) \in W^{1,\infty}$. Assume that a monotonically decreasing influence function $\phi \leq \phi(0) = 1$ is global in the sense that*[2]

$$V_0 < m_0 \int_{D_0}^{\infty} \phi(r)\,\mathrm{d}r, \qquad m_0 := |\rho_0|_1, \tag{1.7}$$

where D_0 and V_0 are the initial diameters of non-vacuum density and velocity. Then $(\rho,\mathbf{u})$ converges to a flock at an exponential rate, namely the support of $\rho(\cdot,t)$ remains within a finite diameter D_∞ whose existence follows from assumption (1.7)

$$\sup_{t\geq 0} D(t) \leq D_\infty \quad \textit{where} \quad m_0 \int_{D_0}^{D_\infty} \phi(s)ds = V_0, \tag{1.8a}$$

and

$$V(t) \leq V_0 e^{-\kappa t} \longrightarrow 0, \qquad \kappa := \begin{cases} m_0\phi_\infty, & \text{CS } \textit{model}, \\ \phi_\infty, & \text{MT } \textit{model}, \end{cases} \quad \phi_\infty := \phi(D_\infty). \tag{1.8b}$$

In particular, if $|\phi|_1 = \infty$, then there is an unconditional flocking *in the sense that (1.8) holds for all finite V_0.*

For the sake of completeness, we provide below an alternative derivation of the exponential alignment in (1.8), as an a priori bound instead of the "propagation along characteristics" argument in [27, Theorem 2.1]. To this end, we extend the scalar argument in [25, Lemma 1.1] to general systems using a projection argument employed in [21, Theorem 2.3]. Fix an arbitrary $\mathbf{w} \in \mathbb{R}^n$ and project the CS model (1.4) on $\mathbf{w}$ to find

$$(\partial_t + \mathbf{u}\cdot\nabla)\langle \mathbf{u}(x,t),\mathbf{w}\rangle = \int \phi(|x-y|)\Big(\langle \mathbf{u}(y,t),\mathbf{w}\rangle - \langle \mathbf{u}(x,t),\mathbf{w}\rangle\Big)\rho(y,t)\mathrm{d}y.$$

It follows that $u_+(t) := \max_{x\in\mathrm{supp}\{\rho(\cdot,t)\}} \langle \mathbf{u}(x,t),\mathbf{w}\rangle$ satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}u_+ &= \int \phi(|x_+-y|)\Big(\langle \mathbf{u}(y,t),\mathbf{w}\rangle - \langle \mathbf{u}(x_+,t),\mathbf{w}\rangle\Big)\rho(y,t)\mathrm{d}y \\ &\leq \min_{x,y\in\mathrm{supp}\,\rho(\cdot,t)} \phi(|x-y|) \int \Big(\langle \mathbf{u}(y,t),\mathbf{w}\rangle - \langle \mathbf{u}(x_+,t),\mathbf{w}\rangle\Big)\rho(y,t)\mathrm{d}y\end{aligned}$$

Similarly, we have the lower bound on $u_-(t) := \min_{x\in\mathrm{supp}\{\rho(\cdot,t)\}} \langle \mathbf{u}(x,t),\mathbf{w}\rangle$

$$\frac{\mathrm{d}}{\mathrm{d}t}u_- \geq \min_{x,y\in\mathrm{supp}\{\rho(\cdot,t)\}} \phi(|x-y|) \int \Big(\langle \mathbf{u}(y,t),\mathbf{w}\rangle - \langle \mathbf{u}(x_-,t),\mathbf{w}\rangle\Big)\rho(y,t)\mathrm{d}y$$

The difference of the last two inequalities implies

$$\frac{\mathrm{d}}{\mathrm{d}t}|u_+(t)-u_-(t)| \leq -\phi(D_\infty)m_0|u_+(t)-u_-(t)|, \qquad \phi(D_\infty) = \min_{x,y\in\mathrm{supp}\{\rho(\cdot,t)\}} \phi(|x-y|).$$

It follows that the CS velocity diameter, $V(t) = \sup_{|\mathbf{w}|=1} |u_+(t)-u_-(t)|$, satisfies the bound (1.8b) with $\kappa = m_0\phi_\infty$. The same argument follows for MT model with $\kappa = \phi_\infty$, independently of m_0.

[2] We let $|\cdot|_p$ denote the usual L^p norm.

1.2. Critical thresholds

Theorem 1.1 raises the problem whether solutions to the hydrodynamic model (1.4) remain smooth for all time. This question was addressed in [4,27], proving that if the compactly supported initial data stay below a certain critical threshold in configuration space then initial smoothness propagates and, as a result, the corresponding strong solutions will flock. Recall the finite-time blow-up of compactly supported density in the presence of *local* pressure [17,23] and even in the presence of global Poisson forcing [18]. In both cases, a positive lower bound on the (potential of the) forcing—the pressure, the Poisson forcing, etc., over the *finite* $\mathrm{supp}\{\rho(\cdot,t)\}$ leads to finite-time blow up. In contrast, here the non-local character of the influence function ϕ guarantees global regularity, at least for sub-critical initial data. This type of conditional regularity for Eulerian dynamics depending on a *critical threshold* in the configuration space was advocated in a series of papers [9,12, 14–16,28]. Here, we pursue this approach to derive sharp critical thresholds for the propagation of the regularity of flocking hydrodynamics.

1.3. Vacuum and the finite horizon alignment

According to (1.7), if the influence function is global in the sense that $\int^{\infty}\phi(r)\,\mathrm{d}r=\infty$, then the alignment dynamics (1.4) admits *unconditional* flocking in the sense that (1.8) holds for all V_0's. This holds for both the symmetric CS model and non-symmetric MT model [21, proposition 2.9]. In this case, the alignment in (1.4) is active *throughout* $\mathbb{R}^n$, inside and outside $\mathrm{supp}\{\rho(\cdot,t)\}$. Indeed, one has a global lower bound on the alignment action for all $x\in\mathbb{R}^n$ [27, proposition 6.1],

$$(\phi*\rho)(x,t)\geq m_0\phi(d(x,t)+D_\infty)>0,\qquad d(x,t)=\mathrm{dist}\{x,\mathrm{supp}\{\rho(\cdot,t)\}\}$$

The flocking behavior of such a global approach was pursued in [27].

Another possible approach to study (1.4) is to focus on a specific initial configuration with finite velocity variation $V_0<\infty$. Then, since $\mathrm{supp}\{\rho(\cdot,t)\}$ cannot grow beyond a maximal diameter of size D_∞ dictated by (1.8a), it follows that the alignment term on the right of the underlying conservative formulation (1.3) is

$$\phi(|x-y|)(\mathbf{u}(y,t)-\mathbf{u}(x,t))\rho(x,t)\rho(y,t)\equiv 0,\qquad |x-y|>D_\infty,$$

independently of the values of $\{\phi(r),\ r>D_\infty\}$. Alternatively, we can fix a compactly support influence function ϕ and view (1.8a) as a restriction on initial velocities whose variation is "not too large", so that they lead to flocking. With either one of these two points of view, the values of $\phi(r)$ for $r>D_\infty$ play no role in the dynamics. We therefore may set $\phi(r)_{|r>D_\infty}\equiv 0$, which in turn sets a *finite horizon* on the action of alignment. Namely, the alignment in (1.4) is still active in the vacuous annulus *outside* $\mathrm{supp}\{\rho(\cdot,t)\}$,

$$A(t):=\{x\mid 0<\mathrm{dist}\{x,\mathrm{supp}\{\rho(\cdot,t)\}\}<D_\infty\},$$

and (1.4) applies in $\mathrm{supp}\{\rho(\cdot,t)\}\cup A(t)$,

$$\left.\begin{cases}\rho_t+\nabla\cdot(\rho\mathbf{u})=0,\\ \mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}=\dfrac{\alpha(x,t)}{\phi*\rho}\displaystyle\int\phi(|x-y|)(\mathbf{u}(y)-\mathbf{u}(x))\rho(y)\mathrm{d}y\end{cases}\right\}\ \mathrm{dist}\{x,\mathrm{supp}\{\rho(\cdot,t)\}\}<D_\infty. \tag{1.9a}$$

However, since $\phi(|x-y|)\rho(y)$ is supported for y's in the intersection $y\in Y_x(t):=\mathrm{supp}\{\rho(\cdot,t)\}\cap B_{D_\infty}(x)$, it implies the alignment bound

$$\left|\int\phi(|x-y|)(\mathbf{u}(y,t)-\mathbf{u}(x,t))\rho(y,t)\,\mathrm{d}y\right|\leq V(t)\cdot|\rho(\cdot,t)|_\infty\times\int_{y\in Y_x(t)}\phi(|x-y|)\,\mathrm{d}y.$$

It follows that the alignment on the right of (1.9a) approaches zero, as $x\in A(t)$ approaches the "horizon" boundary $\mathrm{dist}\{x,\mathrm{supp}\{\rho(\cdot,t)\}\}=D_\infty$ and $\mathrm{vol}(Y_x(t))\to 0$. In particular, $(\phi*\rho)(x,t)\equiv 0$ beyond the horizon $\mathrm{dist}\{x,\mathrm{supp}\{\rho(\cdot,t)\}\}>D_\infty$, where the momentum equation is reduced to inviscid pressureless equations, $\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}=0$. Accordingly, (1.9a) can be complemented with *constant* far-field boundary conditions, in agreement with [27, Remarks 2.8 & 6.6],

$$\mathbf{u}(x,t)\equiv\mathbf{u}_\infty,\quad \text{for }\ \mathrm{dist}\{x,\mathrm{supp}\{\rho(\cdot,t)\}\}>D_\infty. \tag{1.9b}$$

2. Cucker–Smale hydrodynamics: global regularity and fast alignment

2.1. Global regularity

We begin by recalling the one-dimensional Cucker–Smale model for $(\rho,u):(\mathbb{R},\mathbb{R}_+)\mapsto(\mathbb{R}_+,\mathbb{R})$,

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ u_t + u u_x = \int\limits_{\mathbb{R}} \phi(|x-y|)(u(y,t)-u(x,t))\rho(y,t)\mathrm{d}y \end{cases} \qquad (x,t) \in (\mathbb{R}, \mathbb{R}_+). \tag{2.1}$$

In [4], it was proved that (2.1) has a global classical solution if and only if the initial data satisfy

$$\partial_x u_0(x) \geq -(\phi * \rho_0)(x), \quad \text{for all } x \in \mathbb{R}. \tag{2.2}$$

Condition (2.2) separates the space of initial configurations into two distinct regimes: a sub-critical regime of initial data satisfying $\partial_x u_0(x) \geq -\phi * \rho_0(x), \forall x \in \mathrm{supp}(\rho_0)$, which guarantees *global* smooth solutions; and a supercritical regime of initial conditions such that $\partial_x u_0(x_0) \leq -\phi * \rho_0(x_0)$ for some $x_0 \in \mathbb{R}$, which leads to a finite-time blowup. This is a typical one-dimensional example for the critical threshold behavior. Condition (2.2) provides a sharp improvement to the earlier critical threshold results in [13,22,27]. Recent results in [8,24] prove the global regularity of (2.1) for singular kernels $\phi(|x|) = |x|^{-(1+\alpha)}$ for $\alpha \in (0,2)$ *independent* of any finite critical threshold. Singularity helps!

A first attempt to extend the study of critical threshold to the *two-dimensional* CS model was derived in [27]. Here, we improve this result with a simplified derivation of a sharper critical threshold condition, leading to an alignment decay of order $\mathrm{e}^{-\kappa t}$. We recall (1.8b), which sets $\kappa = m_0\phi_\infty$ in the present case of the CS model.

Theorem 2.1 *(Critical threshold for 2D Cucker–Smale hydrodynamics). Consider the two-dimensional CS model*

$$\left.\begin{cases} \rho_t + \nabla\cdot(\rho\mathbf{u}) = 0, \\ \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \int \phi(|x-y|)(\mathbf{u}(y,t)-\mathbf{u}(x,t))\rho(y,t)\mathrm{d}y \end{cases}\right\} \quad x \in \mathbb{R}^2, t \geq 0, \tag{2.3}$$

subject to initial conditions, $(\rho_0, \mathbf{u}_0) \in (L^1_+(\mathbb{R}^2), W^{1,\infty}(\mathbb{R}^2))$, with compactly supported density, $D_0 < \infty$, and such that the variation of the initial velocity satisfies the strengthened bound

$$V_0 \leq m_0 \cdot \min\left\{|\phi|_1, \frac{\phi_\infty^2}{4|\phi'|_\infty}\right\}, \qquad V_0 = \max_{x,y\in\mathrm{supp}(\rho_0)} |\mathbf{u}_0(x) - \mathbf{u}_0(y))|, \quad \phi_\infty = \phi(D_\infty). \tag{2.4}$$

Assume that the following critical threshold condition holds.

(i) *The initial velocity divergence satisfies*

$$\mathrm{div}\,\mathbf{u}_0(x) \geq -\phi * \rho_0(x) \quad \textit{for all} \quad x \in \mathbb{R}^2. \tag{2.5}$$

(ii) *Let $S = \frac{1}{2}\{(\partial_j u_i + \partial_i u_j)\}$ denote the symmetric part of the velocity gradient with eigenvalues $\mu_i = \mu_i(S)$. Then the initial* spectral gap *$\eta_{S_0} := \mu_2(S_0) - \mu_1(S_0)$ is bounded*

$$\max_x |\eta_{S_0}(x)| \leq \frac{1}{2} m_0\phi_\infty, \qquad \eta_S = \mu_2(S(x,t)) - \mu_1(S(x,t)). \tag{2.6}$$

Then the class of such sub-critical initial conditions (2.5), (2.6) admits a classical solution $(\rho(\cdot,t), \mathbf{u}(\cdot,t)) \in C(\mathbb{R}^+; L^\infty \cap L^1(\mathbb{R}^2)) \times C(\mathbb{R}^+; \dot{W}^{1,\infty}(\mathbb{R}^2))$ with large-time hydrodynamics flocking behavior (1.8b), $\max\limits_{x,y\in\mathrm{supp}(\rho(\cdot,t))} |\mathbf{u}(x,t) - \mathbf{u}(y,t)| \lesssim \mathrm{e}^{-\kappa t}$.

Before turning to the proof of Theorem 2.1, we comment on its assumptions.

Remark 2.1 *(On the critical threshold (2.5), (2.6)).* Theorem 2.1 recovers the one-dimensional critical threshold (2.2). It amplifies the same theme of critical threshold required for global regularity of other *two-dimensional* Eulerian dynamics found in restricted Euler–Poisson equations [15], rotational Euler equations [16], etc., namely if the initial divergence is “not too negative”, as in (2.5), and the initial spectral gap is “not too large”, as in (2.6), then global regularity persists for all times. In particular, since $\eta_S = \sqrt{(\partial_1 u_1 - \partial_2 u_2)^2 + (\partial_1 u_2 + \partial_2 u_1)^2}$, we find that both (2.5), (2.6) hold if

$$|\partial_j u_i(x,0)| \leq \frac{1}{4\sqrt{2}} m_0\phi_\infty.$$

Remark 2.2 *(On the finite variation (2.4)).* Observe that (2.4) places a restriction on the size of V_0 even in the case of unconditional flocking, $|\phi|_1 = \infty$. Specifically, recall that V_0 dictates the maximal diameter of the flock in (1.8a), and thus (2.4) amounts to

$$\int\limits_{D_0}^{D_\infty} \phi(s)\,\mathrm{d}s \leq \frac{\phi^2(D_\infty)}{4\max_{s\leq D_\infty}|\phi'(s)|}. \tag{2.7}$$

Since the term on the left is increasing while the term on the right is decreasing as functions of D_∞, it follows that (2.7) is satisfied for diameters D_∞ up to some maximal finite size, which means that the condition made in (2.4) is met for finite $V_0 = m_0 \int^{D_\infty} \phi(s)\,\mathrm{d}s$ depending on the influence function ϕ. This finite restriction on V_0 can probably be improved but, unlike in the one-dimensional case, it cannot be completely removed. In fact, since $V_0 \le (\mu_2(S_0) + \omega_0)D_\infty$, the bound sought in (2.4) places a purely two-dimensional restriction on the size of the *initial vorticity*.

Remark 2.3 *(On the finite horizon).* Observe that in the case of alignment with a finite horizon, the critical threshold (2.5) requires that $\operatorname{div}\mathbf{u}_0(x) \ge 0$ for $\operatorname{dist}\{x, \operatorname{supp}\{\rho_0\}\} > D_\infty$. This is precisely the critical threshold condition that rules out finite time blow-up in the pressureless equations [26], which is satisfied when prescribing a far-field constant velocity (1.9b). In this case, the critical threshold (2.5) needs to be verified within the finite horizon $\operatorname{dist}\{x, \operatorname{supp}\{\rho_0\}\} < D_\infty$.

Proof. Our purpose is to show that the derivatives $\{\partial_j u_i\}$ are uniformly bounded. We proceed in four steps.

Step #1: the dynamics of $\operatorname{div}\mathbf{u} + \phi * \rho$. Differentiation of (1.1) implies that the 2×2 velocity gradient matrix, $M_{ij} := \partial_j u_i$, satisfies

$$M_t + \mathbf{u} \cdot \nabla M + M^2 = -(\phi * \rho)M + R, \qquad R_{ij} := \partial_j \phi * (\rho u_i) - u_i \partial_j \phi * \rho. \tag{2.8}$$

The entries of the residual matrix $\{R_{ij}\}$ can be bounded by the commutator estimate [27, proposition 4.1] in terms of $V(t) = \sup_{\operatorname{supp}(\rho)} |u_i(x,t) - u_i(y,t)| \le V_0\,\mathrm{e}^{-\kappa t}$,

$$|R_{ij}| = \left| \int_{\mathbb{R}^n} \partial_j \phi(|x-y|)(u_i(y,t) - u_i(x,t))\rho(y,t)\mathrm{d}y \right| \le |\phi'|_\infty m_0 V_0\,\mathrm{e}^{-\kappa t}, \qquad \kappa = m_0 \phi_\infty.$$

The first step is to bound the divergence: taking the trace of (2.8), we find that $\mathsf{d} := \nabla \cdot \mathbf{u}$ satisfies

$$\mathsf{d}_t + \mathbf{u} \cdot \nabla \mathsf{d} + \operatorname{Tr} M^2 = -(\phi * \rho)\mathsf{d} + \operatorname{Tr} R.$$

Expressed in terms of the material derivative along particle path, $X' := (\partial_t + \mathbf{u} \cdot \nabla)X$, we have $\mathsf{d}' + \operatorname{Tr} M^2 = -(\phi * \rho)\mathsf{d} + \operatorname{Tr} R$. We now make a key observation that $\operatorname{Tr} R$ is in fact an exact derivative along the particle path. Indeed, as in [4] we invoke the mass equation,

$$\operatorname{Tr} R = \phi * \nabla \cdot (\rho \mathbf{u}) - \mathbf{u} \cdot \nabla \phi * \rho = -(\phi * \rho)_t - \mathbf{u} \cdot \nabla \phi * \rho = -(\phi * \rho)',$$

and we end up with

$$(\mathsf{d} + \phi * \rho)' + \operatorname{Tr} M^2 = -(\phi * \rho)\mathsf{d}. \tag{2.9}$$

To proceed, we express $\operatorname{Tr} M^2 \equiv \dfrac{\mathsf{d}^2 + \eta_M^2}{2}$ in terms of the *spectral gap*, $\eta_M := \lambda_2(M) - \lambda_1(M)$, associated with the eigenvalues of M,

$$(\mathsf{d} + \phi * \rho)' = -\frac{1}{2}\eta_M^2 - \frac{1}{2}\mathsf{d}(\mathsf{d} + 2\phi * \rho). \tag{2.10}$$

We need to follow the dynamics of the spectral gap η_M. To this end, one may try to use the *spectral dynamics* associated with M, [14]: by (2.8) the λ_i's satisfy

$$\lambda_i' + \lambda_i^2 = -(\phi * \rho)\lambda_i + \langle \boldsymbol{\ell}_i, R\mathbf{r}_i \rangle, \qquad i = 1, 2,$$

where $\{\boldsymbol{\ell}_i, \mathbf{r}_i\}$ are the left- and right-hand-side eigenvectors associated with λ_i, normalized such that $\langle \boldsymbol{\ell}_i, \mathbf{r}_i \rangle = 1$. Taking the difference of these two equations shows that the spectral gap $\eta_M = \lambda_2 - \lambda_1$, satisfies the transport equation

$$\eta_M' + (\mathsf{d} + \phi * \rho)\eta_M = \langle \boldsymbol{\ell}_2, R\mathbf{r}_2 \rangle - \langle \boldsymbol{\ell}_1, R\mathbf{r}_1 \rangle.$$

Here one faces the difficulty that arises with the term on the right, namely, even with the control of the entries $\{R_{ij}\}$, we may still encounter an ill-conditioned M with $|\boldsymbol{\ell}_i| \cdot |\mathbf{r}_i| \gg 1$ so that the magnitude of this term is left unchecked. To circumvent this difficulty, we proceed along the lines argued in [26]: we split M into its symmetric and antisymmetric parts $M = S + \Omega$ and use the identity[3]

[3] Equating the trace of M^2 with that of $S^2 + \Omega^2 + S\Omega + \Omega S$ we find $\operatorname{Tr} M^2 = \operatorname{Tr} S^2 - 2\omega^2$. Using $\operatorname{Tr} X^2 = \frac{1}{2}(\mathsf{d}^2 + \eta_X^2)$ with $X = M$ on the left and $X = S$ on the right implies (2.11).

$$\eta_M^2 \equiv \eta_S^2 - 4\omega^2, \qquad M = S + \Omega, \ \ \Omega := \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}, \tag{2.11}$$

where ω is the *scaled* vorticity[4] $\omega = \frac{1}{2}(\partial_1 u_2 - \partial_2 u_1)$. Expressed in terms of η_S, the trace dynamics (2.10) now reads

$$(\mathsf{d} + \phi * \rho)' = \frac{1}{2}(4\omega^2 - \eta_S^2) - \frac{1}{2}\mathsf{d}(\mathsf{d} + 2\phi * \rho).$$

This calls for the introduction of the new "natural" variable $\mathsf{e} = \mathsf{d} + \phi * \rho$, satisfying

$$\mathsf{e}' = \frac{1}{2}\left((\phi * \rho)^2 + 4\omega^2 - \eta_S^2 - \mathsf{e}^2\right). \tag{2.12}$$

Our purpose is to show that $\{x \mid \mathsf{e}(x,t) \geq 0\}$ is invariant region of the dynamics (2.12).

Step #2: bounding the spectral gap η_S. Consider the dynamics of the symmetric part of (2.8)

$$S' + S^2 = \omega^2 \mathbb{I}_{2\times 2} - (\phi * \rho)S + R_{\mathrm{sym}}, \qquad R_{\mathrm{sym}} = \frac{1}{2}(R + R^\top).$$

The spectral dynamics of its eigenvalues, $\mu_2(S) \geq \mu_1(S)$, is governed by

$$\mu_i' + \mu_i^2 = \omega^2 - (\phi * \rho)\mu_i + \langle \mathbf{s}_i, R_{\mathrm{sym}}\mathbf{s}_i \rangle \tag{2.13}$$

driven by the *orthonormal* eigenpair $\{\mathbf{s}_1, \mathbf{s}_2\}$ of the symmetric S. Taking the difference, we find that $\eta_S := \mu_2(S) - \mu_1(S) \geq 0$ satisfies

$$\eta_S' + \mathsf{e}\eta_S = q, \qquad \mathsf{e} = \mathsf{d} + \phi * \rho. \tag{2.14}$$

This is the *same* dynamics found with η_M, except that the different residual on the right-hand side of (2.14), given by

$$q := \langle \mathbf{s}_2, R_{\mathrm{sym}}\mathbf{s}_2 \rangle - \langle \mathbf{s}_1, R_{\mathrm{sym}}\mathbf{s}_1 \rangle,$$

is now controlled by the size of $\{R_{ij}\}$: since $\mathbf{s}_i$ are normalized,

$$|q(\cdot,t)| \leq 2\max_{ij}|R_{ij}(\cdot,t)| \leq 2|\phi'|_\infty m_0 V_0 \mathrm{e}^{-\kappa t}, \qquad \kappa = m_0\phi_\infty. \tag{2.15}$$

Hence, as long as $\mathsf{e}(\cdot,t)$ remains positive then η_S remain uniformly bounded

$$|\eta_S(x,t)| \leq \max_x|\eta_S(x,0)| + 2\frac{|\phi'|_\infty}{\phi_\infty}V_0 < \max_x|\eta_S(x,0)| + \frac{1}{2}m_0\phi_\infty < m_0\phi_\infty \tag{2.16}$$

The first inequality on the right-hand side follows from the integration of (2.14)–(2.15); the second follows from the V_0-bound in (2.4) and the third from the assumed bound on η_{S_0} in (2.6).

Step #3: the invariance of $\mathsf{e}(\cdot,t) \geq 0$. We return to (2.12): expressed in terms of $c(x,t) := \sqrt{(\phi * \rho)^2 - \eta_S^2}$, we have

$$\mathsf{e}' \geq \frac{1}{2}\left(c^2(x,t) - \mathsf{e}^2\right), \qquad c(x,t) = \sqrt{(\phi * \rho)^2 - \eta_S^2}. \tag{2.17}$$

Observe that $c(\cdot)$ is well defined in $\mathbb{R}$: the upper bound (2.16) and the lowerbound $\phi * \rho \geq m_0\phi_\infty$ imply that as long as $\mathsf{e} \geq 0$, the right term on the right of (2.17) remains boundedly positive

$$c(x,t) \geq \sqrt{m_0^2\phi_\infty^2 - \max_x \eta_S^2(x,t)} \geq c_{\min} > 0.$$

Since $\mathsf{e}' \geq \frac{1}{2}(c_{\min}^2 - \mathsf{e}^2) = \frac{1}{2}(c_{\min} - \mathsf{e})(c_{\min} + \mathsf{e})$, it follows that e is increasing whenever $\mathsf{e} \in (-c_{\min}, c_{\min})$ and in particular, if $\mathsf{e}_0 \geq 0$ then $\mathsf{e}(x,t)$ remains positive at later times. Thus, if the initial data are *sub-critical* in the sense that (2.5) holds

$$\mathsf{e}_0 = \operatorname{div}\mathbf{u}_0(x) + \phi * \rho_0(x) \geq 0,$$

then $\mathsf{e}(\cdot,t) \geq 0$ and $\eta_S(\cdot,t)$ remains bounded.

Step #4: an upper-bound of $\mathsf{e}(\cdot,t)$. The lower-bound $\mathsf{e} \geq 0$ implies that the vorticity is bounded. Indeed, the anti-symmetric part of (2.8) yields that the vorticity $\omega = \frac{1}{2}\operatorname{Tr} JM$ satisfies

$$\omega' + \mathsf{e}\,\omega = \frac{1}{2}\operatorname{Tr} JR, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.18}$$

[4] The use of such scaling simplifies the computation below.

hence

$$|\omega|' \leq -\mathrm{e}|\omega| + \frac{1}{2}|q|, \qquad |q(\cdot,t)| \leq 2|\phi'|_\infty m_0 V_0 \mathrm{e}^{-\kappa t}, \quad \kappa = m_0\phi_\infty, \tag{2.19}$$

and we end up with same upper bound on ω as with η_S,

$$|\omega(x,t)| \leq \omega_{\max}, \qquad \omega_{\max} := \max_x |\omega_0| + \frac{1}{2}m_0\phi_\infty. \tag{2.20}$$

Returning to (2.12), we have (recall $\phi \leq 1$)

$$\mathrm{e}' \leq \frac{1}{2}\Big((\phi * \rho)^2 + 4\omega^2 - \mathrm{e}^2\Big) \leq \frac{1}{2}\Big(m_0^2 + 4\omega_{\max}^2 - \mathrm{e}^2\Big),$$

which implies that $\mathrm{e}(x,t) \leq \mathrm{e}_{\max} < \infty$. The uniform bound on e implies that $\operatorname{div} u$ is uniformly bounded, $|\operatorname{div}\mathbf{u}| \leq |\mathrm{e}|_\infty + |\phi * \rho|_\infty \leq \mathrm{e}_{\max} + m_0$, and, together with the bound on the spectral gap (2.16), it follows that the symmetric part $\{S_{ij}\}$ is bounded. Finally, together with the vorticity bound (2.20), it follows that $\{\partial_j u_i\}$ are uniformly bounded, which completes the proof. □

Remark 2.4. Observe that the region of sub-critical configuration leading to global regularity becomes *larger* for $|\omega_0| \gg 1$, in agreement with the statements made in [6,16] that rotation prevents or at least delays finite-time blow-up. Specifically, if $|\omega_0(\cdot)| \geq \omega_{\min} > 0$, then one can set a larger lower barrier $c = \sqrt{(\phi * \rho)^2 + 4\omega_{\min}^2 - \eta_S^2}$ in (2.17) leading to the improved threshold $\operatorname{div}\mathbf{u}_0 > -\phi * \rho_0 - \omega_{\min}$. In particular, if ω is large enough so that $4\omega^2 - \eta_S^2 > 0$, that is if M has complex-valued eigenvalues, then the invariance of the positivity of e follows at once from the fact that (2.12) is dominated by $\mathrm{e}' \geq \frac{1}{2}\big((\phi * \rho)^2 - \mathrm{e}^2\big)$. As in the 2D restricted Euler–Poisson equations [15], the difficulty lies within the case of real eigenvalues.

Remark 2.5. The proof of Theorem 2.1 reveals two main aspects. First, the *commutator structure* of the alignment term on the right of $(2.3)_2$, expressed as $[\phi*, u](\rho)$, leads to the 'residual terms' R_{ij} with exponentially decaying bound. The role of the commutator structure was highlighted in our recent work [24]. Second, of spectral dynamics [12,14,15] are used to trace the propagation of regularity for the remaining, non-residual terms in (2.8).

2.2. Fast alignment

We extend the one-dimensional arguments of [24] that show an exponentially rapid convergence towards a *flocking state*, consisting of a constant 2-vector velocity $\bar{\mathbf{u}} \in \mathbb{R}^2$ and a traveling density profile $\bar{\rho}(x,t) = \rho_\infty(x - t\bar{\mathbf{u}})$. We only indicate the main aspects in the passage to the present system. We start by noting that the positivity of e implies more than the mere boundedness of the spectral gap η_S and the vorticity ω. Indeed, (2.14) and (2.19) imply that these quantities follow the exponential decay of q in (2.15)

$$|\eta_S(\cdot,t)|_\infty + |\omega(\cdot,t)|_\infty \lesssim \mathrm{e}^{-\kappa t}.$$

This shows that, modulo rapidly decaying error terms $E(t)$ of order $E(t) \lesssim \mathrm{e}^{-\kappa t}$, equation (2.12), which governs e, takes the form

$$\mathrm{e}_t + \mathbf{u}\cdot\nabla\mathrm{e} = \frac{1}{2}\Big(\mathrm{h}^2 - \mathrm{e}^2\Big) + E(t), \qquad \mathrm{h} := \phi * \rho$$

Moreover, convolving the mass equation with ϕ, we find

$$\mathrm{h}_t + \mathbf{u}\cdot\nabla\mathrm{h} = \int \nabla\phi(|x-y|)\cdot(\mathbf{u}(x,t) - \mathbf{u}(y,t))\rho(y,t)\mathrm{d}y. \tag{2.21}$$

Observe that the quantity on the right of rapidly decaying, being upper-bounded by $\lesssim |\phi'|_\infty V(t) \lesssim \mathrm{e}^{-\kappa t}$. Hence, the difference $\mathrm{d} = \mathrm{e} - \mathrm{h}$ satisfies

$$\mathrm{d}_t + \mathbf{u}\cdot\nabla\mathrm{d} = -\frac{1}{2}(\mathrm{e}+\mathrm{h})\mathrm{d} + E(t).$$

The positivity of $\mathrm{e}+\mathrm{h}$ then implies the rapid decay of the divergence, $|\operatorname{div}\mathbf{u}(\cdot,t)|_\infty \lesssim \mathrm{e}^{-\kappa t}$. The exponential decay of the divergence, the vorticity and the spectral gap imply that $|\partial_j u_i(\cdot,t)|_\infty \lesssim \mathrm{e}^{-\kappa t}$. Let $\bar{\mathbf{u}}$ be a large-time limiting value of $\mathbf{u}(\cdot,t)$. The mass equation reads

$$\rho_t + \bar{\mathbf{u}}\cdot\nabla\rho = -\mathrm{d}\rho + (\bar{\mathbf{u}} - \mathbf{u})\cdot\nabla\rho.$$

The term on the right is rapidly decaying because d and $(\bar{\mathbf{u}} - \mathbf{u})$ are, and one concludes along the lines of [25], that there exists a traveling density profile such that $\rho(x,t) - \rho_\infty(x - t\bar{\mathbf{u}}) \to 0$.

3. Motsch–Tadmor hydrodynamics

In this section, we study the flocking hydrodynamics that arises from the MT model (1.5) with $\kappa = \phi_\infty$. We begin by recalling the one-dimensional case

$$\begin{aligned} &\rho_t + (\rho u)_x = 0, \qquad (x,t) \in (\mathbb{R}, \mathbb{R}_+) \\ &u_t + u u_x = \int \frac{\phi(|x-y|)}{(\phi * \rho)(x,t)}(u(y,t) - u(x,t))\rho(y,t)\mathrm{d}y. \end{aligned} \tag{3.1}$$

System (3.1) was recently studied in [1], as the hydrodynamic description for agent-based model of "emotional contagion", and in [10] in the context of stable swarming. In [4], it was proved that (3.1) has a global classical solution for sub-critical initial data such that

$$\partial_x u_0(x) \geq -\sigma_+(V_0) \ \text{ for all } \ x \in \mathbb{R}, \tag{3.2}$$

for a certain critical curve $\sigma_+ \geq 0$. We now make a precise statement of the critical threshold for both the one- and two-dimensional MT model.

Theorem 3.1 *(Critical threshold for 2D Motsch–Tadmor hydrodynamics). Consider the two-dimensional MT model in* $(x,t) \in (\mathbb{R}^2, \mathbb{R}_+)$,

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = \displaystyle\int a(x,y,t)(\mathbf{u}(y,t) - \mathbf{u}(x,t))\rho(y,t)\mathrm{d}y, \qquad a(x,y,t) := \frac{\phi(|x-y|)}{(\phi * \rho)(x,t)}, \end{cases} \tag{3.3}$$

subject to initial conditions $(\rho_0, \mathbf{u}_0) \in (L^1, W^{1,\infty}(\mathbb{R}^2))$, *with compactly supported density,* $D_0 < \infty$ *and initial velocity of finite variation*

$$V_0 \leq m_0 \cdot \min\left\{ |\phi|_1, \frac{\phi_\infty^2}{4|\phi'|_\infty(1+2\phi_\infty)} \right\}, \qquad \phi_\infty = \phi(D_\infty). \tag{3.4}$$

Assume that the following critical threshold condition holds.

(i) *The initial velocity divergence satisfies*

$$\operatorname{div} \mathbf{u}_0(x) \geq -1 \quad \text{for all} \quad x \in \mathbb{R}^2. \tag{3.5}$$

(ii) *Then the initial* spectral gap $\eta_{S_0} := \mu_2(S_0) - \mu_1(S_0)$ *is bounded*

$$\max_x \left|\eta_{S_0}(x)\right| \leq \frac{1}{2}, \qquad \eta_S = \mu_2(S(x,t)) - \mu_1(S(x,t)). \tag{3.6}$$

Then the class of such sub-critical initial conditions (3.5), (3.6) give rise to a classical solution $(\rho(t), \mathbf{u}(t) \in C(\mathbb{R}^+; L^\infty(\mathbb{R}^2)) \times C(\mathbb{R}^+; \dot{W}^{1,\infty}(\mathbb{R}^2))$ *with large-time hydrodynamics flocking behavior (1.8b)* $\max_{x \in \operatorname{supp}(\rho)} |\mathbf{u}(x,t) - \mathbf{u}(y,t)| \lesssim \mathrm{e}^{-\kappa t}$.

Remark 3.1. In the case of finite horizon alignment encoded in (1.9) with $\alpha = \phi * \rho$, the critical thresholds (3.5), (3.6) can be restricted to the finite set $\operatorname{dist}\{x, \operatorname{supp}\{\rho_0\}\}$.

Proof. As before, we trace the dynamics of $M = \partial_j u_i$,

$$M_t + \mathbf{u} \cdot \nabla M + M^2 = -M + R, \tag{3.7}$$

where the entries of the residual matrix $\{R_{ij}\}$ are given by

$$R_{ij}(x,t) := \int_{y \in \mathbb{R}^2} \partial_j a(x,y,t)(u_i(y,t) - u_i(x,t))\rho(y,t)\mathrm{d}y, \qquad a(x,y,t) = \frac{\phi(|x-y|)}{(\phi * \rho)(x,t)}$$

Expressed in terms of the operator $A(w) := \int_y a(x,y,t)w(y)\mathrm{d}y$, the entries of R have the commutator structure $R_{ij} = \partial_j[A, u_i](\rho)$ which can be estimated by the commutator bound [27, proposition 7.1] in terms of $V(t) = \sup_{\operatorname{supp}(\rho)} |u_i(x,t) - u_i(y,t)|$,

$$|R_{ij}(x,t)| = \left|\partial_j[A, u_i](\rho)\right| \leq \frac{|\phi'|_\infty}{\phi_\infty} V_0 \,\mathrm{e}^{-\kappa t}, \qquad \kappa = \phi_\infty.$$

We now proceed as before. As a first step, we follow the dynamics of the $\mathrm{d} = \operatorname{div} \mathbf{u}$: taking the trace of (3.7), we find

$$\mathrm{d}' + \frac{1}{2}(\mathrm{d}^2 + \eta_S^2) = \omega^2 - \mathrm{d} + r, \qquad r := \mathrm{Tr}\, R \le 2\frac{|\phi'|_\infty}{\phi_\infty} V_0. \tag{3.8}$$

This calls for the introduction of a new variable $\mathrm{e} := \mathrm{d} + 1$, where the last equation recast into the Ricaccti's form

$$\mathrm{e}' = \frac{1}{2}\Big(1 - \eta_S^2 + 2r - \mathrm{e}^2\Big) + \omega^2. \tag{3.9}$$

Our purpose is to show that the $\{x \mid \mathrm{e}(x,t) \ge 0\}$ is invariant of the dynamics (3.9) and to this end we need to bound the spectral gap η_S.

The second step is to follow the spectral dynamics associated with the symmetric part of (3.7)

$$\mu_i'(S) + \mu_i^2(S) = \omega^2 - \mu_i(S) + \langle \mathbf{s}_i, R_{\mathrm{sym}}\mathbf{s}_i \rangle.$$

Taking the difference and recalling that $\mathbf{s}_i$ are the normalized eigenvectors of S we find the dynamics of the spectral gap,

$$\eta_S' + \mathrm{e}\eta_S = q, \qquad |q| \le 2\max|R_{ij}(x,t)| \le 2\frac{|\phi'|_\infty}{\phi_\infty} V_0\, \mathrm{e}^{-\kappa t}. \tag{3.10}$$

It follows that as long as $\mathrm{e}(\cdot,t)$ is positive, then

$$|\eta_S(x,t)| \le \max_x |\eta_{S_0}(x)| + 2\frac{|\phi'|_\infty}{\phi_\infty^2} V_0 < \frac{1}{2}, \tag{3.11}$$

and therefore $c := \sqrt{1 - \eta_S^2 + 2r}$ has the lower bound $c(x,t) \ge c_{\min} > 0$, where

$$\max_x |\eta_{S_0}(x)| + \Big(2\frac{|\phi'|_\infty}{\phi_\infty^2} + 4\frac{|\phi'|_\infty}{\phi_\infty}\Big) V_0 \le 1 - c_{\min}^2 < 1$$

This inequality follows from the assumed bounds on V_0 in (3.4) and on the initial spectral gap (3.6), and the bound of r in (3.8). As a final step, we return to (3.9) to find, $\mathrm{e}' \ge \frac{1}{2}(c_{\min}^2 - \mathrm{e}^2)$, which guarantees that if the critical threshold (3.5) holds, i.e. if $\mathrm{e}_0 \ge 0$, then $\mathrm{e}(x,t) \ge 0$ at a later time. Moreover, since $\mathrm{e}(\cdot,t) \ge 0$, the vorticity equation, $\omega' + \mathrm{e}\omega = \frac{1}{2}\mathrm{Tr}\, JR$, shows that $|\omega(\cdot,t)|$ remains bounded in terms of $\max_x |R_{ij}(x,t)| \lesssim r_{\max} < \infty$. The transport equation (3.9) implies

$$\mathrm{e}' \le \frac{1}{2}\Big(1 + 2r + 2\omega^2 - \mathrm{e}^2\Big) \le \frac{1}{2}\Big(\frac{3}{2} + 2\omega_{\max}^2 - \mathrm{e}^2\Big),$$

and a uniform upper bound of $\mathrm{e}(\cdot,t) \le \mathrm{e}_{\max} < \infty$ follows. □

Remark 3.2. In the one-dimensional case, $\eta_S = \omega \equiv 0$ and the dynamics of $\mathrm{e} = \mathrm{d} + 1$ in (3.9) simplifies into $\mathrm{e}' = \frac{1}{2}(1 + 2r - \mathrm{e}^2)$. Hence, the variation bound (3.4) can be related to

$$V_0 < m_0 \min\left\{ |\phi|_1\, , \frac{1}{4}\frac{\phi_\infty}{|\phi'|_\infty} \right\}$$

so that $1 + 2r \ge c_{\min} > 0$ and $\mathrm{e}' > \frac{1}{2}(c_{\min} - \mathrm{e}^2)$ implies global smoothness under the critical threshold condition $\partial_x u_0(x) \ge -1$.

Remark 3.3. One can follow the argument in section 2.2 to conclude that the same rapid alignment holds for MT model. Indeed, the MT model enhances the convergence *rate* towards a limiting flocking state.

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