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TOPOLOGICALLY BASED FRACTIONAL DIFFUSION AND EMERGENT DYNAMICS WITH SHORT-RANGE INTERACTIONS*

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Abstract. We introduce a new class of models for emergent dynamics. It is based on a new communication protocol which incorporates two main features: short-range kernels which restrict the communication to local metric balls, and anisotropic communication kernels, adapted to the local density in these balls, which form *topological neighborhoods*. We prove flocking behavior—the emergence of global alignment for regular, nonvacuous solutions of the n-dimensional models based on short-range topological communication. Moreover, global regularity (and hence unconditional flocking) of the one-dimensional model is proved via an application of a De Giorgi-type method. To handle the *nonsymmetric* singular kernels that arise with our topological communication, we develop a new analysis for *local* fractional elliptic operators (interesting in its own right) encountered in the construction of our class of models.

Key words. flocking, alignment, collective behavior, emergent dynamics, fractional diffusion, Cucker–Smale, Motsch–Tadmor

AMS subject classifications. 92D25, 35Q35, 76N10

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1. Introduction and statement of main results.

1.1. Emergent dynamics: Long-range and short-range kernels. A fascinating aspect of collective dynamics is self-organization, in which higher order patterns emerge from an underlying dynamics driven by short-range interactions. This type of collective dynamics is found in a wide variety of biological, social, and technological contexts. We investigate this phenomenon in the context of canonical models for flocking and swarming. A key feature in these models is *alignment*, where a crowd described as a continuum with density $\rho(t,\mathbf{x}) : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}_+$ aligns its macroscopic velocity, $\mathbf{u}(t,\mathbf{x}) : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$, over the local neighborhoods $\mathscr{N}(\mathbf{x})$,

$$
\begin{cases} \rho_t + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) = 0, \\ \mathbf{u}_t + \mathbf{u} \cdot \nabla_{\mathbf{x}} \mathbf{u} = \displaystyle\int_{\mathscr{N}(\mathbf{x})} \phi(\mathbf{x},\mathbf{y})(\mathbf{u}(t,\mathbf{y}) - \mathbf{u}(t,\mathbf{x}))\rho(t,\mathbf{y})\,\mathrm{d}\mathbf{y}. \end{cases} \tag{1.1}
$$

The dynamics is subject to prescribed initial conditions, $(\rho_0, \mathbf{u}_0)$, with two main configurations: either compactly supported density $diam\,\{supp\,\rho_0\} \leqslant D_0$ in $\mathbb{R}^n$, or

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over the torus $\mathbb{T}^n$. System (1.1) corresponds to the large-crowd description of a discrete crowd, consisting of $N \gg 1$ agents (of birds, insects, fish, robots, etc.) which align their microscopic velocities, $\{\mathbf{v}_i(t)\}_{i=1}^N \in \mathbb{R}^n$,

$$\dot{\mathbf{v}}_i = \sum_{j\in\mathscr{N}(\mathbf{x}_i)} \phi\big(\mathbf{x}_i(t),\mathbf{x}_j(t)\big)(\mathbf{v}_j(t)-\mathbf{v}_i(t)), \qquad \dot{\mathbf{x}}_i = \mathbf{v}_i. \tag{1.2}$$

Different models distinguish themselves with different choices of communication kernels, $\phi(\cdot,\cdot) \geqslant 0$, which dictate the neighborhoods $\mathscr{N}(\mathbf{x}) := \{\mathbf{y}\,|\,\phi(\mathbf{x},\mathbf{y}) > 0\}$. The most notable examples found in the literature [34, 1, 44, 56, 3, 20, 21, 38] employ radial kernels depending on the *metric distance*

$$\phi(\mathbf{x},\mathbf{y}) = \varphi(|\mathbf{x}-\mathbf{y}|); \tag{1.3}$$

that is, communication is taking place in balls, $\mathscr{N}(\mathbf{x}) = B_{R_0}(\mathbf{x})$, where R_0 is the diameter of $supp\,\varphi$:

$$\begin{cases}\rho_t + \nabla_{\mathbf{x}}\cdot(\rho\mathbf{u}) = 0,\\ \mathbf{u}_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} = \displaystyle\int_{B_{R_0}(\mathbf{x})} \varphi(|\mathbf{x}-\mathbf{y}|)(\mathbf{u}(t,\mathbf{y})-\mathbf{u}(t,\mathbf{x}))\rho(t,\mathbf{y})\,\mathrm{d}\mathbf{y}.\end{cases} \tag{1.4}$$

The communication kernels are in general unknown: their approximate shape is either derived empirically [16, 2, 15, 14, 19, 11] or learned from the data [8, 36] or postulated based on phenomenological arguments [57, 5, 4]. Since the precise form of the communication kernel is in general not known, it is therefore imperative to understand how general φ's affect the large-time, large-crowd dynamics. It is here that we make a distinction between *long-range* and *short-range* interactions.

Long-range interactions. Here, the support of φ is large enough, $R_0 \gg 1$, so that every part of the crowd is in direct communication with every other part. In particular, if φ satisfies

$$\text{a "fat tail" condition:} \qquad \int^\infty \varphi(r)\,\mathrm{d}r = \infty, \tag{1.5}$$

then $supp\,\rho(t,\cdot)$ remains within a finite diameter $D_\infty < \infty$, and consequently, the alignment dynamics (1.4) enforces the the crowd to "aggregate" around a limiting velocity, $\mathbf{u}_\infty \in \mathbb{R}^n$. The flocking behavior in this case of long-range interactions is captured by the statement "smooth solutions must flock" [53, 30]; namely, if $(\rho(t,\cdot),\mathbf{u}(t,\cdot)) \in L^\infty \times W^{1,\infty}$ is a global strong solution of (1.4),(1.5) subject to compactly supported initial data $(\rho_0,\mathbf{u}_0)$, then there exists $\eta > 0$ (depending on D_∞) such that $\mathbf{u}(t,\cdot)$ flocks towards a limiting velocity $\mathbf{u}_\infty$,

$$\max_{\mathbf{x}} |\mathbf{u}(t,\mathbf{x})-\mathbf{u}_\infty| \lesssim e^{-\eta t} \to 0, \qquad \mathbf{u}_\infty = \frac{\mathbf{P}_0}{M_0}, \quad (M_0,\mathbf{P}_0) := \int (1,\mathbf{u}_0)\rho_0(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{1.6}$$

The unconditional flocking asserted in (1.6) is rooted in the corresponding statement for the discrete dynamics (1.2) with long-range interactions (1.3), (1.5) [12, 20, 21, 28, 27, 26, 39].

The conditional statement for long-range interactions shifts the burden of proving their flocking behavior to the regularity theory. Here we make a further distinction between bounded and singular φ's.

For *bounded kernels*, global regularity in dimension $n = 1, 2$ holds if the initial configuration satisfies certain threshold conditions [53, 13, 30]. Global regularity (and hence flocking behavior) of (1.4) for any dimension but for small data in higher order Sobolev spaces, $\|\mathbf{u}\|_{H^{s+1}} < \varepsilon_0(\|\rho_0\|_{H^s})$, was proved in [25]. The regularity and flocking behavior of (1.4) with *singular kernels* $\varphi(r) = r^{-\beta}$ was studied in [43] for weakly singular kernels, $0 < \beta < n$ (consult [40, 41, 42] for the corresponding discrete case), and in [51, 49, 50, 23] for strongly singular kernels, $\beta = n + \alpha$, $0 < \alpha < 2$. In the latter case, the system (1.4) is endowed with a fractional parabolic diffusion structure which enabled us to prove, at least in the one-dimensional case, *unconditional flocking behavior*, independent of any initial threshold. We quote here our main result of [51, 50] which will be echoed in the statements of this present paper: for the system (1.4) with strongly singular kernel, $\varphi(r) = r^{-(n+\alpha)}, 0 < \alpha < 2$, on $\mathbb{T}$, any nonvacuous initial data gives rise to a unique global solution, $(\rho, u) \in L^\infty([0,\infty); H^{s+\alpha} \times H^{s+1})$, $s \geqslant 3$, which converges to a flocking traveling wave,

$$\|u(t,\cdot) - u_\infty\|_{H^s} + \|\rho(t,\cdot) - \rho_\infty(\cdot - tu_\infty)\|_{H^{s-1}} \lesssim e^{-\eta t}, \quad t > 0, \qquad u_\infty := \frac{P_0}{M_0}.$$

The question of regularity (and hence flocking) for strongly singular kernels $\varphi(r) = r^{-(n+\alpha)}$ in dimensions $n > 1$ is open, with the exceptions of recent small initial data results in [48] for Hölder spaces, $|\mathbf{u}_0 - \mathbf{u}_\infty|_\infty \lesssim (1 + \|\rho_0\|_{W^{3,\infty}} + \|\mathbf{u}_0\|_{W^{3,\infty}})^{-n}$ with $2/3 < \alpha < 3/2$, and in [22] for small Besov data $\|\mathbf{u}_0\|_{B^{2-\alpha}_{n,1}} + \|\rho_0 - 1\|_{B^1_{n,1}} \leqslant \varepsilon$ with $\alpha \in (1, 2)$.

Short-range interactions. The class of singular kernels $\varphi(r) = r^{-\beta}$ offers a communication framework which emphasizes short-range interactions over long-range interactions, yet their global support still reflects global communication. In particular, strongly singular kernels, $n < \beta < n + 2$, demonstrate hydrodynamic flocking for thinner tails than those sought in (1.5), yet their infinite support still maintains global direct communication over all *supp* $\rho(t,\cdot)$.

This brings us back to the original question alluded to at the beginning—namely, understanding self-organization driven by a *purely local communication protocol*. This is the question we address in our present work in the context of general alignment (1.1) with short-range singular communication kernels[1]

$$\frac{\mathbb{1}_{|\mathbf{x}-\mathbf{y}|<R_0}}{|\mathbf{x}-\mathbf{y}|^{n+\alpha}} \lesssim \phi(\mathbf{x},\mathbf{y}) \lesssim \frac{\mathbb{1}_{|\mathbf{x}-\mathbf{y}|<2R_0}}{|\mathbf{x}-\mathbf{y}|^{n+\alpha}}, \quad 0 < \alpha < 2. \tag{1.7}$$

It provides a first fundamental step in our understanding of emergent phenomena in collective dynamics driven by short-range communication kernels.

It has been an open question whether the emergence of hydrodynamic flocking survives the cut-off localization in (1.7). The situation is analogous to the scenario of a discrete crowd with short-range communication, (1.2), which may fail to flock due to finite-time loss of graph connectivity associated with the time-dependent adjacency matrix $\{\phi(\mathbf{x}_i(t), \mathbf{x}_j(t))\}$ [39, sec. 2.2]. The subtle issue in the discrete case is that a short-range *time-dependent covering*, $\cup_{i=1}^N \mathscr{N}_i(t)$, may be unstable for a finite N, and one needs to *assume* the persistence of connectivity for all time [31] or at least close enough to a constant state—so close that it does not allow connectivity to be lost [55, 32]. At the level of hydrodynamic description (1.1), lack of connectivity manifests

[1]Here and throughout $\mathbb{1}_S$ denotes the characteristic function of a set S, and $A \lesssim B$ means $A/B < C$, where C is a fixed constant.

itself as "thinning" of crowd density inside *supp* $\rho(t,\cdot)$ and eventually creating vacuous subregions in which the flow does not exert any alignment on its neighborhood. In this case, the dynamics (1.1) is reduced to inviscid Burgers-type blowup [54], thereby demonstrating the necessity of the no-vacuum assumption. This brings us to our first main result, asserting that smooth nonvacuous solutions of alignment dynamics associated with a general class of *short-range* singular kernels, (1.7), must flock.

THEOREM 1.1 (smooth solutions must flock—singular symmetric kernels). *Let* $(\rho(t,\cdot),\mathbf{u}(t,\cdot))$ *be a global strong solution of the alignment dynamics* (1.1) *with short-range symmetric kernel* (1.7)*, over the torus* $\mathbb{T}^n$*. Assume that*

$$\eta(t) := \int^t \rho_-^2(s)\,ds \stackrel{t\to\infty}{\longrightarrow} \infty, \qquad \rho_-(t) := \min_{\mathbf{x}} \rho(t,\mathbf{x}). \tag{1.8}$$

Then there is convergence towards flocking (*with the average velocity* $\mathbf{u}_\infty = \frac{\mathbf{P}_0}{M_0}$)

$$\int_{\mathbb{T}^n} |\mathbf{u}(t,\mathbf{x}) - \mathbf{u}_\infty|^2 \rho(t,\mathbf{x})\,\mathrm{d}\mathbf{x} \leqslant \frac{1}{2M_0} e^{-\eta(t)}. \tag{1.9}$$

Note that any positive lower bound on the density is impossible in the open space if finite mass is assumed. So, periodic conditions are more natural for the setting. Compactness is also important for the proof, which is presented in section 3 below. Theorem 1.1 provides a general framework for the flocking of alignment dynamics driven by short-range singular communication kernels, under the assumption that the global solution is nonvacuous. Here, the precise decay rate of the density $\min \rho(t,\cdot)$ is at the heart of matter: according to Theorem 1.1, unconditional flocking is achieved under the lower bound

$$\rho(t,\cdot) \gtrsim \frac{1}{\sqrt{1+t}}. \tag{1.10}$$

The difficulty is that verification of such a priori lower bound seems out of reach. To address this difficulty, we now introduce a new *topological* short-range communication protocol which tames the required decay rate of the density by adapting itself to sub-regions with thinner densities. Moreover, the new protocol is more realistic in various behavioral experiments than the purely metric one, as we will discuss in the next section.

1.2. A new paradigm for collective dynamics: Topological kernels. We introduce a new communication protocol based on the principle that *information between agents spreads faster in regions of lower density.* To realize this principle we consider a communication kernel of the form

$$\phi(\mathbf{x},\mathbf{y}) = \varphi(|\mathbf{x}-\mathbf{y}|) \times \frac{1}{\mathrm{d}_\rho^n(\mathbf{x},\mathbf{y})}, \tag{1.11a}$$

which depends on two main features.

(i) *Metric distances.* $\varphi(r)$ reflects the dependence on metric distance in $\mathbb{R}^n$ (and, respectively, in $\mathbb{T}^n$), $r(\mathbf{x},\mathbf{y}) = |\mathbf{x}-\mathbf{y}|$. For the metric part of the communication, we use the short-range singular kernel

$$\varphi(r) = \frac{h(r)}{r^\alpha}, \qquad \mathbb{1}_{r<R_0} \lesssim h(r) \lesssim \mathbb{1}_{r<2R_0}, \quad 0<\alpha<2. \tag{1.11b}$$

The smooth cut-off $h(r)$ guarantees that communication is localized in balls of radius $\leqslant 2R_0$.

(ii) *Topological distances.* For any two parts of the crowd at two different locations $\mathbf{x}, \mathbf{y} \in supp\, \rho(t, \cdot)$, we fix an intermediate region of communication $\Omega(\mathbf{x}, \mathbf{y}) \subset \mathbb{R}^n$ (or $\subset \mathbb{T}^n$). In the one-dimensional case, it is taken simply as the closed interval $\Omega(x, y) = [x, y]$; in the multidimensional case, we choose a conical region outlined in section 2.1. Then, $\mathrm{d}_\rho(\mathbf{x}, \mathbf{y})$ reflects the dependence on the "mass" as a *topological measure of a distance* between the crowd at $\mathbf{x}$ and $\mathbf{y}$; specifically,

$$\mathrm{d}_\rho(\mathbf{x}, \mathbf{y}) := \left[\int_{\Omega(\mathbf{x},\mathbf{y})} \rho(t, \mathbf{z})\, \mathrm{d}\mathbf{z}\right]^{\frac{1}{n}} \quad \text{with } \Omega(\mathbf{x}, \mathbf{y}) \text{ given in (2.3).} \tag{1.11c}$$

Remark 1.2 (Why topological distances?). To motivate the so-called topological distances (1.11c) we refer to the underlying discrete setup (1.2). The discrete configuration of N agents is captured by the empirical distribution $\mu_t(\mathbf{x}, \mathbf{v}) = \frac{1}{N}\sum_k \delta_{\mathbf{x}_k(t)}(\mathbf{x}) \otimes \delta_{\mathbf{v}_k(t)}(\mathbf{v})$. Then $\mu_t(\Omega(\mathbf{x}_i, \mathbf{x}_j))$ amounts to counting the (discrete) crowd in the region of communication $\Omega(\mathbf{x}_i, \mathbf{x}_j)$, and we set the discrete distance to be

$$\mathrm{d}_N(\mathbf{x}_i, \mathbf{x}_j) := \big(\mu_t(\Omega(\mathbf{x}_i, \mathbf{x}_j))\big)^{\frac{1}{n}} = \left(\frac{\#\{\mathbf{x}_k \,|\, \mathbf{x}_k \in \Omega(\mathbf{x}_i, \mathbf{x}_j)\}}{N}\right)^{\frac{1}{n}}.$$

The dependence of the communication kernel (1.11a) on $\mathrm{d}_N^{-n}(\mathbf{x}_i, \cdot)$ indicates that the agent at $\mathbf{x}_i$ places a strong preference of communication with its nearest agents, $\{\mathbf{x}_j \,|\, \mathrm{d}_N(\mathbf{x}_i, \mathbf{x}_j) \sim N^{-\frac{1}{n}}\}$, over the increased interference in communication with agents farther away, $\{\mathbf{x}_j \,|\, \mathrm{d}_N(\mathbf{x}_i, \mathbf{x}_j) \lesssim 1\}$. The net effect of probing low density neighborhoods using such singular kernels is communication dictated by the number of nearest agents rather than geometric proximity [29, 6, 7]. Letting $N \to \infty$ recovers the topological distance (1.11c) in the continuum setup, $\mathrm{d}_N(\mathbf{x}, \mathbf{y}) \stackrel{N\to\infty}{\longrightarrow} \mathrm{d}_\rho(\mathbf{x}, \mathbf{y})$. Thus, the corresponding alignment dynamics (1.1), (1.11) is a continuum realization of the same paradigm—namely, enhancing communication in regions of low density by invoking the "density of closest neighbors" as the proper continuum substitute for the "number of closest neighbors." Accordingly, we refer to $\mathrm{d}_\rho(\mathbf{x}_i, \mathbf{x}_j)$ as *topological (quasi-)distance.* This is consistent with the established terminology in experimental literature, which refers to such topological communication in flocking birds [16, 2, 15, 14] and in human interaction in pedestrian dynamics [46].

Noting that $\mathrm{d}_\rho(\mathbf{x}, \mathbf{y}) \gtrsim c(\rho)|\mathbf{x} - \mathbf{y}|$, it follows that $\phi(\mathbf{x}, \mathbf{y})$ is singular of order $n + \alpha$, $\phi(\mathbf{x}, \mathbf{y}) \lesssim \mathbb{1}_{|\mathbf{x}-\mathbf{y}| \leqslant 2R_0} |\mathbf{x} - \mathbf{y}|^{-(n+\alpha)}$. Thus, the topological kernel (1.11) belongs to the general class of short-range kernels (1.7). It reflects short-range communication (of diameter $\leqslant 2R_0$), maintaining finite amplitude $\{\mathbf{y} \mid \phi(\mathbf{x}, \mathbf{y}) \gtrsim 1\}$ within active topological neighborhoods

$$\mathscr{N}(\mathbf{x}) = \{\mathbf{y} \in B_{2R_0}(\mathbf{x}) \mid \mathrm{d}_\rho(\mathbf{x}, \mathbf{y}) < c_0\},$$

where c_0 is an empirical constant indicating perception ability of the agents. The kernel is nonconvolutive, and though ϕ is symmetric, $\phi(\mathbf{x}, \mathbf{y}) = \phi(\mathbf{y}, \mathbf{x})$, the full kernel that appears in the alignment term, $K(\mathbf{x}, \mathbf{y}, t) := \phi(\mathbf{x}, \mathbf{y})\rho(\mathbf{y})$, is not. The proper notion of the nonsymmetric (strongly) singular alignment action on the right-hand side of (1.1), $\mathscr{C}_\phi(\rho, f) = \int \phi(\mathbf{x}, \mathbf{y})(f(\mathbf{y}) - f(\mathbf{x}))\rho(\mathbf{y})\, \mathrm{d}\mathbf{y}$, is discussed in section 2.2. This brings us to our second main result.

THEOREM 1.3 (flocking of short-range topological kernels). *Let* $(\rho, \mathbf{u})$ *be a global smooth solution of the topological model* (1.1), (1.11) *on* $\mathbb{T}^n$. *Assume that the density* $\rho(t,\cdot)$ *satisfies*

$$\rho(t,\mathbf{x}) \geqslant \frac{c}{1+t}. \tag{1.12}$$

Then the solution aligns with $\mathbf{u}_\infty$ *with at least a root-logarithmic rate*

$$|\mathbf{u}(t) - \mathbf{u}_\infty|_\infty \lesssim \frac{c}{\sqrt{\ln t}}. \tag{1.13}$$

The proof of Theorem 1.3, given in section 3.2 below, traces the propagation of information between the extreme values of (the components of) $\mathbf{u}(t,\cdot)$, which are most susceptible to breakup since they can no longer rely on distant communication. Instead, we introduce a new method of sliding averages, in which we measure how far $\mathbf{u}(t,\mathbf{x})$ deviates from its average over the *local* balls $B(\mathbf{x}, r)$, $r \leqslant R_0$, using a density-weighted Campanato class. For some algebraic sequence of times $t_n \to \infty$, these deviations are proved to be small. At the same time, we show that, overwhelmingly, $\mathbf{u}(t,\mathbf{x})$ stays close to its extreme values near the critical points where these values are attained. To achieve this, we estimate the conditional probability of an unlikely event of $\mathbf{u}$ being far from its extremes, in terms of the mass-measure $\mathrm{d}m_t = \rho\,\mathrm{d}\mathbf{x}$: it is here that the topological-based alignment in (1.11a) plays a key role. We end up with a (finite) overlapping chain of nonvacuous balls to connect any two points, and by chain estimates, the fluctuations of $\mathbf{u}(t,\cdot)$ are shown to decay uniformly in time. This explains the emergence of global alignment from short-range interactions, which, to the best of our knowledge, is the first result of its kind.

In closing this section, a couple of remarks are in order.

Remark 1.4 (a comparison with Motsch–Tadmor scaling). It is instructive to compare the topological kernel (1.11), which we rewrite as

$$\phi(\mathbf{x},\mathbf{y}) = \varphi(|\mathbf{x}-\mathbf{y}|) \times \frac{1}{m_t(\Omega(\mathbf{x},\mathbf{y}))}, \qquad m_t(\Omega) := \int_\Omega \rho(t,\mathbf{z})\,\mathrm{d}\mathbf{z},$$

with the Motsch–Tadmor scaling [38] with local $\varphi(r) = \mathbb{1}_{r<R_0}$,

$$\phi(\mathbf{x},\mathbf{y}) = \varphi(|\mathbf{x}-\mathbf{y}|) \times \frac{1}{m_t(B_{R_0}(\mathbf{x}))}.$$

In the former, the pairwise interaction between two "agents" depends on the density in an intermediate region of communication; in the latter, the communication of each "agent" depends on how thin the crowd is in its own metric neighborhood.

1.3. Global regularity: Drift-diffusion beyond symmetric kernels. As in the case of long-range communication, Theorem 1.3 shifts the "burden" of proving flocking with short-range topological kernels to the question of existence: do (1.1), (1.11) admit global smooth solutions with lower-bounded density $\rho(t,\cdot) \gtrsim (1+t)^{-1}$? In section 4, which is at the heart of matter and occupies the bulk of this paper, we provide an affirmative answer for the one-dimensional model over $\mathbb{T}$, thus providing a first example of unconditional flocking. The question of nonvacuous global regularity in dimension $n > 1$ remains open.

To elaborate further on the required regularity of (ρ, u), we note that both density and momentum equations in (1.1) fall under a general class of *parabolic drift-diffusion* equations,

$$u_t + \mathbf{b} \cdot \nabla_{\mathbf{x}} u = \int K(\mathbf{x}, \mathbf{y}, t)(u(\mathbf{y}) - u(\mathbf{x}))\, \mathrm{d}\mathbf{y} + f,$$

with (a priori) rough coefficients, $\mathbf{b}$, and with a proper singular local kernels

$$\frac{\mathbb{1}_{|\mathbf{x}-\mathbf{y}|<R_0}}{|\mathbf{x}-\mathbf{y}|^{1+\alpha}} \lesssim K(\mathbf{x}, \mathbf{y}, t) \lesssim \frac{\mathbb{1}_{|\mathbf{x}-\mathbf{y}|<2R_0}}{|\mathbf{x}-\mathbf{y}|^{1+\alpha}}.$$

Regularity theory for equations of this type had a rapid development in recent years due to breakthroughs in understanding of the nonlocal structure of the fractional Laplacian; see Caffarelli et al. [9, 10], Silverstre et al. [52, 47], Mikulevicius and Pragarauskas [37], and local jump processes in Chen et al. [17] and the references therein. Any of these regularity results requires, however, the symmetry of the kernel $K(\cdot, \cdot, t)$, which we lack in the present framework; thus, the velocity $\mathbf{u}$ in our topological model (1.1) is governed by drift-diffusion associated with kernel $K(\mathbf{x}, \mathbf{y}) = \phi(\mathbf{x}, \mathbf{y})\rho(\mathbf{y})$; while $\phi(\cdot, \cdot)$ is symmetric, K is not. Similarly, the same dynamics expressed in terms of the momentum, $\mathbf{m} := \rho\mathbf{u}$, or the density (consult (4.10) and, respectively, (4.9)) encounters the nonsymmetric kernel $K(\mathbf{x}, \mathbf{y}) = \phi(\mathbf{x}, \mathbf{y})\rho(\mathbf{x})$.

Lack of symmetry in the K-kernels associated with the topological communication (1.11) poses a fundamental difficulty which prevents us from using the known results about the regularizing effect in such transport-diffusion. Instead, we adapt the De Giorgi method to settle the Hölder regularity of $\rho(t, \cdot)$ in the critical case $\alpha = 1$ (section 4.4.2) and employ fractional Schauder estimates to address the $\alpha > 1$ case (section 4.4.1). Together with the propagation of higher order regularity proved in section 4.3, we arrive at our third main regularity result stated below.

THEOREM 1.5 (global regularity of 1D topological model). *Consider the one-dimensional system* (1.1) *on* $\mathbb{T}$ *with short-range topological kernel* (1.11) *and singularity of order* $1 \leqslant \alpha < 2$. *Any nonvacuous initial data* $(\rho_0, u_0) \in H^{s+\alpha} \times H^{s+1}$, $s \geqslant 3$, *admits a unique global in time solution,* (ρ, u), *in the class*

$$\rho \in C_w(\mathbb{R}^+; H^{s+\alpha}) \cap L^2_{\mathrm{loc}}(\mathbb{R}^+; H^{s+1+\frac{\alpha}{2}}),$$
$$u \in C_w(\mathbb{R}^+; H^{s+1}) \cap L^2_{\mathrm{loc}}(\mathbb{R}^+; H^{s+1+\frac{\alpha}{2}}),$$

which flocks $|u(t, \cdot) - u_\infty|_\infty \to 0$.

Here, C_w designates the space of weakly continuous function. Let us note that the density-enstrophy is expected to persist in a more natural, stronger regularity space $L^2_t H^{s+\alpha+\frac{\alpha}{2}}_x$ with $\alpha > 1$, yet proving this would involve rather technical fractional energy estimates directly in $H^{s+\alpha}$, which we will postpone to future work.

Remark 1.6. What distinguishes the 1D setup is a conservation law, $e_t + (ue)_x = 0$, of the first-order quantity $e = u_x + \int \phi(x, y)(\rho(y) - \rho(x))\, \mathrm{d}x$; while this is known for the metric kernels $\phi = \varphi(|x - y|)$ [13, 49, 23], it is remarkable that the same conservation law still survives for the *anisotropic* topological kernels $\varphi(|x-y|)\mathrm{d}_\rho(x, y)$. In section 4.1 we show that it enforces the parabolic character of the 1D mass equation $\rho_t + (u\rho)_x = 0$, and in section 4.2 we show that it implies the lower-bound $\rho(t, \cdot) \gtrsim (1 + t)^{-1}$ sought in (1.12).

1.4. Notation. The following notation is used throughout the text: $|f|_p$ stands for the classical L^p-norm, $1 \leqslant p \leqslant \infty$, $\|f\|_X$ stands for all other norms such as H^s, etc., and $[f]_\gamma$, $0<\gamma<1$, stands for the Hölder seminorm. The use of the following brackets is adopted:

$$\langle f,g\rangle = \int_{\mathbb{T}^n} f(\mathbf{x})g(\mathbf{x})\,\mathrm{d}\mathbf{x}, \quad \langle f,g\rangle_\rho = \int_{\mathbb{T}^n} f(\mathbf{x})g(\mathbf{x})\rho(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

We denote $\delta_{\mathbf{z}} f(\mathbf{x}) = f(\mathbf{x}+\mathbf{z}) - f(\mathbf{x})$. For Sobolev spaces of fractional order, $H^s(\mathbb{T}^n)$, $0<s<1$, we always adopt the Gagliardo definition, which states

$$\|f\|_{H^s}^2 = \int_{\mathbb{T}^n} |\delta_{\mathbf{z}} f(\mathbf{x})|^2 \phi_s(\mathbf{z})\,\mathrm{d}\mathbf{z}, \tag{1.14}$$

where

$$\phi_s(\mathbf{z}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} \frac{1}{|\mathbf{z}+2\pi\mathbf{k}|^{n+2s}}.$$

Considering f periodically extended to $\mathbb{R}^n$ the above is the same as

$$\|f\|_{H^s}^2 = \int_{\mathbb{R}^n} |\delta_{\mathbf{z}} f(\mathbf{x})|^2 \frac{\mathrm{d}\mathbf{z}}{|\mathbf{z}|^{n+2s}}.$$

We sometimes may use the latter for the benefit of a more explicitly defined kernel.

2. The new protocol: Short-range topological diffusion.

In what follows we restrict ourselves to the periodic domain $\mathbb{T}^n$. This choice is motivated by the fact that the density in (1.1) quantifies parabolicity of the equation. With finite mass $M<\infty$ such parabolicity cannot be controlled uniformly on the open space. In this section we elaborate on the basic ingredients which are involved in the short-range singular topological alignment model (1.1), (1.11),

$$\begin{cases} \rho_t + \nabla\cdot(\rho\mathbf{u}) = 0, \\ \mathbf{u}_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} = \displaystyle\int_{\mathbb{T}^n} \phi(\mathbf{x},\mathbf{y})(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}, \end{cases} \tag{2.1a}$$

where ϕ is the topological kernel given by

$$\phi(\mathbf{x},\mathbf{y}) = \frac{h(|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|^\alpha} \times \frac{1}{\mathrm{d}_\rho^n(\mathbf{x},\mathbf{y})}, \quad \mathbb{1}_{r<R_0} \lesssim h(r) \lesssim \mathbb{1}_{r<2R_0}, \quad 0<\alpha<2. \tag{2.1b}$$

Here, the first component of the kernel is quantified in terms of metric distance $|\mathbf{x}-\mathbf{y}|$, and the second involves the topological "distance" $\mathrm{d}_\rho(\mathbf{x},\mathbf{y})$ between $\mathbf{x}$ and $\mathbf{y}$, defined by the mass located in the intermediate region of communication $\Omega(\mathbf{x},\mathbf{y})$

$$\mathrm{d}_\rho(\mathbf{x},\mathbf{y}) = \left[\int_{\Omega(\mathbf{x},\mathbf{y})} \rho(t,\mathbf{z})\,\mathrm{d}\mathbf{z}\right]^{\frac{1}{n}}.$$

The region of communication enclosed between $\mathbf{x}$ and $\mathbf{y}$ is outlined in section 2.1 below. Observe that in the absence of pressure each component u of $\mathbf{u}$ satisfies the maximum principle, $\min u_0 \leqslant u(t,\cdot) \leqslant \max u_0$, and that for all global regular solutions, $u \in L^1_{\mathrm{loc}}W^{1,\infty}$, the density remains nonvacuous, $\rho_0(x)>0 \rightsquigarrow \rho(t,\mathbf{x})>0$ for all $t\geqslant 0$; hence we may assume that the density ρ is a nonvacuous kinematic quantity satisfying

$$0<c(t)\leqslant\rho(t,\mathbf{x})\leqslant C(t)<\infty, \quad \mathbf{x}\in\mathbb{T}^n. \tag{2.2}$$

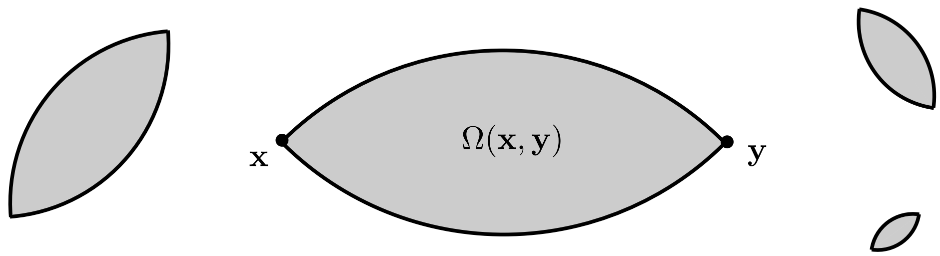

Fig. 1. *Communication domains between agents.*

Note that although the distance function d_ρ is not a proper metric (except for the one-dimensional case where it accumulates the mass along the interval $[x, y]$), it defines an equivalent topology on $\mathbb{T}^n$ such that $\mathrm{d}_\rho(\mathbf{x},\mathbf{y}) \geqslant c|\mathbf{x}-\mathbf{y}|$, and all the distances are bounded by the total mass M. Moreover, since $\Omega(\mathbf{x},\mathbf{y}) = \Omega(\mathbf{y},\mathbf{x})$, the topological distance is symmetric: $\mathrm{d}_\rho(\mathbf{x},\mathbf{y}) = \mathrm{d}_\rho(\mathbf{y},\mathbf{x})$.

2.1. Region of communication. The topological distance $\mathrm{d}_\rho(\mathbf{x},\mathbf{y})$ requires us to specify a domain of communication, $\Omega(\mathbf{x},\mathbf{y})$, which is probed by agents located at $\mathbf{x}$ and $\mathbf{y}$. In the one-dimensional case, it is simply the closed interval, $\Omega(x, y) = [x, y]$. In the multidimensional case, it is reasonably argued that the "intermediate environment" between agents could be an n-dimensional region inside the ball enclosed by $\mathbf{x}$ and $\mathbf{y}$—namely $B(\frac{\mathbf{x}+\mathbf{y}}{2}, r)$ with radius $r := \frac{|\mathbf{x}-\mathbf{y}|}{2}$. For example, one can simply set $\Omega(\mathbf{x},\mathbf{y})$ to be that ball. As we shall see below, however, the fine structure of the local regions of communication, $\Omega(\mathbf{x}_i,\mathbf{x}_j)$, is important in order to retain unconditional flocking. To this end, we set a more restrictive *conical* region $\Omega(\mathbf{x},\mathbf{y})$; see Figure 1. First, we consider two basic locations $\mathbf{x} = (-1, 0, \dots, 0)$ and $\mathbf{y} = (1, 0, \dots, 0)$ and set the region of revolution generated by a parabolic arch connecting $\mathbf{x}$ and $\mathbf{y}$:

$$\Omega_0 := \{\mathbf{z} = (a, \mathbf{z}_-) \mid |\mathbf{z}_-| < 1 - a^2, -1 \leqslant a \leqslant 1\}.$$

For an arbitrary pair of points $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$, let $\Omega(\mathbf{x},\mathbf{y})$ denote the region scaled and translated from Ω_0:

$$\Omega(\mathbf{x},\mathbf{y}) := \{\mathbf{z} \mid |\mathbf{z}-\mathbf{z}_-| < 1 - r^2a^2\}, \qquad r = \frac{|\mathbf{x}-\mathbf{y}|}{2}, \tag{2.3}$$

where $\mathbf{z}_- := \mathbf{z}(a)$ is the projection of $\mathbf{z}$ on the diameter $\{\mathbf{z}_-(a) = \frac{\mathbf{x}+\mathbf{y}}{2} + \frac{a}{2}(\mathbf{y}-\mathbf{x}),\ -1 \leqslant a \leqslant 1\}$ connecting $\mathbf{x}$ and $\mathbf{y}$.

Observe that at the tips, $\Omega(\mathbf{x},\mathbf{y})$ has the opening of $\frac{\pi}{2}$. For subsequent analysis, it can be replaced by any angle $< \pi$, calibrated according to a particular application.[2] It is crucial, however, that the region of communication is not locally smooth near the tips $\mathbf{x},\mathbf{y}$ (see Claim 3.1 below), which excludes the ball $B(\frac{\mathbf{x}+\mathbf{y}}{2}, r)$ with conical opening of $90°$.

2.2. Topological kernels and the operators they define. A distinctive feature of the alignment term on the right-hand side of (2.1a) is that it admits a (formal)

[2]Thus, for example, (2.3) can be enlarged to $\Omega(\mathbf{x},\mathbf{y}) := \{\mathbf{z} \mid |\mathbf{z}-\mathbf{z}_-|^\gamma < 1 - r^2a^2\}$ for any $0 < \gamma < 2$.

commutator structure [49]

$$\int_{\mathbb{T}^n} \phi(\mathbf{x},\mathbf{y})(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} = \mathscr{L}_\phi(\rho\mathbf{u}) - \mathscr{L}_\phi(\rho)\mathbf{u} := \mathscr{C}_\phi(\mathbf{u},\rho),$$

where $\mathscr{L}_\phi$ is the integral operator given formally by

$$\mathscr{L}_\phi(f) := p.v.\int_{\mathbb{T}^n} \phi(\mathbf{x},\mathbf{y})(f(\mathbf{y})-f(\mathbf{x}))\,\mathrm{d}\mathbf{y}. \tag{2.4}$$

Strong solutions to the system (1.1) satisfy energy equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^n} \rho|\mathbf{u}|^2\,\mathrm{d}\mathbf{x} = -\int_{\mathbb{T}^n} \phi(\mathbf{x},\mathbf{y})|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|^2\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}, \tag{2.5a}$$

which will be a key component in establishing alignment. We note in passing that in view of the symmetry of the kernel ϕ, we have conservation of mass and momentum:

$$M(t) = \int_{\mathbb{T}^n} \rho(t,\mathbf{x})\,\mathrm{d}\mathbf{x} \equiv M_0, \quad \mathbf{P}(t) = \int_{\mathbb{T}^n} \rho\mathbf{u}(t,\mathbf{x})\,\mathrm{d}\mathbf{x} \equiv \mathbf{P}_0.$$

Hence, the rate of decay of the energy of the left-hand side of (2.5a) is the same rate of decay of the fluctuations

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^{2n}} |\mathbf{u}(t,\mathbf{x})-\mathbf{u}(t,\mathbf{y})|^2\rho(t,\mathbf{x})\rho(t,\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} = 2M_0\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^n} \rho|\mathbf{u}|^2\,\mathrm{d}\mathbf{x}. \tag{2.5b}$$

Since we have the Galilean invariance $\mathbf{u} \to \mathbf{u}(\mathbf{x}+t\mathbf{U},t)-\mathbf{U}$ and $\rho \to \rho(\mathbf{x}+t\mathbf{U},t)$, we may assume that $\mathbf{P}(t) = \mathbf{P}_0 = 0$.

We note that care has to be taken to properly define the singular integral operators $\mathscr{L}_\phi f(\mathbf{x})$ and the corresponding commutator

$$\mathscr{C}_\phi(f,g) = \int_{\mathbb{T}^n} \phi(\mathbf{x},\mathbf{y})(f(\mathbf{y})-f(\mathbf{x}))g(\mathbf{y})\,\mathrm{d}\mathbf{y} \tag{2.6}$$

for strongly singular kernels $\alpha \geqslant 1$. Our immediate goal below is therefore to develop formal definitions and initial facts about the operator $\mathscr{L}_\phi$ in multidimensional settings (more details specific to the 1D situation will follow in section 2.3). Due to the nonconvolutive and anisotropic nature of the kernel, most of the standard facts do not apply and will need to be readdressed. Our plan is to define $\mathscr{L}_\phi f$ as a distribution first. Then we state a formal justification of pointwise evaluations of $\mathscr{L}_\phi f(\mathbf{x})$ and the commutator $\mathscr{C}_\phi(f,g)$, so as to justify the fundamental bookkeeping of energy/enstrophy fluctuations in (2.5). Technicalities of the proofs will be collected in the appendices.

Definition 2.1 (the topologically based fractional diffusion). *With the kernel given by* (2.1b) *we define an operator* $\mathscr{L}_\phi : H^{\alpha/2} \to H^{-\alpha/2}$ *by the following action: for any* $f \in H^{\alpha/2}$ *and* $g \in H^{\alpha/2}$

$$\langle \mathscr{L}_\phi f, g\rangle = -\frac{1}{2}\int_{\mathbb{T}^{2n}} \phi(\mathbf{x},\mathbf{y})(f(\mathbf{x})-f(\mathbf{y}))(g(\mathbf{x})-g(\mathbf{y}))\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}. \tag{2.7}$$

Note that formally such action could be obtained from (2.4), if (2.4) made sense pointwise, by the usual symmetrization. Clearly, from the Gagliardo definition of $H^{\alpha/2}$, (1.14), we have

$$|\langle \mathscr{L}_\phi f, g\rangle| \lesssim \|f\|_{H^{\alpha/2}}\|g\|_{H^{\alpha/2}}.$$

Due to the symmetry of the kernel, the operator $\mathscr{L}_\phi$ is clearly self-adjoint, and its range is in $H_0^{-\alpha/2}$ (here subscript 0 means mean-free distributions). By the standard operator theory this implies the following statement.

LEMMA 2.2. *The restricted operator* $\mathscr{L}_\phi : H_0^{\alpha/2} \to H_0^{-\alpha/2}$ *is invertible.*

Proof. Clearly, $c_0\|f\|^2_{H_0^{\alpha/2}} \leqslant -\langle \mathscr{L}_\phi f, f\rangle \leqslant C_0\|f\|^2_{H_0^{\alpha/2}}$. Hence, $\|\mathscr{L}_\phi f\|_{H^{-\alpha/2}} \geqslant c\|f\|_{H^{\alpha/2}}$, which shows that the operator has closed range and is injective. If the range is not all of $H_0^{-\alpha/2}$, then there is a $g \in H_0^{\alpha/2}$ for which $\langle \mathscr{L}_\phi f, g\rangle = 0$ for all $f \in H^{\alpha/2}$. Taking $f = g$, we arrive at a contradiction. Thus, $\mathscr{L}_\phi$ is invertible. ∎

In what follows we will need to be able to evaluate the action of the operator pointwise. In the range $0 < \alpha < 1$ such evaluation presents no problem as long as $f \in C^1$. The rigorous argument goes by "unwinding" the symmetric defining formula (2.7). To demonstrate it, let us denote by $L_\phi f(\mathbf{x})$ the integral on the right-hand side of (2.4). Clearly, $L_\phi f \in C(\mathbb{T}^n)$. Let us fix a point $\mathbf{x}_0 \in \mathbb{T}^n$. Let g be the standard nonnegative Friedrichs mollifier supported on the ball of radius 1. Denote $g_\varepsilon = \frac{1}{\varepsilon^n} g((\mathbf{x}-\mathbf{x}_0)/\varepsilon)$. It suffices to show that

$$\langle \mathscr{L}_\phi f, g_\varepsilon\rangle \to L_\phi f(\mathbf{x}_0).$$

Since, for $0 < \alpha < 1$, $L_\phi f(x)$ is a continuous function, we can break up the integral without ambiguity:

$$\begin{aligned}\langle \mathscr{L}_\phi f, g_\varepsilon\rangle &= -\frac{1}{2}\int_{\mathbb{T}^{2n}} (f(\mathbf{x}) - f(\mathbf{y}))(g_\varepsilon(\mathbf{x}) - g_\varepsilon(\mathbf{y}))\phi(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\ &= \int_{\mathbb{T}^{2n}} (f(\mathbf{y}) - f(\mathbf{x}))g_\varepsilon(\mathbf{x})\phi(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} = \langle L_\phi f, g_\varepsilon\rangle \to L_\phi f(\mathbf{x}_0).\end{aligned}$$

The higher case $1 \leqslant \alpha < 2$ is more subtle. Let us show that when ρ and f are smooth, the element $\mathscr{L}_\phi f \in H^{-\alpha/2}$ gains regularity. Formally, this first step is necessary to even discuss pointwise values $\mathscr{L}_\phi f(\mathbf{x})$. So, let us make the following observation:

$$\nabla_{\mathbf{x}} \mathrm{d}_\rho(\mathbf{x}+\mathbf{z},\mathbf{x}) = \frac{1}{\mathrm{d}_\rho^{n-1}(\mathbf{x}+\mathbf{z},\mathbf{x})}\int_{\Omega(\mathbf{x}+\mathbf{z},\mathbf{x})} \nabla\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_{\partial\Omega(\mathbf{x}+\mathbf{z},\mathbf{x})} \vec{\nu}\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}. \tag{2.8}$$

Clearly, if $|\nabla\rho|_\infty < \infty$, then $|\nabla_{\mathbf{x}}\mathrm{d}_\rho(\mathbf{x}+\mathbf{z},\mathbf{x})| \leqslant C|\nabla\rho|_\infty|\mathbf{z}|$ with C depending on a standing hypothesis on the density (2.2). Next, we rewrite the defining formula (2.7) in terms of the difference operator $\delta_{\mathbf{z}} f(\mathbf{x}) := f(\mathbf{x}+\mathbf{z}) - f(\mathbf{x})$,

$$\begin{aligned}\langle \mathscr{L}_\phi f, g\rangle &= -\frac{1}{2}\int_{\mathbb{T}^{2n}} \delta_{\mathbf{z}} f(\mathbf{x})\delta_{\mathbf{z}} g(\mathbf{x})\phi(\mathbf{x},\mathbf{x}+\mathbf{z})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}\\ &= -\frac{1}{2}\int_0^1\int_{\mathbb{T}^{2n}} \delta_{\mathbf{z}} f(\mathbf{x})\nabla g(\mathbf{x}+\theta\mathbf{z})\cdot\mathbf{z}\;\phi(\mathbf{x},\mathbf{x}+\mathbf{z})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}\,\mathrm{d}\theta.\end{aligned}$$

Integrating by parts and recalling (1.11), $\phi(\mathbf{x}+\mathbf{z},\mathbf{x}) = \frac{h(\mathbf{z})}{|\mathbf{z}|^\alpha} \times \mathrm{d}_\rho^{-1}(\mathbf{x}+\mathbf{z},\mathbf{x})$, we obtain

$$\begin{aligned}\langle \mathscr{L}_\phi f, g\rangle &= \frac{1}{2}\int_0^1\int_{\mathbb{T}^{2n}} \delta_{\mathbf{z}}\nabla f(\mathbf{x})\cdot\mathbf{z} g(\mathbf{x}+\theta\mathbf{z})\phi(\mathbf{x},\mathbf{x}+\mathbf{z})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}\,\mathrm{d}\theta\\ &\quad+ \frac{1}{2}\int_0^1\int_{\mathbb{T}^{2n}} \delta_{\mathbf{z}} f(\mathbf{x}) g(\mathbf{x}+\theta\mathbf{z})\delta_{\mathbf{z}}\rho(\mathbf{x})\frac{\nabla\mathrm{d}_\rho(\mathbf{x}+\mathbf{z},\mathbf{x})\cdot\mathbf{z}}{|\mathbf{z}|^\alpha\mathrm{d}_\rho^{n+1}(\mathbf{x}+\mathbf{z},\mathbf{x})}h(\mathbf{z})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}\,\mathrm{d}\theta.\end{aligned}$$

Note that the singularity of the kernels appearing inside both integrals is of order $n+\alpha-1$ now. With additional use of smoothness of other quantities we obtain

$$|\langle \mathscr{L}_\phi f, g\rangle| \lesssim (\|f\|_{C^2} + \|f\|_{C^1}\|\rho\|_{C^1})|g|_\infty.$$

This is of course not an optimal bound, but it shows that the regularity of $\mathscr{L}_\phi f$ improves. One can continue in a similar fashion. Assuming $g = \partial_x^k h$, for some $h \in L^\infty$, one obtains

$$|\langle \mathscr{L}_\phi f, \partial_x^k h\rangle| \lesssim C(\|f\|_{C^{k+2}}, \|\rho\|_{C^{k+1}})|h|_\infty.$$

Thus, $\mathscr{L}_\phi f \in (C^{-k})^* \subset C^{k-\varepsilon}$ for any $\varepsilon > 0$.

Lemmas A.1 and A.2 stated in Appendix A make a formal justification for representation formulas (2.4) and (2.6), which are to be understood in the principal value sense. They come with estimates that will be crucial in the proof of the global regularity in 1D; see section 4.

In what follows the density function ρ of course depends on time, and so does the kernel. However, we will suppress the time variable for notational brevity.

2.3. Leibnitz rules and coercivity. In this section we develop basic product rules and coercivity estimates for the operator $\mathscr{L}_\phi$. We restrict ourselves to the one-dimensional case both for notational simplicity and for its use in the proof of regularity asserted in Theorem 1.5.

We start with basic product formulas for the derivative of $\mathscr{L}_\phi f$ provided f and ρ are smooth.

First, let us observe that (2.8) in the 1D case takes a simple form:

$$\partial_x \mathrm{d}_\rho(x+z,x) = \delta_z\rho(x)\,\mathrm{sgn}(z). \tag{2.9}$$

Formally the Leibnitz rule reads

$$(\mathscr{L}_\phi f)' = \mathscr{L}_\phi(f') + \mathscr{L}_{\phi'} f, \tag{2.10}$$

where

$$\mathscr{L}_{\phi'}(f) = -\int_{\mathbb{T}} \frac{h(z)}{|z|^\alpha \mathrm{d}_\rho^2(x,x+z)}\delta_z\rho(x)\,\mathrm{sgn}(z)\delta_z f(x)\,\mathrm{d}z.$$

The symmetric kernel ϕ' is of the same order $1+\alpha$. So, we can make sense of the integral in the same way as we did for $\mathscr{L}_\phi$. Rigorous justification of (2.10) follows by proving (2.10) in its weak formulation. So, for any $g \in C^\infty$, we have

$$\begin{aligned}
\langle (\mathscr{L}_\phi f)', g\rangle &= -\langle \mathscr{L}_\phi f, g'\rangle \\
&= \frac{1}{2}\int \delta_z f(x)\delta_z g'(x)\phi(\mathrm{d}_\rho(x+z,x),z)\,\mathrm{d}x\,\mathrm{d}z \\
&= -\frac{1}{2}\int \delta_z f'(x)\delta_z g(x)\phi(\mathrm{d}_\rho(x+z,x),z)\,\mathrm{d}x\,\mathrm{d}z \\
&\quad -\frac{1}{2}\int \delta_z f(x)\delta_z g(x)\partial_x\phi(\mathrm{d}_\rho(x+z,x),z)\,\mathrm{d}x\,\mathrm{d}z \\
&= \langle \mathscr{L}_\phi(f'), g\rangle + \langle \mathscr{L}_{\phi'} f, g\rangle.
\end{aligned}$$

Continuing in the same fashion, we obtain

$$(\mathscr{L}_\phi f)^{(n)} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \mathscr{L}_{\phi^{(k)}} f^{(n-k)}. \tag{2.11}$$

We can now discuss the coercivity property of the operator $\mathscr{L}_\phi$. In tune with the fact that $\mathscr{L}_\phi$ puts α-derivatives on f, it is natural to expect that $\mathscr{L}_\phi f \in H^s$ if and only if $f \in H^{s+\alpha}$. For the topological kernels, however, this is a delicate result, the details of which are presented in the following lemma.

LEMMA 2.3. *For any $s \geqslant 3$ and $1 \leqslant \alpha < 2$ one has the following bounds:*

$$\begin{aligned} \|\mathscr{L}_\phi f\|_{H^s}^2 &\lesssim \|f\|_{H^{s+\alpha}}^2 + \|f\|_{H^{s+\frac{\alpha}{2}}}^N + \|\rho\|_{H^{s+\frac{\alpha}{2}}}^N + 1, \\ \|\mathscr{L}_\phi f\|_{H^s}^2 &\gtrsim \|f\|_{H^{s+\alpha}}^2 - \|f\|_{H^{s+\frac{\alpha}{2}}}^N - \|\rho\|_{H^{s+\frac{\alpha}{2}}}^N - 1, \end{aligned} \tag{2.12}$$

where $N = N(n, \alpha, s)$, and $\lesssim$ denotes up to a constant depending on $(\min \rho)^{-1}$ and $\max \rho$.

Proof. According to Lemma B.3 the commutator satisfies

$$|(\mathscr{L}_\phi f)^{(s)} - \mathscr{L}_\phi(f^{(s)})|_2^2 \lesssim \|f\|_{H^{s+\frac{\alpha}{2}}}^N + \|\rho\|_{H^{s+\frac{\alpha}{2}}}^N + 1.$$

To deduce that we simply observe that all the dependencies on $|\rho'|_\infty, |f'|_\infty, \dots, |f^{(k-1)}|_\infty$, $|\rho^{(k-1)}|_\infty$ translate into $H^{s+\frac{\alpha}{2}}$-norms by the Sobolev embedding. So, it remains to estimate the top term $|\mathscr{L}_\phi(f^{(s)})|_2^2$.

Let us denote for simplicity $f^{(s)} = g$. We "freeze" the density in the topological distance as follows:

$$\begin{aligned} \mathscr{L}_\phi g(x) &= \frac{1}{\rho(x)} \int_{\mathbb{T}} \frac{h(|z|)}{|z|^{1+\alpha}} \delta_z g(x) \, \mathrm{d}z + \int_{\mathbb{T}} \frac{h(|z|)}{|z|^{1+\alpha}} \left(\frac{1}{\frac{1}{|z|}\int_{[0,z]} \rho(x+\xi)\, \mathrm{d}\xi} - \frac{1}{\rho(x)} \right) \delta_z g(x) \, \mathrm{d}z \\ &= J_1 + J_2. \end{aligned}$$

The first integral represents the truncated fractional Laplacian. We clearly have

$$|J_1|_2^2 \sim \|g\|_{H^\alpha}^2.$$

As for J_2 we estimate

$$\left| \frac{1}{\frac{1}{|z|}\int_{[0,z]} \rho(x+\xi)\, \mathrm{d}\xi} - \frac{1}{\rho(x)} \right| \lesssim |z| |\nabla \rho|_\infty,$$

and with that

$$\begin{aligned} |J_2(x)| \lesssim |\nabla \rho|_\infty \int_{\mathbb{T}} \frac{h(|z|)}{|z|^\alpha} |\delta_z g(x)| \, \mathrm{d}z &= |\nabla \rho|_\infty \int_{\mathbb{T}} \frac{h(|z|)}{|z|^{\frac{\alpha+1}{2}}} |\delta_z g(x)| \frac{\mathrm{d}z}{|z|^{\frac{\alpha-1}{2}}} \\ &\lesssim |\nabla \rho|_\infty \|g\|_{H^{\alpha/2}} \lesssim \|f\|_{H^{s+\frac{\alpha}{2}}}^N + \|\rho\|_{H^{s+\frac{\alpha}{2}}}^N. \end{aligned}$$

Putting together the obtained estimates proves the lemma. ∎

3. Smooth solutions must flock. The goal of this section will be to prove that any global, nonvacuous smooth solution to the topological model (1.1) aligns to its average velocity vector $\mathbf{u}_\infty$, which can be determined from the conservation of momentum and mass: $\mathbf{u}_\infty = \mathbf{P}_0/M_0$.

3.1. Flocking for local symmetric kernels. Let us first cast the question of flocking in the general setting (1.7), which includes both metric (1.3) and topological kernels (1.11a), as well as other singular ϕ's localized along the diagonal. In other words, at this point we do not specify any fine structure of the kernel near the singularity. We recast the fundamental energy balance relation (2.5), valid for *any* singular symmetric kernel, via our definition (2.7):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^{2n}} |\mathbf{u}(t,\mathbf{x})-\mathbf{u}(t,\mathbf{y})|^2\rho(t,\mathbf{x})\rho(t,\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} &= -2M_0\int_{\mathbb{T}^{2n}}\langle\mathscr{C}_\phi(\mathbf{u},\rho),\mathbf{u}\rangle_\rho\,\mathrm{d}\mathbf{y} \\ &= -2M_0\int_{\mathbb{T}^{2n}}\phi(\mathbf{x},\mathbf{y})|\mathbf{u}(t,\mathbf{x})-\mathbf{u}(t,\mathbf{y})|^2\rho(t,\mathbf{x})\rho(t,\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}.\end{aligned} \tag{3.1}$$

The main technical aspect of deriving a proper Grönwall differential inequality from (3.1) consists of obtaining lower-bounds of the *enstrophy* on the right-hand side of (3.1) for short-range ϕ's.

It is clear that a *necessary condition* for flocking $|\mathbf{u}(t,\cdot)-\mathbf{u}_\infty|\to 0$ requires the density to be bounded away from vacuum, or else the flow may break apart into two or more separate "islands" traveling in their own velocity which is disconnected from the influence of others. Indeed, when $\rho(\cdot,t)$ vanishes on a compact set, the momentum equation (1.1) is reduced to the pressureless Burgers system $\mathbf{u}_t+\mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u}=0$, which in turn leads to a finite-time blowup; see [54]. Precisely how far from vacuum the density must be in order to fulfill an alignment dynamics for general local kernels ϕ is asserted in (1.8). This brings us to the proof of our first main result.

Proof of Theorem 1.1. We begin by setting up the general Hilbert structure for a variational formulation of the problem. Let us denote by L^2_ρ the space of $L^2(\mathbb{T}^n)$-fields $\mathbf{u}$ with scalar product given by

$$\langle\mathbf{u},\mathbf{v}\rangle_\rho=\int_{\mathbb{T}^n}\mathbf{u}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\rho(t,\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Note that the metric of the space L^2_ρ changes in time. Next, we consider the family of eigenvalue problems parametrized by time: we seek eigenpairs, $\kappa(t)$ and $\mathbf{u}(t,\cdot)$,

$$\int_{\mathbb{T}^n}\phi(\mathbf{x},\mathbf{y})(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\rho(t,\mathbf{y})\,\mathrm{d}\mathbf{y}=\kappa(t)\mathbf{u}(\mathbf{x}),\qquad \mathbf{u}\in\mathscr{U}^\alpha_\rho:=L^2_\rho\cap H^{\alpha/2}. \tag{3.2}$$

Note that the left-hand side is precisely the action of the commutator $\mathscr{C}_\phi(\mathbf{u},\rho)$, which, for any fixed smooth ρ and any symmetric kernel satisfying (1.7), maps $H^{\alpha/2}$ into $H^{-\alpha/2}$. Moreover, the symmetric definition of $\mathscr{L}_\phi$ (2.7) yields that $-\mathscr{C}_\phi(\mathbf{u},\rho)$ is nonnegative, $-(\mathscr{C}_\phi(\mathbf{u},\rho),\mathbf{u})\geqslant 0$. Hence $\kappa_1=0$ is the minimal eigenevalue corresponding to the constant solution $\mathbf{u}\equiv\mathbf{1}$, and this allows us to seek the *second* minimal eigenvalue as a solution to the variational problem[3]

$$\kappa_2(t)=\inf_{\mathbf{u}\in\mathscr{U}^\alpha_\rho}\frac{-\langle\mathscr{C}_\phi(\mathbf{u}-\overline{\mathbf{u}},\rho),\mathbf{u}-\overline{\mathbf{u}}\rangle_\rho}{|\mathbf{u}-\overline{\mathbf{u}}|^2_{L^2_\rho}},\quad \overline{\mathbf{u}}:=\frac{\int\mathbf{u}\rho}{\int\rho}\ \text{ so that }\ \langle\mathbf{u}-\overline{\mathbf{u}},\mathbf{1}\rangle_\rho=0 \tag{3.3}$$

[3]By symmetry $\overline{\mathbf{u}}=\mathbf{u}_\infty:=\mathbf{P}_0/M_0$, but we keep the separate notation of $\overline{\mathbf{u}}$ to signify orthogonality to the 0-eigenspace spanned by $\mathbf{1}$.

or, stated explicitly in terms of $|\mathbf{u}-\overline{\mathbf{u}}|^2_{L^2_\rho} = \frac{1}{2M_0}\int_{\mathbb{T}^{2n}} |\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x})|^2\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}$,

$$\kappa_2(t) = 2M_0 \times \inf_{\mathbf{u}\in\mathscr{U}^\alpha_\rho} \frac{\displaystyle\int_{\mathbb{T}^{2n}} \phi(\mathbf{x},\mathbf{y})|\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x})|^2\rho(t,\mathbf{y})\rho(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}}{\displaystyle\int_{\mathbb{T}^{2n}} |\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|^2\rho(t,\mathbf{x})\rho(t,\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}}. \tag{3.4}$$

Since the numerator with $\phi(\mathbf{x},\mathbf{y}) \simeq |\mathbf{x}-\mathbf{y}|^{-(n+\alpha)}\mathbb{1}_{r<R_0}(|\mathbf{x}-\mathbf{y}|)$ is equivalent for the $H^{\alpha/2}$-norm, the existence follows classically by compactness. This links the enstrophy on the right-hand side of (3.1) to $\kappa_2(t)$, in complete analogy to the discrete case in which the coercivity of the discrete enstrophy is dictated by the Fiedler number; consult [39, sec. 2.2].

We can now state an alignment estimate in terms of the shrinking L^2_ρ-diameter of the velocity, given by

$$V_2[\mathbf{u},\rho](t) := \int_{\mathbb{T}^{2n}} |\mathbf{u}(t,\mathbf{x})-\mathbf{u}(t,\mathbf{y})|^2\rho(t,\mathbf{x})\rho(t,\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}. \tag{3.5}$$

By (3.1), (3.4) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}V_2[\mathbf{u},\rho](t) \leqslant -\kappa_2(t)V_2[\mathbf{u},\rho](t). \tag{3.6}$$

The implication of (3.6) is of course the bound

$$2M_0\int_{\mathbb{T}^n} |\mathbf{u}(t,\mathbf{x})-\mathbf{u}_\infty|^2\rho(t,\mathbf{x})\,\mathrm{d}\mathbf{x} = V_2[\mathbf{u},\rho](t) \leqslant V_2[\mathbf{u}_0,\rho_0]\exp\left\{-\int_0^t \kappa_2(s)\,\mathrm{d}s\right\}. \tag{3.7}$$

Consequently, the solution aligns in the L^2_ρ-distance sense if $\int_0^\infty \kappa_2(s)\,\mathrm{d}s = \infty$. It is here that we use the assumed lower-bound on the density, $\rho(t,\cdot) \gtrsim \rho_-(t)$, the assumed singularity of our kernel $\phi(\mathbf{x},\mathbf{y}) \gtrsim |\mathbf{x}-\mathbf{y}|^{-(n+\alpha)}\mathbb{1}_{|\mathbf{x}-\mathbf{y}|<R_0}$, and by the uniform upper-bound of the density, $|\mathbf{u}-\overline{\mathbf{u}}|_{L^2_\rho} \lesssim |\mathbf{u}|_{L^2}$, in order to bound the spectral gap

$$\kappa_2(t) \geqslant c\rho_-^2(t)\inf_{\mathbf{u}\in\mathscr{U}^\alpha_\rho} \frac{\displaystyle\int_{|\mathbf{x}-\mathbf{y}|<R_0} \frac{|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{n+\alpha}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}}{|\mathbf{u}|_2^2}, \qquad c := \frac{2M_0}{C^2}. \tag{3.8}$$

Technically, the infimum still depends on time since it is taken over the orthogonal complement of the line spanned by $\rho(t)$, denoted $[\rho(t)]$, in the classical $L^2(\mathbb{T}^n)$. We now have to show that this infimum still stays bounded away from zero. Geometrically this is due to the fact that the space $[\rho(t)]^\perp$ does not come close to the span of constants $\mathbb{R}^n$ in the sense of Hausdorff distance. It is more straightforward to argue by contradiction, however.

Suppose there is a sequence of times $t_k \in \mathbb{R}^+$ and $\mathbf{u}_k \in L^2_{\rho(t_k)} \cap H^{\alpha/2}$ such that $|\mathbf{u}_k|_2 = 1$ yet the homogeneous local $H^{\alpha/2}$-norm tends to zero:

$$\int_{|\mathbf{x}-\mathbf{y}|<R_0} \frac{|\mathbf{u}_k(\mathbf{x})-\mathbf{u}_k(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{n+\alpha}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \to 0. \tag{3.9}$$

Note that the latter, in particular, implies compactness of the sequence $\{\mathbf{u}_k\}_k$ in L^2. Hence, up to a subsequence, $\mathbf{u}_k \to \mathbf{u}_*$ strongly in L^2 and weakly in $H^{\alpha/2}$. By the weak lower-semicontinuity and (3.9), we conclude that $\mathbf{u}_* \in \mathbb{R}^n$ is a constant field, with $|\mathbf{u}_*| = 1$ due to $|\mathbf{u}_k|_2 \to |\mathbf{u}_*|_2$.

At the same time, since $\rho(t) > 0$ and $\int \rho(t_k, \mathbf{x})\,\mathrm{d}\mathbf{x} = M_0$, there exists a weak* limit of a further subsequence $\rho(t_k) \to \mu$, where μ is a positive Radon measure on $\mathbb{T}^n$ with nontrivial total mass $\mu(\mathbb{T}^n) = M_0$ (since $M_0 = \langle \rho(t_k), 1\rangle \to \langle \mu, 1\rangle$). We now reach a contradiction if we prove the limit

$$0 = \int_{\mathbb{T}^n} \mathbf{u}_k(\mathbf{x})\rho(t_k, \mathbf{x})\,\mathrm{d}\mathbf{x} \to \int_{\mathbb{T}^n} \mathbf{u}_*\,\mathrm{d}\mu = M_0\mathbf{u}_*.$$

To prove the claimed limit note that the assumed uniform upper-bound of the density implies

$$\begin{aligned}\int_{\mathbb{T}^n} \mathbf{u}_k(\mathbf{x})\rho(t_k, \mathbf{x})\,\mathrm{d}\mathbf{x} &- \int_{\mathbb{T}^n} \mathbf{u}_*\,\mathrm{d}\mu(\mathbf{x}) \\ &= \int_{\mathbb{T}^n} \mathbf{u}_k(\mathbf{x})\rho(t_k, \mathbf{x})\,\mathrm{d}\mathbf{x} - M_0\mathbf{u}_* = \int_{\mathbb{T}^n} (\mathbf{u}_k(\mathbf{x}) - \mathbf{u}_*)\rho(t_k, \mathbf{x})\,\mathrm{d}\mathbf{x},\end{aligned}$$

and the latter is clearly bounded by $C|\mathbf{u}_k - \mathbf{u}_*|_2 \to 0$. We conclude that

$$\int \kappa_2(s)\,\mathrm{d}s \geqslant c\eta(t) = c\int^t \rho_-^2(s)\,\mathrm{d}s \to \infty,$$

and the result follows from (3.7). ∎

3.2. Flocking with short-range topological kernels. We now turn our attention to the topological communication kernel (1.11) and prove our main result, which improves the general Theorem 1.1 to include a more natural condition on the density.

Proof of Theorem 1.3. Let us fix a coordinate i and aim to prove (1.13) for u_i. We denote $u = u_i$ for notational simplicity. Using the Galilean invariance, we can lift u if necessary and assume that $u(t) > 0$. Note that the extrema of $u(t)$, denoted $u_+(t)$ and $u_-(t)$, are monotonically decreasing and increasing, respectively.

We will make frequent use of the mass measure denoted

$$\mathrm{d}m_t = \rho(t, \mathbf{z})\,\mathrm{d}\mathbf{z}.$$

STEP 1: **Flattening near extremes**. Let $\mathbf{x}_+(t)$ be a point of maximum for $u(t,\cdot)$ and $\mathbf{x}_-(t)$ a point of minimum. Let us fix a time-dependent $\delta(t) > 0$ to be specified later, and consider the sets

$$G_\delta^+(t) = \{u < u_+(t)(1 - \delta(t))\}, \quad G_\delta^-(t) = \{u > u_-(t)(1 + \delta(t))\}.$$

The effect of flattening is expressed in terms of conditional expectations of the above sets in the balls $B(\mathbf{x}_\pm(t), R_0)$ with respect to the mass measure. Let us denote

$$\mathbb{E}_t[A|B] = \frac{m_t(A \cap B)}{m_t(B)}.$$

We show that

$$\int_0^\infty \delta(t)\mathbb{E}_t[G_\delta^\pm(t)|B(\mathbf{x}_\pm(t), R_0)]\,\mathrm{d}t < \infty. \tag{3.10}$$

We focus on the "+" case, as the "−" case is entirely similar. To this end, let us compute the equation pointwise at the critical point $(t, \mathbf{x}_+(t))$ utilizing the Rademacher theorem: $(\partial_t u)(t, \mathbf{x}_+(t)) = \partial_t u_+(t)$ a.e.,

$$\partial_t u_+(t) = \int \phi(\mathbf{x}_+(t), \mathbf{y})(u(\mathbf{y}) - u_+(t))\rho(\mathbf{y})\, \mathrm{d}\mathbf{y}.$$

At point $(\mathbf{x}_+(t), t)$ we estimate on the alignment term with the use of the following observation:

$$c_0 \frac{\mathbb{1}_{r<R_0}(|\mathbf{x}-\mathbf{y}|)}{\mathrm{d}_\rho^n(\mathbf{x}, \mathbf{y})} \leqslant \phi(\mathbf{x}, \mathbf{y}) \tag{3.11}$$

for some $c_0 > 0$. Thus, we have

$$\begin{aligned}
-\partial_t u_+(t) &= \int \phi(\mathbf{x}_+, \mathbf{y})(u_+(t) - u(\mathbf{y}))\rho(\mathbf{y})\, \mathrm{d}\mathbf{y} \\
&\geqslant c_0 \int_{B(\mathbf{x}_+, R_0)} \frac{1}{\mathrm{d}_\rho^n(\mathbf{x}_+, \mathbf{y})} (u_+(t) - u(\mathbf{y}))\rho(\mathbf{y})\, \mathrm{d}\mathbf{y}, \\
&\geqslant \frac{c_0}{m_t(B(\mathbf{x}_+(t), R_0))} \int_{G_\delta^+(t)\cap B(\mathbf{x}_+(t), R_0)} (u_+(t) - u(\mathbf{y}))\rho(\mathbf{y})\, \mathrm{d}\mathbf{y} \quad (\text{since } \Omega(\mathbf{x}_+, \mathbf{y}) \subset B(\mathbf{x}_+, R_0)) \\
&\geqslant \frac{c_0 \delta(t) u_+(t)}{m_t(B(\mathbf{x}_+(t), R_0))} \int_{G_\delta^+(t)\cap B(\mathbf{x}_+(t), R_0)} \rho(\mathbf{y})\, \mathrm{d}\mathbf{y} \\
&= c_0 \delta(t) u_+(t) \mathbb{E}_t[G_\delta^+(t) | B(\mathbf{x}_+(t), R_0)].
\end{aligned}$$

The result follows by integration:

$$c_0 \int_0^\infty \delta(t) \mathbb{E}_t[G_\delta^+(t) | B(\mathbf{x}_+(t), R_0)]\, \mathrm{d}t \leqslant \ln \frac{u_+(0)}{\lim_{t\to\infty} u_+(t)} \leqslant \ln \frac{u_+(0)}{u_-(0)}.$$

STEP 2: **Campanato estimates**. In this next step we obtain proper Campanato estimates that measure deviation of u from its average values in terms of global enstrophy.

We denote the averages with respect to mass measure by

$$u_{\mathbf{x},r} = \frac{1}{m_t(B(\mathbf{x}, r))} \int_{B(\mathbf{x},r)} u(t, \mathbf{z})\, \mathrm{d}m_t(\mathbf{z}).$$

Fix $\mathbf{x}_* \in \mathbb{T}^n$. By the Hölder inequality, we have the following estimate:

$$\int_{|\mathbf{x}-\mathbf{x}_*|<\frac{r}{10}} |u(\mathbf{x}) - u_{\mathbf{x}_*,r}|^2 \rho(\mathbf{x})\, \mathrm{d}\mathbf{x} \leqslant \int_{\substack{|\mathbf{x}-\mathbf{x}_*|<\frac{r}{10} \\ |\mathbf{y}-\mathbf{x}_*|<r}} \frac{1}{m_t(B(\mathbf{x}_*, r))} |u(\mathbf{x}) - u(\mathbf{y})|^2 \rho(\mathbf{x})\rho(\mathbf{y})\, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{x}.$$

At this point we recall that the communication domain $\Omega(\mathbf{x}, \mathbf{y})$ in (2.3) has corner tips of opening $\frac{\pi}{2}$ degrees. Hence, we can make the following geometric observation.

CLAIM 3.1. *If* $|\mathbf{x} - \mathbf{x}_*| < \frac{1}{10} r$ *and* $|\mathbf{y} - \mathbf{x}_*| < r$, *then* $\Omega(\mathbf{x}, \mathbf{y}) \subset B(\mathbf{x}_*, r)$.

In other words, if $\mathbf{y}$ is in a ball and $\mathbf{x}$ is close enough to the center of that ball, then the domain $\Omega(\mathbf{x}, \mathbf{y})$ is entirely enclosed in the ball also; see Figure 2. This implies that $m_t(B(\mathbf{x}_*, r)) \geqslant m_t(\Omega(\mathbf{x}, \mathbf{y})) = \mathrm{d}_\rho^n(\mathbf{x}, \mathbf{y})$. We thus can further estimate, with the use of (3.11),

$$\begin{aligned}
\int_{|\mathbf{x}-\mathbf{x}_*|<\frac{r}{10}} |u(\mathbf{x}) - u_{\mathbf{x}_*,r}|^2 \rho(\mathbf{x})\, \mathrm{d}\mathbf{x} &\leqslant \int_{|\mathbf{x}-\mathbf{y}|<\frac{11}{10} r} \frac{1}{\mathrm{d}_\rho^n(\mathbf{x}, \mathbf{y})} |u(\mathbf{x}) - u(\mathbf{y})|^2 \rho(\mathbf{x})\rho(\mathbf{y})\, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{x} \\
&\leqslant \int_{\mathbb{T}^2} \phi(\mathbf{x}, \mathbf{y}) |u(\mathbf{x}) - u(\mathbf{y})|^2 \rho(\mathbf{x})\rho(\mathbf{y})\, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{x}.
\end{aligned}$$

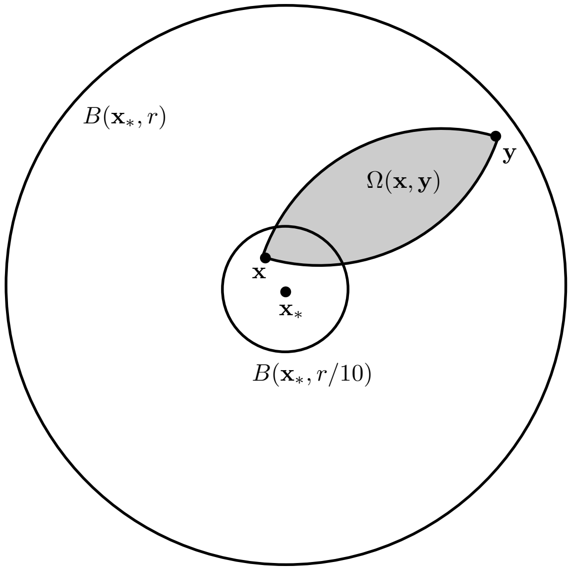

FIG. 2. $\Omega(\mathbf{x},\mathbf{y})$ *is trapped in the outer ball if* $\mathbf{x}$ *is close to the center.*

The energy balance (3.1) (see also (2.5)) yields the space-time bound on the (components of) enstrophy on the right-hand side

$$\int_0^\infty \int_{\mathbb{T}^{2n}} \phi(\mathbf{x},\mathbf{y})|u(\mathbf{x})-u(\mathbf{y})|^2\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \leqslant \frac{1}{2}\int_{\mathbb{T}^n}\rho_0|\mathbf{u}_0|^2\,\mathrm{d}\mathbf{x} < \infty;$$

hence we conclude with a time bound on the Campanato seminorm,

$$\int_0^\infty [u]_\rho^2\,\mathrm{d}t < \infty, \qquad [u]_\rho^2 := \sup_{\mathbf{x}_*\in\mathbb{T}^n, r<\frac{R_0}{2}} \int_{|\mathbf{x}-\mathbf{x}_*|<\frac{r}{10}} |u(\mathbf{x})-u_{\mathbf{x}_*,r}|^2\rho(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{3.12}$$

Combined with (3.10) we have obtained

$$I = \int_0^\infty \Big(\delta(t)\mathbb{E}_t[G_\delta^\pm(t)|B(\mathbf{x}_\pm(t),R_0)] + [u(t)]_\rho^2\Big)\,\mathrm{d}t < \infty.$$

Clearly, for $A = e^{2I}$ we have

$$\int_T^{T^A} \frac{\mathrm{d}t}{t\ln t} = 2I \quad \text{for all} \quad T > 0.$$

Hence, for any $T > 1$ we can find a $t \in [T, T^A]$ such that

$$\begin{aligned} &[u(t)]_\rho^2 < \frac{1}{t\ln t}, \\ &\mathbb{E}_t[G_\delta^+(t)|B(\mathbf{x}_+(t),R_0)] + \mathbb{E}_t[G_\delta^-(t)|B(\mathbf{x}_-(t),R_0)] < \frac{1}{\delta(t)t\ln t}. \end{aligned} \tag{3.13}$$

In view of the assumed lower bound on the density, this implies in particular that

$$\sup_{\mathbf{x}_*,\, r<\frac{R_0}{2}} \int_{|\mathbf{x}-\mathbf{x}_*|<\frac{r}{10}} |u(\mathbf{x})-u_{\mathbf{x}_*,r}|^2\,\mathrm{d}\mathbf{x} \leqslant \frac{1}{\ln t}. \tag{3.14}$$

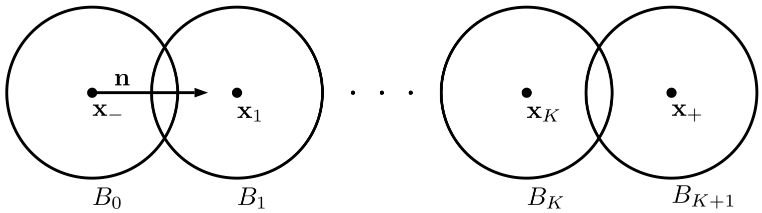

FIG. 3.

STEP 3: **Sliding averages**. Let us assume that $t \in [T, T^A]$ is a time fixed above. We will now reconnect the two averages $u_{\mathbf{x}_+,r}$ and $u_{\mathbf{x}_-,r}$ sliding along the line connecting $\mathbf{x}_+$ and $\mathbf{x}_-$ and show that the variation of those averages is small.

Denote the direction vector $\mathbf{n} = \frac{\mathbf{x}_+ - \mathbf{x}_-}{|\mathbf{x}_+ - \mathbf{x}_-|}$ and define a sequence of *overlapping* balls, $B_k = B(\mathbf{x}_k, \frac{r}{10}),\ k = 0, \dots, K$, with centers given by $\mathbf{x}_k = \mathbf{x}_- + \frac{19r}{100} k\mathbf{n}$, starting at $\mathbf{x}_-$ and ending, with $K = [\frac{|\mathbf{x}_+ - \mathbf{x}_-|}{19r/100}]$, at $\mathbf{x}_{K+1} = \mathbf{x}_+$; see Figure 3.

The Chebyshev inequality, followed by (3.14) applied to the ball centered at $\mathbf{x}_* = \mathbf{x}_0$, yields

$$|\{\mathbf{x} \in B_0 \cap B_1 : |u(\mathbf{x}) - u_{\mathbf{x}_0,r}| > \eta\}| \leqslant \frac{1}{\eta^2} \int_{B_0} |u(\mathbf{x}) - u_{\mathbf{x}_0,r}|^2 \, \mathrm{d}\mathbf{x} \leqslant \frac{1}{\eta^2 \ln t}.$$

We now fix scale $r := R_0/4$. Noticing that $|B_k \cap B_{k+1}| = cR_0^n$ for all $k \leqslant K$ and some dimensional $c > 0$, we set $\eta = \frac{2}{\sqrt{c_0 R_0^n \ln t}}$ so that

$$|\{\mathbf{x} \in B_0 \cap B_1 : |u(\mathbf{x}) - u_{\mathbf{x}_0,r}| > \eta\}| \leqslant \frac{1}{4} |B_0 \cap B_1|.$$

Applying the same argument to the variation around the averaged value $u_{\mathbf{x}_1,r}$, centered at $\mathbf{x}_* = \mathbf{x}_1$, we obtain

$$|\{\mathbf{x} \in B_0 \cap B_1 : |u(\mathbf{x}) - u_{\mathbf{x}_1,r}| > \eta\}| \leqslant \frac{1}{4} |B_0 \cap B_1|.$$

Consequently the complements of the two sets must have a point in common in $B_0 \cap B_1$:

$$\{\mathbf{x} \in B_0 \cap B_1 : |u(\mathbf{x}) - u_{\mathbf{x}_0,r}| \leqslant \eta\} \cap \{\mathbf{x} \in B_0 \cap B_1 : |u(\mathbf{x}) - u_{\mathbf{x}_1,r}| \leqslant \eta\} \neq \emptyset,$$

which implies that

$$|u_{\mathbf{x}_0,r} - u_{\mathbf{x}_1,r}| \leqslant 2\eta.$$

Continuing in the same manner, we obtain the same bound for all consecutive averages:

$$|u_{\mathbf{x}_k,r} - u_{\mathbf{x}_{k+1},r}| \leqslant 2\eta.$$

Hence,

$$|u_{\mathbf{x}_-,r} - u_{\mathbf{x}_+,r}| \leqslant 2(K+1)\eta \lesssim \frac{1}{\sqrt{\ln t}}. \tag{3.15}$$

Note that $K \leqslant 400\pi/R_0$, so it is bounded by an absolute constant. Furthermore, in view of (3.13), we can estimate

$$u_{\mathbf{x}_+,r} \geqslant \frac{1}{m_t(B(\mathbf{x}_+,r))}\int_{B(\mathbf{x}_+,r)\backslash G_\delta^+} u_+(t)(1-\delta(t))\,\mathrm{d}m_t$$
$$\geqslant u_+(t)(1-\delta(t))(1-\mathbb{E}_t[G_\delta^+(t)|B(\mathbf{x}_+(t),R_0)]) \geqslant u_+(t)(1-\delta(t))\left(1-\frac{1}{\delta(t)t\ln t}\right).$$

Hence,

$$u_+(t) - u_{\mathbf{x}_+,r}(t) \lesssim \delta(t) + \frac{1}{\delta(t)t\ln t} \lesssim \frac{1}{\sqrt{t\ln t}}$$

if we set $\delta(t) = \frac{1}{\sqrt{t\ln t}}$. A similar estimate holds for the bottom average. In conjunction with (3.15) these imply

$$|u_+(t) - u_-(t)| \lesssim \frac{1}{\sqrt{\ln t}}.$$

To conclude the proof we note that by the maximum principle

$$|u_+(T^A) - u_-(T^A)| \lesssim \frac{1}{\sqrt{\ln t}} \sim \frac{1}{\sqrt{\ln(T^A)}}.$$

Since T is arbitrary, this finishes the proof. ∎

4. Global well-posedness in 1D. In this section we develop a more complete theory of one-dimensional topological models and provide the proof of Theorem 1.5. In 1D the system takes the form

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ u_t + uu_x = \mathscr{C}_\phi(u,\rho), \quad \phi(x,y) = \dfrac{h(|x-y|)}{|x-y|^\alpha} \times \dfrac{1}{\mathrm{d}_\rho(x,y)}, \end{cases} \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{T}, \tag{4.1}$$

where

$$\mathrm{d}_\rho(x,y) = \left|\int_x^y \rho(t,z)\,\mathrm{d}z\right|.$$

The distinct feature of the one-dimensional models with convolution metric kernels $\phi(x-y)$ is an extra conservation law:

$$e_t + (ue)_x = 0, \qquad e := u_x + \mathscr{L}_\phi \rho. \tag{4.2}$$

The derivation of the conservative "e"-equation is straightforward with either smooth or singular *radial* kernels [13, 49]. It plays a key role in the regularity and hence unconditional flocking of the 1D alignment with metric-based communication [13, 49, 51]. Its role as a measure of disorder of the limiting flock was explored in [35]. A priori, there is no reason for (4.2) to hold in our case: the derivation of such a law stumbles upon the difficulty that the operator $\mathscr{L}_\phi$ does *not* commute with derivatives. Nevertheless, *it is remarkable that the law* (4.2) *still survives for anisotropic topological kernels*. To make our analysis rigorous we need to develop calculus of the operator

$\mathscr{L}_\phi$ and collect several analytical facts before we can proceed. This will be done in section 2.3.

Once we justify (4.2), we can proceed in section 4.1 to the regularity of the 1D solution along the lines of [49, 50]. Since the topological kernels lack translation invariance, we need to revisit the question of propagation of regularity in section 4.3 and Hölder regularization of the density in sections 4.4.1 and 4.4.2.

The proof will be split into several stages. First, before we enter into the technicalities of the argument, we develop necessary tools to work with the operator $\mathscr{L}_\phi$ itself. This will be done in the next section. Second, we establish a priori estimates on the density that are necessary to sustain uniform parabolicity and conclude the alignment; see section 4.2. Third, we prove a propagation of regularity result, Proposition 4.4, which states that if one can propagate some modulus of continuity of the density, then one can propagate any higher order regularity for both u and ρ. Fourth, we show how to gain a Hölder modulus of continuity from several sources. In the case $1 < \alpha < 2$ we reduce the problem to a known Schauder estimate for fractional singular operators. For the case $\alpha = 1$, we employ the De Giorgi method along the lines of Caffarelli, Chan, and Vasseur [9] with significant upgrades related to the presence of a drift, source, and asymmetry of the kernel involved. We also treat the system as truly nonlinear (see also [24]) and highlight scaling properties of the system which become very important; see (4.34)–(4.35).

Finally, the alignment claim follows directly from Theorem 1.3. Indeed, the lower-bound on the density (1.12) requires the rate which will be established for any regular solutions in Lemma 4.3 below.

First we note that the existence and uniqueness of local solutions of (4.1) can be deduced from the estimates we perform below when treated as a priori. We state the result here for our future reference.

THEOREM 4.1. *Let $1 \leqslant \alpha < 2$ and $s \geqslant 3$. For any initial data $u_0 \in H^{s+1}(\mathbb{T})$, $\rho_0 \in H^{s+\alpha}(\mathbb{T})$, with no vacuum $\rho_0(x) > 0$, there exists a unique solution to the system* (4.1) *on a time interval* $[0, T)$ *on which it will remain nonvacuous and belonging to the class*

$$
\begin{aligned}
&u \in C_w([0,T), H^{s+1}) \cap L^2([0,T), H^{s+1+\frac{\alpha}{2}}),\\
&\rho \in C_w([0,T), H^{s+\alpha}) \cap L^2([0,T), H^{s+1+\frac{\alpha}{2}}).
\end{aligned}
\tag{4.3}
$$

Incidentally, local well-posedness in any dimension n can be established too; we refer the reader to [45] for details.

4.1. An additional conservation law. The conservative "e"-equation (4.2) is at the heart of the matter for the 1D regularity theory, along the lines of [49, 50, 51, 23]. We derive it with the use of the product formula (2.10).

LEMMA 4.2 (the conservation law of e). *For any solution to the topological model in class* (4.3) *the following conservation law holds:*

$$
e_t + (ue)_x = 0, \qquad e = u_x + \mathscr{L}_\phi \rho.
$$

Proof. Differentiating the velocity equation and using the product rule (2.10), we obtain

$$
u'_t + u'u' + uu'' = \mathscr{L}_\phi((u\rho)') - u'\mathscr{L}_\phi(\rho) - u(\mathscr{L}_\phi(\rho))' + \mathscr{L}_{\phi'}(u\rho).
\tag{4.4}
$$

The finite difference in the integral representation of the last term is given by

$$u(y)\rho(y) - u(x)\rho(x) = \int_x^y (u\rho)'(\zeta)d\zeta = -\int_x^y \rho_t(\zeta)d\zeta = -\partial_t \mathrm{d}_\rho(x,y)\,\mathrm{sgn}(y-x).$$

Recalling the formula for the distance $\mathrm{d}_\rho(x,y) = \left|\int_x^y \rho(t,z)\,\mathrm{d}z\right|$, we obtain

$$\int_x^y \rho_t(\zeta)d\zeta = \partial_t \mathrm{d}_\rho(x,y)\,\mathrm{sgn}(y-x).$$

Thus,

$$\mathscr{L}_{\phi'}(u\rho) = -\int \partial_t \mathrm{d}_\rho(x,y)\,\mathrm{sgn}(y-x)\phi'(x,y)\,\mathrm{d}y.$$

Noting the relationship

$$\partial_t \mathrm{d}_\rho(x,y)\,\mathrm{sgn}(y-x)\phi'(x,y) = \partial_t\phi(x,y)(\rho(y)-\rho(x)),$$

we obtain $\mathscr{L}_{\phi'}(u\rho) = -\int \partial_t\phi(x,y)(\rho(y)-\rho(x))\,\mathrm{d}y$. Putting it together with the $\mathscr{L}_\phi((u\rho)')$ term, we obtain

$$\mathscr{L}_\phi((u\rho)') + \mathscr{L}_{\phi'}(u\rho) = -\partial_t\mathscr{L}_\phi(\rho).$$

Grouping together terms in (4.4), we arrive at

$$(u' + \mathscr{L}_\phi(\rho))_t + u'(u' + \mathscr{L}_\phi(\rho)) + u(u' + \mathscr{L}_\phi(\rho))' = 0,$$

which is precisely the law (4.2). ∎

Paired with the continuity equation we find that the ratio $q = e/\rho$ satisfies the transport equation

$$q_t + uq_x = 0.$$

Starting from a sufficiently smooth initial condition with ρ_0 away from vacuum, we can assume that $|q(t)|_\infty = |q_0|_\infty < \infty$. This gives the a priori pointwise bound

$$|e(t,x)| \lesssim \rho(t,x). \tag{4.5}$$

The argument can be bootstrapped to higher order derivatives (see [49, sec. 2]) as follows. The next order quantity $q_1 = q_x/\rho$ is again transported:

$$(q_1)_t + u(q_1)_x = 0. \tag{4.6}$$

Solving for $e'(t,\cdot)$, we obtain another a priori pointwise bound:

$$|e'(t,x)| \lesssim |\rho'(t,x)| + \rho(t,x). \tag{4.7}$$

Continuing in the same manner, $q_2 = (q_1)_x/\rho$, etc., we obtain

$$|e^{(k)}(t,x)| \lesssim |\rho^{(k)}(t,x)| + \cdots + \rho(t,x), \qquad k = 0,1,2\ldots. \tag{4.8}$$

Using e allows one to rewrite the continuity equation in parabolic form:

$$\rho_t + u\rho_x + e\rho = \rho\mathscr{L}_\phi(\rho). \tag{4.9}$$

Similarly, one can write the equation for the momentum $m = \rho u$:

$$m_t + um_x + em = \rho\mathscr{L}_\phi(m). \tag{4.10}$$

With a priori bounds on the density established in the next section, we can view (4.9)–(4.10) as a fractional parabolic system with rough drift and bounded force, which opens the possibility for applying some of the tools recently developed for such equations.

4.2. Bounds on the density. Let us first make one trivial remark: if $e_0 = 0$, then the continuity equation becomes a pure drift-diffusion, and hence by the maximum principle the density remains within the confines of its initial bounds:

$$\underline{\rho}_0 \leqslant \rho(t,x) \leqslant \bar{\rho}_0. \tag{4.11}$$

In general, however, the e-quantity introduces a Riccati term that needs to be controlled by the singularity of the kernel. First, we establish a bound from below.

LEMMA 4.3. *Let (ρ, u) be a smooth solution of the topological model* (4.1)*, with $1 \leqslant \alpha < 2$, subject to initial density ρ_0 away from vacuum, $0 < \underline{\rho}_0 \leqslant \rho_0(x) \leqslant \bar{\rho}_0 < \infty$. Then the density obeys the following bounds for all time:*

$$\frac{c}{1+t} \leqslant \rho(t,x) \leqslant \bar{\rho}(M_0, |q_0|_\infty, \phi), \qquad x \in \mathbb{T},\ t \geqslant 0. \tag{4.12}$$

Proof. Let us recall that the continuity equation can be rewritten as

$$\rho_t + u\rho_x = -q\rho^2 + \rho \mathscr{L}_\phi(\rho). \tag{4.13}$$

Let ρ_- and x_- denote the minimum value of ρ and a point where such value is achieved. Invoking Lemma A.1 to justify the pointwise evaluation, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_- \geqslant -|q_0|_\infty \rho_-^2 + \rho_- \int_{\mathbb{T}} \phi(x_-, y)(\rho(y,t) - \rho_-)\,\mathrm{d}y \geqslant -|q_0|_\infty \rho_-^2.$$

The lower bound in (4.12) follows.

Evaluating the mass equation at extreme maximum, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_+ \leqslant |q_0|_\infty \rho_+^2 + \rho_+ \int_{|z|<R_0} \frac{1}{M_0|z|^\alpha}(\rho(t, x_+ + z) - \rho_+)\,\mathrm{d}z.$$

Let us further reduce the region of integration to $\varepsilon < |z| < R_0$ for any fixed $\varepsilon > 0$. By choosing ε small enough, we can ensure that

$$\int_{\varepsilon<|z|<R_0} \frac{1}{|z|^\alpha} > 2|q_0|_\infty M_0.$$

Then for that fixed ε we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_+ \leqslant -|q_0|_\infty \rho_+^2 + C\rho_+.$$

The result follows. ∎

4.3. Continuation of solutions. Our goal in this section is to establish a general continuation result that relies on the uniform Hölder continuity of the density. The latter will be justified in section 4.4.

PROPOSITION 4.4. *Consider a local solution to a topological model with $1 \leqslant \alpha < 2$ given by Theorem* 4.1*. Suppose there are constants $\underline{\rho}, \bar{\rho} > 0$ such that*

$$\underline{\rho} \leqslant \rho(t,x) \leqslant \bar{\rho}, \quad (t,x) \in [0,T) \times \mathbb{T}. \tag{4.14}$$

Furthermore, suppose that ρ is uniformly Hölder on $[0,T)$, i.e., there exists $\gamma > 0$ such that

$$|\rho(t,x+z) - \rho(t,x)| \leqslant C|z|^\gamma, \qquad (t,x,z) \in [0,T) \times \mathbb{T} \times \mathbb{T}. \tag{4.15}$$

Then the solution remains uniformly in the Sobolev classes $(u,\rho) \in H^{s+1} \times H^{s+\alpha}$ on $[0,T)$ and, hence, can be continued beyond T.

Proof. We split the proof into five steps. In steps 1–2 we establish control over derivatives of the density up to order s. Remarkably, this can be done independently of the momentum equation. Such estimates provide bounds on the velocity derivatives up to order $s-1$. In step 3 we develop energy estimates for $\rho^{(s+1)}$, a necessary step before tackling $|u^{(s)}|_\infty$, which is done in step 4. Finally, energy estimates on $u^{(s+1)}$ will finalize the argument with the help of coercivity estimates (2.12). All the pointwise estimates we used in the proof are presented in Appendix B.

STEP 1: **Control over $|\rho'|_\infty$.** Let us differentiate (4.13):

$$\partial_t\rho' + u\rho'' + u'\rho' + e'\rho + e\rho' = \rho'\mathscr{L}_\phi\rho + \rho\mathscr{L}_{\phi'}\rho + \rho\mathscr{L}_\phi\rho'. \tag{4.16}$$

Using again $u' = e - \mathscr{L}_\phi\rho$, we rewrite

$$\partial_t\rho' + u\rho'' + e'\rho + 2e\rho' = 2\rho'\mathscr{L}_\phi\rho + \rho\mathscr{L}_{\phi'}\rho + \rho\mathscr{L}_\phi\rho'.$$

Evaluating at a point x where $|\rho'|$ achieves its maximum and multiplying by ρ', we obtain

$$\partial_t|\rho'|^2 + e'\rho\rho' + 2e|\rho'|^2 = 2|\rho'|^2\mathscr{L}_\phi\rho + \rho'\rho\mathscr{L}_{\phi'}\rho + \rho\rho'\mathscr{L}_\phi\rho'. \tag{4.17}$$

In view of (4.5) and (4.7) we can bound

$$|e'\rho\rho' + 2e|\rho'|^2| \leqslant C(|\rho'|^2 + |\rho'|).$$

Thus,

$$\partial_t|\rho'|^2 \leqslant C(|\rho'|^2 + |\rho'|) + 2|\rho'|^2\mathscr{L}_\phi\rho + \rho'\rho\mathscr{L}_{\phi'}\rho + \rho\rho'\mathscr{L}_\phi\rho'. \tag{4.18}$$

Let us note in passing that Lemma A.1 justifies pointwise evaluation of all operators involved. Due to the bound from below on ρ, the last term provides dissipation. Indeed, let us note the identity

$$\rho'(x)\delta_z\rho'(x) = -\frac{1}{2}|\delta_z\rho'(x)|^2 + \frac{1}{2}((\rho'(x+z))^2 - (\rho'(x))^2).$$

Since x is a point of maximum, we can see that the second difference is negative. Thus,

$$\rho\rho'\mathscr{L}_\phi\rho' \leqslant -c_1\mathrm{D}_\alpha\rho'(x), \tag{4.19}$$

where

$$\mathrm{D}_\alpha\rho'(x) = \int_{\mathbb{R}} \frac{|\delta_z\rho'(x)|^2}{|z|^{1+\alpha}} h(z)\,\mathrm{d}z.$$

Let us recall the nonlinear estimate on $\mathrm{D}_\alpha\rho'(x)$ obtained in Constantin and Vicol [18], which will play a crucial role in what follows:

$$\mathrm{D}_\alpha\rho'(x) \geqslant C\frac{|\rho'(x)|^{2+\alpha}}{|\rho|_\infty^\alpha} - c|\rho'|_2^2. \tag{4.20}$$

Here the $-c|\rho'|_2^2$ appears when we complement the cutoff function h to the full unity. Given a uniform bound on $|\rho|_\infty$, we further estimate, keeping half of the dissipation as is for subsequent usage,

$$\mathrm{D}_\alpha\rho'(x) \geqslant \frac{1}{2}\mathrm{D}_\alpha\rho'(x) + C|\rho'(x)|^{2+\alpha} - c|\rho'|_2^2. \tag{4.21}$$

Because of the second term in (4.21), all powers of ρ below $2+\alpha$ which appear in (4.18) are absorbed. So in particular at this stage we can rewrite (4.18) as

$$\partial_t|\rho'|^2 \leqslant C + 2|\rho'|^2\mathscr{L}_\phi\rho + \rho'\rho\mathscr{L}_{\phi'}\rho - \frac{1}{2}\mathrm{D}_\alpha\rho'(x) - c|\rho'(x)|^{2+\alpha}. \tag{4.22}$$

We now invoke the estimates on the operators $\mathscr{L}_\phi\rho$ and $\mathscr{L}_{\phi'}\rho$ obtained in Lemma B.1 and Lemma B.2, respectively. We have, knowing that by our assumption the density is uniformly Hölder continuous,

$$|\rho'|^2_\infty|\mathscr{L}_\phi\rho(x)| \lesssim r^{1-\frac{\alpha}{2}}|\rho'|^2_\infty\sqrt{\mathrm{D}_\alpha[\rho'](x)} + r^{\gamma-\alpha}|\rho'|^2_\infty + r^{2-\alpha}|\rho'|^4_\infty. \tag{4.23}$$

Let us fix a small $\varepsilon > 0$ to be determined later and define $r = \frac{\varepsilon}{|\rho'|_\infty}$. Then the above is bounded by

$$\begin{aligned}&\lesssim \varepsilon^{1-\frac{\alpha}{2}}|\rho'|^{1+\frac{\alpha}{2}}_\infty\sqrt{\mathrm{D}_\alpha[\rho'](x)} + c_\varepsilon|\rho'|^{2+\alpha-\gamma}_\infty + \varepsilon^{2-\alpha}|\rho'|^{2+\alpha}_\infty\\ &\lesssim \varepsilon^{1-\frac{\alpha}{2}}|\rho'|^{2+\alpha}_\infty + \varepsilon^{1-\frac{\alpha}{2}}\mathrm{D}_\alpha[\rho'](x) + \varepsilon|\rho'|^{2+\alpha}_\infty + C_\varepsilon + \varepsilon^{2-\alpha}|\rho'|^{2+\alpha}_\infty.\end{aligned}$$

For ε sufficiently small we can see that all these terms except for the free constant get absorbed into the dissipation term in view of (4.21). Continuing to the next term, in view of Lemma B.2, and with the same choice of scale r, we obtain

$$|\rho'|_\infty|\mathscr{L}_{\phi'}\rho(x)| \lesssim r^{1-\frac{\alpha}{2}}|\rho'|^2_\infty\sqrt{\mathrm{D}_\alpha[\rho'](x)} + r^{\gamma-\alpha}|\rho'|^2_\infty + r^{2-\alpha}|\rho'|^4_\infty,$$

which exactly repeats (4.23).

Going back to (4.22), we arrive at

$$\partial_t|\rho'|^2 \leqslant c_1 - c_2\mathrm{D}_\alpha\rho'. \tag{4.24}$$

This finishes the proof of control over ρ'.

STEP 2: **Control over $|\rho^{(s)}|_\infty$ and $|u^{(s-1)}|_\infty$.** We now establish uniform control over the maximal allowed derivative of ρ in the L^∞ metric. Note that $H^{s+\alpha}$ embeds into $W^{s,\infty}$ for the range in question $1 \leqslant \alpha < 2$. So, initially and on the local time interval $[0,T)$ we have the density in $W^{s,\infty}$ class nonuniformly at the moment. Once this step is accomplished, we obtain automatically a uniform bound on $u^{(s-1)}$. Indeed, by Lemma B.4,

$$|u^{(s-1)}|_\infty \leqslant |e^{(s-2)}|_\infty + |(\mathscr{L}_\phi\rho)^{(s-2)}|_\infty \lesssim |\rho^{(s-2)}|_\infty + \sqrt{\mathrm{D}_\alpha[\rho^{(s-1)}]} + |\rho^{(s-1)}|_\infty \lesssim C + |\rho^{(s)}|_\infty.$$

We will argue by induction. The initial hypothesis was established in the previous step. Let us now assume that we have a uniform control over $|\rho^{(k-1)}|_\infty$ for $2 \leqslant k \leqslant s$ and obtain control over $|\rho^{(k)}|_\infty$.

Differentiating the continuity equation k times and expanding, we obtain

$$\partial_t\rho^{(k)} + u\rho^{(k+1)} + \sum_{l=1}^{k} c_{k,l}u^{(l)}\rho^{(k+1-l)} + u^{(k+1)}\rho = 0. \tag{4.25}$$

Evaluating at the maximum of $|\rho^{(k)}|$ and multiplying by $\rho^{(k)}$ the term $u\rho^{(k+1)}$ drops out. In what follows we replace all u's with the corresponding e-expression. So, let us consider the endpoint case first:

$$\rho^{(k)}u^{(k+1)}\rho = \rho^{(k)}e^{(k)}\rho - \rho^{(k)}(\mathscr{L}_\phi\rho)^{(k)}\rho.$$

By the induction hypothesis and (4.8) we have

$$|e^{(k)}| \lesssim |\rho^{(k)}| + C,$$

and so

$$|\rho^{(k)} e^{(k)} \rho| \lesssim |\rho^{(k)}|^2 + C.$$

Next, we have

$$(\mathscr{L}_\phi \rho)^{(k)} = \mathscr{L}_\phi(\rho^{(k)}) + [(\mathscr{L}_\phi \rho)^{(k)} - \mathscr{L}_\phi(\rho^{(k)})].$$

For the dissipation, we have, as usual, in view of the nonlinear maximum estimate and the fact that $\rho^{(k-1)}$ is under control,

$$\rho^{(k)} \mathscr{L}_\phi(\rho^{(k)}) \lesssim -\mathrm{D}_\alpha[\rho^{(k)}](x) - |\rho^{(k)}|_\infty^{2+\alpha}. \tag{4.26}$$

For the commutator we encounter a cubic term of top order in the case $k = 2$. Therefore, we use (B.5) with small ε:

$$|\rho''[(\mathscr{L}_\phi \rho)'' - \mathscr{L}_\phi(\rho'')]| \lesssim |\rho''|_\infty \sqrt{\mathrm{D}_\alpha[\rho''](x)} + \varepsilon|\rho''|_\infty^3 + C_\varepsilon \lesssim |\rho''|_\infty^2 + \varepsilon \mathrm{D}_\alpha[\rho''](x) + \varepsilon|\rho''|_\infty^3 + C_\varepsilon.$$

In view of (4.26), the terms $\varepsilon \mathrm{D}_\alpha[\rho''](x) + \varepsilon|\rho''|_\infty^3$ are absorbed by dissipation. For general $k \geqslant 3$, we use (B.4) by replacing $\sqrt{\mathrm{D}_\alpha[\rho^{(k-1)}](x)} \lesssim |\rho^{(k)}|_\infty$:

$$|\rho^{(k)}(\mathscr{L}_\phi f)^{(k)} - \mathscr{L}_\phi(f^{(k)})(x)| \lesssim |\rho^{(k)}|_\infty \sqrt{\mathrm{D}_\alpha[\rho^{(k)}](x)} + |\rho^{(k)}|_\infty^2 + 1 \lesssim \varepsilon \mathrm{D}_\alpha[\rho^{(k)}](x) + c_\varepsilon |\rho^{(k)}|_\infty^2 + 1.$$

Again, the dissipation term is absorbed.

Next, let us look into intermediate terms, $1 \leqslant l \leqslant k$,

$$\rho^{(k)} u^{(l)} \rho^{(k+1-l)} = \rho^{(k)} e^{(l-1)} \rho^{(k+1-l)} - \rho^{(k)} (\mathscr{L}_\phi \rho)^{(l-1)} \rho^{(k+1-l)}.$$

Since $l - 1 \leqslant k - 1$, we have all $e^{(l-1)}$ uniformly bounded; hence,

$$|\rho^{(k)} e^{(l-1)} \rho^{(k+1-l)}| \lesssim |\rho^{(k)}|^2 + C.$$

Finally for the remaining terms $\rho^{(k)}(\mathscr{L}_\phi \rho)^{(l-1)} \rho^{(k+1-l)}$ we appeal to Lemma B.4. So, if $l = k$, by (B.7)

$$|\rho^{(k)}(\mathscr{L}_\phi \rho)^{(k-1)}(x)\rho'| \lesssim |\rho^{(k)}|_\infty (\sqrt{\mathrm{D}_\alpha[\rho^{(k)}](x)} + |\rho^{(k)}|_\infty + 1) \lesssim \varepsilon \mathrm{D}_\alpha[\rho^{(k)}](x) + c_\varepsilon |\rho^{(k)}|_\infty^2 + 1,$$

so this term is taken care of. For $l = k - 1$, if $k = 2$, we estimate, using the more refined bound (B.1),

$$|\rho''(\mathscr{L}_\phi \rho)\rho''| \lesssim \varepsilon|\rho''|_\infty^3 + c_\varepsilon |\rho''|_\infty^2,$$

which is absorbed. And for $k > 2$, we obtain from (B.8)

$$|\rho^{(k)}(\mathscr{L}_\phi \rho)^{(k-2)}\rho''| \lesssim |\rho^{(k)}|_\infty^2 + 1.$$

Finally, for all $1 \leqslant l \leqslant k - 2$, we use (B.9):

$$|\rho^{(k)}(\mathscr{L}_\phi \rho)^{(l-1)} \rho^{(k+1-l)}| \lesssim |\rho^{(k)}|_\infty^2 + 1.$$

We have obtained

$$\partial_t |\rho^{(k)}|_\infty^2 \lesssim |\rho^{(k)}|_\infty^2 + 1,$$

and the result follows.

STEP 3: **Energy estimates for $\rho^{(s+1)}$.** Before going into estimates for the momentum, we take one more intermediate step by establishing that $\rho^{(s+1)} \in L^\infty_t L^2_x \cap L^2_t H^{\alpha/2}_x$. The basic energy estimate for $\rho^{(s+1)}$ is obtained in the standard way. To simplify some computations, let us note the a priori bound

$$\|u\|_{C^{s-1}} \leqslant \|e\|_{C^{s-2}} + \|\mathscr{L}_\phi \rho\|_{C^{s-2}} \lesssim \|\rho\|_{C^{s-2}} + \sqrt{\mathrm{D}_\alpha[\rho^{(s-1)}]} + \|\rho\|_{C^{s-1}} \lesssim C + \|\rho\|_{C^s} \leqslant C.$$

With this and expansion (4.25) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho^{(s+1)}|_2^2 \lesssim |\rho^{(s+1)}|_2^2 + \int_{\mathbb{T}} \rho^{(s+1)} u^{(s)} \rho'' \,\mathrm{d}x + \int_{\mathbb{T}} \rho^{(s+1)} u^{(s+1)} \rho' \,\mathrm{d}x + \int_{\mathbb{T}} \rho^{(s+1)} u^{(s+2)} \rho \,\mathrm{d}x.$$

By replacing the remaining velocities with $e - \mathscr{L}_\phi \rho$ we now estimate each term:

$$\left| \int_{\mathbb{T}} \rho^{(s+1)} u^{(s)} \rho'' \,\mathrm{d}x \right| \lesssim |\rho^{(s+1)}|_1 |\rho^{(s-1)}|_\infty |\rho''|_\infty + \left| \int_{\mathbb{T}} \rho^{(s+1)} (\mathscr{L}_\phi \rho)^{(s-1)} \rho'' \,\mathrm{d}x \right|;$$

applying (B.8) with $k = s+1$,

$$\lesssim |\rho^{(s+1)}|_2 + |\rho^{(s+1)}|_2^2 + \|\rho\|^2_{H^{s+\alpha/2}} \lesssim |\rho^{(s+1)}|_2^2 + \varepsilon \|\rho\|^2_{H^{s+1+\alpha/2}} + c_\varepsilon.$$

The $H^{s+1+\alpha/2}$-norm will be absorbed into dissipation. Next,

$$\int_{\mathbb{T}} \rho^{(s+1)} u^{(s+1)} \rho' \,\mathrm{d}x \lesssim \int_{\mathbb{T}} \rho^{(s+1)} \rho^{(s)} \rho' \,\mathrm{d}x + \int_{\mathbb{T}} \rho^{(s+1)} (\mathscr{L}_\phi \rho)^{(s)} \rho' \,\mathrm{d}x,$$

and applying (B.7),

$$\lesssim |\rho^{(s+1)}|_2^2 + \int_{\mathbb{T}} |\rho^{(s+1)}(x)| \sqrt{\mathrm{D}_\alpha[\rho^{(s+1)}](x)} \,\mathrm{d}x + \|\rho\|^2_{H^{s+\alpha/2}} \lesssim |\rho^{(s+1)}|_2^2 + \varepsilon \|\rho\|^2_{H^{s+1+\alpha/2}} + c_\varepsilon.$$

Finally,

$$\int_{\mathbb{T}} \rho^{(s+1)} u^{(s+2)} \rho \,\mathrm{d}x = \int_{\mathbb{T}} \rho^{(s+1)} e^{(s+1)} \rho \,\mathrm{d}x - \int_{\mathbb{T}} \rho^{(s+1)} (\mathscr{L}_\phi \rho)^{(s+1)} \rho \,\mathrm{d}x.$$

Via (4.8) the first term is bounded by $|\rho^{(s+1)}|_2^2$. As for the second we use commutator estimates

$$\int_{\mathbb{T}} \rho^{(s+1)} (\mathscr{L}_\phi \rho)^{(s+1)} \rho \,\mathrm{d}x \lesssim -\|\rho\|^2_{H^{s+1+\alpha/2}} + \int_{\mathbb{T}} |\rho^{(s+1)}(x)| \sqrt{\mathrm{D}_\alpha[\rho^{(s+1)}](x)} \,\mathrm{d}x$$
$$+ \int_{\mathbb{T}} |\rho^{(s+1)}(x)| \sqrt{\mathrm{D}_\alpha[\rho^{(s)}](x)} \,\mathrm{d}x + |\rho^{(s+1)}|_2^2 \lesssim -\|\rho\|^2_{H^{s+1+\alpha/2}} + \varepsilon \|\rho\|^2_{H^{s+1+\alpha/2}} + c_\varepsilon |\rho^{(s+1)}|_2^2.$$

All the estimates now add up to

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho^{(s+1)}|_2^2 \lesssim -\frac{1}{2}\|\rho\|^2_{H^{s+1+\alpha/2}} + c_\varepsilon |\rho^{(s+1)}|_2^2 + c_\varepsilon.$$

This shows that $\rho^{(s+1)} \in L^\infty_t L^2_x \cap L^2_t H^{\alpha/2}_x$, and the step is complete.

STEP 4: **Control over $|u^{(s)}|_\infty$.** Due to close resemblance of the momentum equation (4.10) to the continuity equation written in parabolic form (4.9), it is easier to work with the momentum variable m. Since all the high order spaces are Banach algebras, establishing control over m is equivalent to establishing control over u:

$$\|u\|_X \lesssim \|m\|_X \left\|\rho^{-1}\right\|_X \lesssim \|m\|_X \|\rho\|_X, \quad \|m\|_X \lesssim \|u\|_X \|\rho\|_X,$$

which applies to $X = H^s, C^s$, etc. Knowing that $\rho \in X$ shows that $\|u\|_X \sim \|m\|_X$. In particular, this is the case for all C^k, $k \leqslant s$.

We do have automatic uniform bound in C^{s-1} as a consequence of the previous step. Indeed, by Lemma B.4,

$$\|m\|_{C^{s-1}} \lesssim \|u\|_{C^{s-1}} \leqslant \|e\|_{C^{s-2}} + \|\mathscr{L}_\phi \rho\|_{C^{s-2}} \lesssim \|\rho\|_{C^{s-2}} + \sqrt{\mathrm{D}_\alpha[\rho^{(s-1)}]} + \|\rho\|_{C^{s-1}}$$
$$\lesssim C + \|\rho\|_{C^s} \leqslant C.$$

So, essentially we need to complete one more step.

Differentiating (4.10) s times, testing with $m^{(s)}$, and evaluating at the maximum, we obtain

$$\partial_t |m^{(s)}|_\infty^2 + m^{(s)} \sum_{l=1}^{s} u^{(l)} m^{(s+1-l)} + (em)^{(s)} m^{(s)} = (\rho \mathscr{L}_\phi m)^{(s)} m^{(s)}.$$

The e-term is under control:

$$|(em)^{(s)} m^{(s)}| \leqslant |(em)^{(s)}|_\infty |m^{(s)}|_\infty \lesssim \|\rho\|_{C^s} \|m\|_{C^s}^2 \lesssim \|m\|_{C^s}^2.$$

Next, using the induction hypothesis,

$$\left| m^{(s)} \sum_{l=1}^{s} u^{(l)} m^{(s+1-l)} \right| \lesssim (|u'| + \cdots + |u^{(s-1)}|) |m^{(s)}|_\infty^2 + |u^{(s)}||m^{(s)}||m'|_\infty \lesssim |m^{(s)}|_\infty^2.$$

So, further argument is reduced to estimating the dissipation term. We have for all $1 \leqslant l \leqslant s$

$$|m^{(s)} \rho^{(l)} (\mathscr{L}_\phi m)^{(s-l)}(x)| \lesssim |m^{(s)}|_\infty |(\mathscr{L}_\phi m)^{(s-l)}(x)|$$

and using Lemma B.4,

$$\lesssim |m^{(s)}|_\infty \left(\sqrt{\mathrm{D}_\alpha[m^{(s)}](x)} + |m^{(s)}|_\infty + C \right) \lesssim c_\varepsilon |m^{(s)}|_\infty^2 + \varepsilon \mathrm{D}_\alpha[m^{(s)}](x) + C.$$

The D_α-term will be absorbed subsequently. So, it comes down to

$$\rho (\mathscr{L}_\phi m)^{(s)} m^{(s)}.$$

As usual, $\mathscr{L}_\phi(m^{(s)}) m^{(s)}$ produces dissipation $\mathrm{D}_\alpha[m^{(s)}](x)$, and all that remains to estimate is the commutator, for which we use Lemma B.3 with $r \sim 1$:

$$|m^{(s)}|_\infty |(\mathscr{L}_\phi m)^{(s)}(x) - \mathscr{L}_\phi(m^{(s)})(x)| \lesssim |m^{(s)}|_\infty \left(\sqrt{\mathrm{D}_\alpha[m^{(s)}](x)} + \sqrt{\mathrm{D}_\alpha[\rho^{(s)}](x)} \right) + |m^{(s)}|_\infty^2$$
$$c_\varepsilon |m^{(s)}|_\infty^2 + \varepsilon \mathrm{D}_\alpha[m^{(s)}](x) + \varepsilon \mathrm{D}_\alpha[\rho^{(s)}](x).$$

It remains to notice that

$$|\mathrm{D}_\alpha[\rho^{(s)}](x)| \lesssim |\rho^{(s+1)}|_q^2 \quad \text{for } q > \frac{2}{2-\alpha}.$$

Since $H^{\alpha/2} \hookrightarrow H^{1/2} \hookrightarrow L^q$, for any $q < \infty$, we have $|\rho^{(s+1)}|_q^2 \in L^1$ by the previous step. We conclude that the term $\mathrm{D}_\alpha[\rho^{(s)}](x)$ is L^1-integrable in time. Thus,

$$\partial_t |m^{(s)}|_\infty^2 \lesssim |m^{(s)}|_\infty^2 - \mathrm{D}_\alpha[m^{(s)}] + f(t),$$

where $f \in L^1([0,T])$. This finishes the step.

STEP 5: **Energy estimates for $u^{(s+1)}$ and conclusion of the proof**. Since the momentum equation is structurally the same as the continuity, this step is entirely similar to Step 4. The use of commutator estimates of Lemma B.3 and Lemma B.4 is identical to $f = m$ due to the fact that at this point we are in the same position in terms of control of m as we were at the beginning of Step 4. We thus conclude

$$m^{(s+1)} \in L_t^\infty L_x^2 \cap L_t^2 H_x^{\alpha/2},$$

and via Banach algebra inequality $\|u\|_X \leqslant \|m\|_X \|1/\rho\|_X \sim \|m\|_X \|\rho\|_X$ for the classes in question, we obtain

$$u^{(s+1)} \in L_t^\infty L_x^2 \cap L_t^2 H_x^{\alpha/2}.$$

To conclude the proof it remains to notice that via the e-quantity, we have $(\mathscr{L}_\phi \rho)^{(s+1)} \in L_t^\infty L_x^2$. Due to (2.12),

$$\|\rho\|_{H^{s+\alpha}}^2 \lesssim C + \|\rho\|_{H^{s+\alpha/2}}^N \lesssim C + \|\rho\|_{H^{s+1}}^N.$$

In Step 3 we established uniform control over $\|\rho\|_{H^{s+1}}$. The proof is finished. ∎

4.4. Hölder regularization of the density. In this section we derive the Hölder regularity of the density; its Hölder *regularization* follows from the fractional diffusion embedded in our topological alignment term. The proof is obtained by various techniques of fractional parabolicity depending on α. Combined with Proposition 4.4, we immediately obtain global existence and conclude Theorem 1.5.

4.4.1. Case $1 < \alpha < 2$ via Schauder. In this particular case the regularization will follow from a kinematic argument based on the Schauder estimates as in [10, 33]. So, we start by rewriting the relation between ρ, u, and e as follows:

$$\partial_x^{-1} \mathscr{L}_\phi \rho = \partial_x^{-1} e - u \in L^\infty. \tag{4.27}$$

In the purely metric case this of course implies $\rho \in C^{1-\alpha}$ immediately. For the topological models the conclusion is not so straightforward and in fact may not even be true up to regularity $1 - \alpha$.

First let us make an observation that $\mathscr{L}_\phi \rho = \partial_x(\mathscr{F}\rho)$, where

$$\mathscr{F}\rho(x) = \int \frac{\mathrm{sgn}(z) \ln \mathrm{d}_\rho(x+z,x)}{|z|^\alpha} h(z)\, \mathrm{d}z.$$

Next, by symmetrization

$$\mathscr{F}\rho(x) = \frac{1}{2} \int \frac{\ln \mathrm{d}_\rho(x+z,x) - \ln \mathrm{d}_\rho(x-z,x)}{|z|^\alpha} \mathrm{sgn}(z) h(z)\, \mathrm{d}z.$$

Now we use the expansion

$$\begin{aligned} &\ln \mathrm{d}_\rho(x+z,x) - \ln \mathrm{d}_\rho(x-z,x) \\ &= [\mathrm{d}_\rho(x+z,x) - \mathrm{d}_\rho(x-z,x)] \int_0^1 \frac{\mathrm{d}\theta}{\theta \mathrm{d}_\rho(x+z,x) + (1-\theta)\mathrm{d}_\rho(x-z,x)}. \end{aligned} \tag{4.28}$$

Next,

$$[\mathrm{d}_\rho(x+z,x) - \mathrm{d}_\rho(x-z,x)]\,\mathrm{sgn}(z) = \int_x^{x+z} \rho(y)\,\mathrm{d}y + \int_x^{x-z} \rho(y)\,\mathrm{d}y = \int_{-z}^{z} \rho(x+w)\,\mathrm{sgn}\,w\,\mathrm{d}w.$$

We can now subtract the total mass from the density without changing the result. However, the function $\rho - M_0$ is a mean-zero function. Hence, $\rho - M_0 = f'$ for some f. Continuing, we obtain

$$[\mathrm{d}_\rho(x+z,x) - \mathrm{d}_\rho(x-z,x)]\,\mathrm{sgn}(z) = \int_{-z}^{z} f'(x+w)\,\mathrm{sgn}(w)\,\mathrm{d}w = f(x+z) + f(x-z) - 2f(x),$$

which is the second order finite difference of f. We thus obtain

$$\mathscr{F}\rho(x) = \int [f(x+z) + f(x-z) - 2f(x)] K(x,z,t)\,\mathrm{d}z,$$

where the kernel $K(x,z,t)$ is given by

$$K(x,z,t) = \frac{h(z)}{|z|^\alpha} \int_0^1 \frac{\mathrm{d}\theta}{\theta \mathrm{d}_\rho(x+z,x) + (1-\theta)\mathrm{d}_\rho(x-z,x)}.$$

It satisfies the following four conditions:

(i) $\frac{\mathbb{1}_{|z|<R_0}}{|z|^{1+\alpha}} \lesssim K(x,z,t) \lesssim \frac{\mathbb{1}_{|z|<2R_0}}{|z|^{1+\alpha}}$;
(ii) $K(x,-z,t) = K(x,z,t)$;
(iii) $|z|^{2+\alpha}|K(x+h,z,t) - K(x,z,t)| \leqslant C|h|$;
(iv) $|\partial_z(|z|^{1+\alpha}K(x,z,t))| \leqslant C|z|^{-1}$.

Here the inequalities involve generic constants which may depend only on the density but not on its derivatives. Indeed, (i) is trivial. As for (iv), we have

$$|z|^{1+\alpha}K(x,z,t) = h(z)|z| \int_0^1 [\theta \mathrm{d}_\rho(x+z,x) + (1-\theta)\mathrm{d}_\rho(x-z,x)]^{-1}\,\mathrm{d}\theta. \tag{4.29}$$

Given that $\mathrm{d}_\rho(x+z,x) \sim |z|$, it is clear that this expression is uniformly bounded by a constant. It will remain so if ∂_z falls on h. The bound gains a negative power $|z|^{-1}$ when ∂_z falls on $|z|$. Next, observe that

$$\partial_z \mathrm{d}_\rho(x \pm z, x) = \rho(x \pm z)\,\mathrm{sgn}(z),$$

which is a uniformly bounded quantity. So, any derivative that falls on the distance inside the expression (4.29) reduces the power of that term by 1, while the rest remains uniformly bounded.

To verify (iii) we can even prove a stronger inequality:

$$|z|^{2+\alpha}|\partial_x K(x,z,t)| \leqslant C.$$

Indeed, in this case we recall (2.9), which implies that $\partial_x \mathrm{d}_\rho(x \pm z, x)$ remains uniformly bounded. So, we have

$$|z|^{2+\alpha}\partial_x K(x,z,t) = h(z)|z|^2 \partial_x \int_0^1 [\theta \mathrm{d}_\rho(x+z,x) + (1-\theta)\mathrm{d}_\rho(x-z,x)]^{-1}\,\mathrm{d}\theta.$$

In view of the above observation, the order of the partial of the entire expression in parentheses is $|z|^{-2}$. This finishes the verification.

So, the initial relation (4.27) can be stated now as a fractional elliptic problem:

$$\int [f(x+z)+f(x-z)-2f(x)]K(x,z,t)\,\mathrm{d}z = g(x) \in L^\infty. \tag{4.30}$$

Under the assumptions (i)–(iv), it is known (see, for example, [10, 33]) that any bounded solution f to (4.30) satisfies $f \in C^{1+\gamma}$ for some positive $\gamma > 0$. This readily implies $\rho \in C^\gamma$ and concludes the argument.

4.4.2. Case $\alpha = 1$ via De Giorgi. In this section we present a regularization result for the case $\alpha = 1$. We state our result more precisely in the following proposition.

PROPOSITION 4.5. *Consider the case $\alpha = 1$. Assume the density is uniformly bounded (4.14). Then there exists a $\gamma > 0$ such that $[\rho]_\gamma \leqslant \frac{C}{t^\gamma}$ for all $t \in (0,T]$. Here C depends on the bounds on the density on $[0,T]$.*

Let us make some preliminary remarks. Our proof is based on blending our model into the settings of Caffarelli, Chan, and Vasseur's work [9], which adopts the method of De Giorgi to nonlocal equations with symmetric kernels. We note, however, that the result of [9] is not directly applicable to our model due to the presence of drift and force in the continuity equation, and in addition we lack symmetry of the kernel. The forced case was considered in a similar situation in Golse, Imbert, and Vasseur's paper [24], where the control over the force is achieved via prescaling of the equation. We will use a similar argumentation here as well. We proceed in five steps.

STEP 1: **Symmetric form of the continuity equation**. Let us recall the continuity equation in parabolic form:

$$\rho_t + u\rho_x = \rho\mathscr{L}_\phi\rho - e\rho. \tag{4.31}$$

To get rid of the ρ prefactor we will perform the following procedure: divide (4.31) by ρ and write evolution equation for the new variable $w = \ln\rho$,

$$w_t + uw_x = \mathscr{L}_\phi e^w - e.$$

Using that

$$e^{w(y)} - e^{w(x)} = (w(y)-w(x))\int_0^1 \rho^\theta(y)\rho^{1-\theta}(x)\,\mathrm{d}\theta,$$

we further rewrite the equation as

$$w_t + uw_x = \mathscr{L}_K w - e, \tag{4.32}$$

where

$$K(x,y,t) = \phi(x,y)\int_0^1 \rho^\theta(y)\rho^{1-\theta}(x)\,\mathrm{d}\theta.$$

In view of the bounds on the density, the new kernel satisfies

$$\frac{\mathbb{1}_{|x-y|<R_0}}{|x-y|^{1+\alpha}} \lesssim K(x,y) \lesssim \frac{\mathbb{1}_{|x-y|<2R_0}}{|x-y|^{1+\alpha}} \tag{4.33}$$

and now is fully symmetric:

$$K(x,y,t) = K(y,x,t).$$

Clearly, Hölder continuity of w is equivalent to that of ρ, so we will work with (4.32) instead.

In what follows we treat the term $-e$ as a passive source. However, we cannot treat u similarly since the derivative u_x that will come up in the truncated energy inequality will have to be recycled back through its connection with e. We therefore first discuss scaling properties of the system.

STEP 2: **Rescaling**. Let us adopt the point of view that our solution (u, ρ) is defined periodically on the real line $\mathbb{R}$. Elementary computation shows that if (u, ρ) is a solution and $R > 0$, then the new pair

$$u_R = u\left(t_0 + \frac{t}{R^\alpha}, x_0 + \frac{x}{R}\right), \quad \rho_R = \rho\left(t_0 + \frac{t}{R^\alpha}, x_0 + \frac{x}{R}\right) \tag{4.34}$$

satisfies the rescaled system

$$\begin{cases} \partial_t \rho_R + R^{1-\alpha}(\rho_R u_R)_x = 0, \\ \quad \partial_t u_R + R^{1-\alpha} u_R u_R' = \displaystyle\int_{\mathbb{R}} \rho_R(y)(u_R(y) - u_R(x))\phi_R(x,y)\,\mathrm{d}y, \end{cases} \tag{4.35}$$

where the new kernel is given by

$$\phi_R(x,y,t) = \frac{1}{R^{1+\alpha}}\phi\left(x_0 + \frac{x}{R}, x_0 + \frac{y}{R}, t_0 + \frac{t}{R^\alpha}\right).$$

Note that for a given bound on the density $c < \rho < C$ on a given time interval I, the new kernel still satisfies

$$\lambda \frac{\mathbb{1}_{|x-y| \leqslant R_0 R}}{|x-y|^{1+\alpha}} \leqslant \phi_R(x,y) \leqslant \Lambda \frac{\mathbb{1}_{|x-y| < 2R_0 R}}{|x-y|^{1+\alpha}}$$

on time interval $R^\alpha(I - t_0)$, and the constants Λ, λ are independent of R. Thus, if $R > 1$, the bound from below holds on a wider space and time intervals. The corresponding e-quantity rescales to

$$e_R = R^{1-\alpha} u_R' + \mathscr{L}_{\phi_R}\rho_R = \frac{1}{R^\alpha} e\left(t_0 + \frac{t}{R^\alpha}, x_0 + \frac{x}{R}\right)$$

and satisfies

$$\partial_t e_R + R^{1-\alpha}(u_R e_R)_x = 0.$$

Hence, e_R/ρ_R is transported, and as a consequence we obtain an priori bound

$$|e_R| \lesssim \frac{1}{R^\alpha}\rho_R \lesssim \frac{1}{R^\alpha}. \tag{4.36}$$

The rescaled continuity equation becomes

$$\partial_t \rho_R + R^{1-\alpha} u_R \rho_R' + e_R \rho_R = \rho_R \mathscr{L}_{\phi_R}\rho_R.$$

The corresponding w-equation reads

$$\partial_t w_R + R^{1-\alpha} u_R w_R' = \mathscr{L}_{K_R} w - e_R,$$

where the kernel K_R satisfies the same bound (4.33) for all $R \geqslant 1$.

So, it is clear that the drift remains under control for $\alpha \geqslant 1$ and is scaling invariant in the case $\alpha = 1$.

STEP 3: **First De Giorgi lemma**. We return to the symmetrized version of the continuity equation (4.32), where the only extra term that prevents us from directly applying [9] is the drift. Since, in addition, the drift is not div-free and nonlinearly depends upon ρ, we will take extra care to maintain the protocol of relation between w and u after rescalings.

First, we start by noting that it suffices to work on time interval $[-3, 0]$ and prove uniform Hölder continuity on $[-1, 0]$. Second, in view of (4.36), if necessary we can rescale the equation by a large $R > 1$ and assume without loss of generality that $|e|_{L^\infty(\mathbb{R}\times[-3,0))} = \varepsilon_0 < 1$, where ε_0 will be determined at a later stage and will in fact depend only on Λ, λ.

The argument of [9] uses rescaling of the form $\omega = \frac{w_R}{C_1} + C_2$, where $R \geqslant 1$, and $0 < C_1 \leqslant C_0 = \max\{1, |w|_\infty\}$, and w is the original solution, and C_2 is a constant which changes from step to step. Let us note that the new quantity ω satisfies

$$\begin{aligned} &\omega_t + u_R \omega_x = \mathscr{L}_{K_R}\omega + f_{R,C_1}, \\ &|f_{R,C_1}|_\infty \leqslant \frac{\varepsilon_0}{RC_1}. \end{aligned} \tag{4.37}$$

To keep control over the source, we therefore impose the following assumption on all rescalings:

$$RC_1 > 1. \tag{4.38}$$

We will now derive a truncated energy inequality for ω.

Let ψ be a Lipschitz function on $\mathbb{R}$. We always assume that our Lipschitz functions have slopes bounded by a universal constant. Testing (4.37) with $(\omega - \psi)_+$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} (\omega-\psi)_+^2 \,\mathrm{d}x - \frac{1}{2}\int (u_R)_x(\omega-\psi)_+^2 \,\mathrm{d}x - \frac{1}{2}\int u_R\psi_x(\omega-\psi)_+ \,\mathrm{d}x \\ = -B_R(\omega, (\omega-\psi)_+) + \int f_{R,C_1}(\omega-\psi)_+ \,\mathrm{d}x, \end{aligned}$$

where

$$B_R(h,g) = \frac{1}{2}\int K_R(x,y)(h(y)-h(x))(g(y)-g(x)) \,\mathrm{d}y \,\mathrm{d}x.$$

Continuing, we obtain

$$(u_R)_x = e_R - \mathscr{L}_{\phi_R}\rho_R = e_R - \mathscr{L}_{K_R} w_R = e_R - C_1 \mathscr{L}_{K_R}\omega.$$

We also note that in view of our assumptions and the maximum principle we have a scaling invariant bound $|u_R\psi_x| \leqslant C$. So, as long as in addition $RC_1 > 1$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} (\omega-\psi)_+^2 \,\mathrm{d}x + B_R(\omega, (\omega-\psi)_+) \leqslant \frac{C_1}{2} B_R(\omega, (\omega-\psi)_+^2) + C(|(\omega-\psi)_+|_1 + |(\omega-\psi)_+|_2^2).$$

Note that the B-term on the right-hand side is cubic, while on the left-hand side it is quadratic. This will help hide the cubic term with the help of the following smallness assumption:

$$|(\omega-\psi)_+|_\infty \leqslant \frac{1}{2C_0}. \tag{4.39}$$

Under this assumption we have

$$B_R(\omega, (\omega-\psi)_+) - \frac{C_1}{2} B_R(\omega, (\omega-\psi)_+^2) = B_{R,\omega}(\omega, (\omega-\psi)_+),$$

where $B_{R,\omega}$ is the bilinear form associated with the kernel

$$K_{R,\omega}(x,y) = K_R(x,y)\left[1 - \frac{C_1}{2}((\omega-\psi)_+(x) + (\omega-\psi)_+(y))\right],$$

which under (4.39) satisfies similar bounds as the original kernel and is symmetric. Continuing with the energy inequality, we write $\omega - \psi = (\omega-\psi)_+ - (\omega-\psi)_-$ and obtain

$$B_{R,\omega}(\omega, (\omega-\psi)_+) = B_{R,\omega}((\omega-\psi)_+, (\omega-\psi)_+) - B_{R,\omega}((\omega-\psi)_-, (\omega-\psi)_+) \\ + B_{R,\omega}(\psi, (\omega-\psi)_+).$$

The first is the main dissipative term for which we have a coercive bound

$$B_{R,\omega}((\omega-\psi)_+, (\omega-\psi)_+) \geqslant c_{\Lambda,C_0} |(\omega-\psi)_+|^2_{H^{1/2}} - |(\omega-\psi)_+|_2^2.$$

For the second we have after cancellations

$$-B_{R,\omega}((\omega-\psi)_-, (\omega-\psi)_+) = 2\int K_{R,\omega}(x,y)(\omega-\psi)_-(y)(\omega-\psi)_+(z)\,\mathrm{d}y\,\mathrm{d}z := P,$$

which is positive and can be dismissed for the application of the first De Giorgi lemma. Finally, as in [9], we obtain

$$|B_{R,\omega}(\psi, (\omega-\psi)_+)| \leqslant \frac{1}{2} B_R((\omega-\psi)_+, (\omega-\psi)_+) + |(\omega-\psi)_+|_1 + |\{\omega-\psi>0\}|.$$

We thus have proved the following energy bound under (4.39) and for any rescaled solution with $RC_1 > 1$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} (\omega-\psi)_+^2\,\mathrm{d}x + |(\omega-\psi)_+|^2_{H^{1/2}} \lesssim |(\omega-\psi)_+|_2^2 + |(\omega-\psi)_+|_1 + |\{\omega-\psi>0\}|.$$

We now recap the first De Giorgi lemma: there exist $\delta > 0$ and $\theta \in (0,1)$ such that any solution ω to (4.37) satisfying

$$\omega(t,x) \leqslant 1 + (|x|^{1/4} - 1)_+ \quad \text{on } \mathbb{R} \times [-2,0]$$

and

$$|\{\omega > 0\} \cap (B_2 \times [-2,0])| \leqslant \delta$$

must have a bound

$$\omega(t,x) \leqslant 1 - \theta.$$

The proof proceeds as in [9] with extra care taken for (4.39). We consider the Lipschitz function

$$\psi_{L_k}(x) = 1 - \theta - \frac{\theta}{2^k} + (|x|^{1/2} - 1)_+.$$

For θ small enough it is clear that $(\omega - \psi_{L_k})_+$ can be made as small as we wish for all $k \in \mathbb{N}$, in particular satisfying (4.39). With θ fixed we can then apply the energy inequality for all terms $(\omega - \psi_{L_k})_+$, and the argument of [9] proceeds.

STEP 4: **The second De Giorgi lemma**. In the second De Giorgi lemma the energy bound is used in a somewhat different way. Here the presence of the drift term requires extra attention as well as condition (4.39). We recall the lemma first. For a $\lambda < 1/3$ we define $\psi_\lambda(x) = ((|x| - \frac{1}{\lambda^4})_+^{1/4} - 1)_+$. Let also F be nonincreasing with $F = 1$ on B_1 and $F = 0$ outside B_2. Define

$$\phi_j = 1 + \psi_\lambda - \lambda^j F, \quad j = 0, 1, 2.$$

The lemma claims that there exist $\mu, \lambda, \gamma > 0$ depending only on Λ such that if

$$\omega(t,x) < 1 + \psi_\lambda(x) \quad \text{on } \mathbb{R} \times [-3, 0]$$

and

$$\begin{aligned} |\{\omega < \phi_0\} \cap B_1 \times (-3,-2)| &\geqslant \mu, \\ |\{\omega > \phi_2\} \cap \mathbb{R} \times (-2,0)| &\geqslant \delta, \end{aligned}$$

then necessarily

$$|\{\phi_0 < \omega < \phi_2\} \cap \mathbb{R} \times (-3,0)| \geqslant \gamma.$$

So, if the function has a substantial weight under ϕ_0 and later over ϕ_2, then it must leave some appreciable weight in between. The proof goes by application of the energy inequality to $(\omega - \phi_1)_+$. However, $(\omega - \phi_1)_+ \leqslant \lambda$ pointwise. Hence, to satisfy (4.39) it is sufficient to pick $\lambda < 1/2C_0$, among further restrictions which come subsequently in the course of the proof. Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} (\omega - \phi_1)_+^2 \,\mathrm{d}x + B_{R,\omega}((\omega - \phi_1)_+, (\omega - \phi_1)_+) + P = -B_{R,\omega}(\phi_1, (\omega - \phi_1)_+) \\ + \int \left(\frac{1}{2} u_R (\phi_1)_x + f_{R,C_1} \right) (\omega - \phi_1)_+ \,\mathrm{d}x.$$

All the terms are exactly the same as in [9] except the last one. To bound the last term we note that $(\omega - \phi_1)_+$ is supported on B_2, where $\phi_1 = 1 + \lambda F$; hence $|(\phi_1)_x|_{L^\infty(B_2)} \leqslant C\lambda$. Furthermore, as noted above, $(\omega - \phi_1)_+ \leqslant \lambda$. Hence,

$$\left| \frac{1}{2} \int u_R (\phi_1)_x (\omega - \phi_1)_+ \,\mathrm{d}x \right| \leqslant C\lambda^2.$$

As for the source term, we obtain the same bound provided $\varepsilon_0 < \lambda$. The resulting bound repeats another estimate on the term $B_{R,\omega}(\phi_1, (\omega - \phi_1)_+)$ and hence blends with the rest of section 4 in [9].

The rest of the proof makes no further direct use of the energy inequality and thus proceeds ad verbatim. The penultimate constant λ ends up being dependent only on Λ and C_0, which are scaling invariant.

STEP 5: **Diminishing oscillation and C^γ regularity**. The first and second De Giorgi lemmas are now being used to prove that any solution with controlled tails on $[-3, 0] \times \mathbb{R}$,

$$-1 - \psi_{\varepsilon,\lambda} \leqslant w \leqslant 1 + \psi_{\varepsilon,\lambda},$$

where

$$\psi_{\varepsilon,\lambda}(x) = \begin{cases} 0 & \text{if } |x| < \lambda^{-4}, \\ [(|x| - \lambda^{-4})^{\varepsilon} - 1]_+ & \text{if } |x| \geqslant \lambda^{-4} \end{cases}$$

satisfies

$$\sup_{[-1,0]\times B_1} w - \inf_{[-1,0]\times B_1} w < 2 - \lambda^*$$

for some $\lambda^* > 0$. The proof proceeds by application of shift-amplitude rescalings of the form

$$w_{k+1} = \frac{1}{\lambda^2}(w_k - (1-\lambda^2)) = \frac{1}{\lambda^{2k}} w + C_k.$$

For our sourced equation this is the worst kind of rescaling since it does not come with a compensated space-time stretching. However, in the argument the number of iterations is limited to $k_0 = |[-3,0] \times B_3|/\gamma$ and hence depends only on Λ. We can prescale the equation in the beginning using $R' > 0$ so large that $\varepsilon_0 = |f_{R'}|_\infty < \lambda^{2k_0} C_0 \leqslant \lambda^{2k_0}$. Hence, on each step of the iteration we have $|f_k| < \lambda$, fulfilling the requirement of Step 4 automatically.

The final iteration consists of the zooming and shifting process:

$$w_1 = w/|w|_\infty,$$
$$w_{k+1} = \frac{1}{1-\lambda^*/4}((w_k)_R - \bar{w_k}),$$

where $\bar{w_k}$ is the average over $[-1,0] \times B_1$. On the first step we still have the bound $|f_1| < \lambda^{2k_0}$. Subsequently, among other restrictions put on R in [9], we set in addition $R(1-\lambda^*/4) > 1$, which preserves the bound $|f| < \varepsilon_0$ for all steps. This finishes the proof.

5. Further extensions and discussion. The class of topological models can be extended within our framework to include generalized topological diffusion of type

$$\phi(\mathbf{x},\mathbf{y}) = \frac{h(|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|^{n+\alpha-\tau}} \times \frac{1}{\mathrm{d}_\rho^\tau(\mathbf{x},\mathbf{y})}, \qquad \tau > 0. \tag{5.1}$$

In fact, this class arises naturally in a hierarchy fashion in commutator estimates proved below in Appendix B. Our main flocking result of Theorem 1.3 extends to all $\tau \geqslant n$. In fact, the most general statement which includes various stronger assumptions on density, and hence, better alignment rates, can be summarized in the following formulation.

THEOREM 5.1. *Let $(\rho, \mathbf{u})$ be a global smooth solution of the topological model with kernel* (5.1). *Assume that the density $\rho(t,\cdot)$ satisfies, for all $t > 0$,*

$$\rho(t,\mathbf{x}) \geqslant \frac{c}{(1+t)^\beta}, \quad 0 \leqslant \beta \leqslant \beta_0 := \min\left\{1, \frac{n}{2n-\tau}\right\}, \tag{5.2}$$

and if $\tau > n+\alpha$, additionally

$$|\rho(t,\cdot)|_{\frac{\tau-n}{\alpha}} < C. \tag{5.3}$$

Then the solution aligns with at least algebraic rate given by

$$|\mathbf{u}(t) - \mathbf{u}_\infty|_\infty = \frac{o(1)}{t^\gamma}, \quad \textit{where} \;\; \gamma = \frac{1}{2}\left(1 - \frac{\beta}{\beta_0}\right). \tag{5.4}$$

One notable application of this more general result is for the 1D case when $e \equiv 0$. Indeed, in this case we have a uniform bound on the density from above and below (see (4.11)), and hence the alignment rate improves to $\gamma = \frac{1}{2}$.

More can be said about the density itself. If $e = 0$, the continuity equation acquires the structure of the u-equation. Along with the maximum principle comes the possibility of applying Theorem 5.1 directly to the continuity equation. The energy law takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho|_2^2 = \int |\rho|^2 \mathscr{L}_\phi \rho \, \mathrm{d}x,$$

which after symmetrizing becomes

$$\int |\rho|^2 \mathscr{L}_\phi \rho \, \mathrm{d}x = -\frac{1}{2} \int \phi(x,y)(\rho(x)+\rho(y))(\rho(x)-\rho(y))^2 \, \mathrm{d}x \, \mathrm{d}y.$$

Since the prefactor $(\rho(x)+\rho(y))$ is uniformly bounded from above and below, this supplies the energy inequality analogous to (2.5a). We have all the ingredients for a direct application of Theorem 5.1 (with $\beta = 0$) to the continuity equation, and we conclude that

$$\left| \rho(t) - \frac{1}{2\pi} M_0 \right|_\infty = \frac{o(1)}{\sqrt{t}}.$$

Remark 5.2 (about $\tau = n$). We make another remark concerning the apparent threshold value of $\tau = n$. Clearly from (5.2), if $\tau \geqslant n$, then $\rho \geqslant \frac{1}{1+t}$ is the weakest assumption under which the theorem holds, while for $\tau < n$ a more stringent bound on ρ is required. This can be explained by the fact that the density on the bottom of ϕ needs to compensate for the density on the top inside the diffusion term. The condition manifests itself even more vividly after taking the limit as $\alpha \to 2$. Such limits are standard in elliptic theory, and so we will not provide many details here. One can verify the following:

$$\lim_{\alpha \to 2} (2-\alpha) \mathscr{L}_\phi f(x) = \nabla \cdot \left(\rho^{-\frac{\tau}{n}} \nabla f \right) := D(f). \tag{5.5}$$

The commutator which would appear in the corresponding limit model reads

$$D(\rho \mathbf{u}) - \mathbf{u} D(\rho) = \frac{1}{\rho^{\gamma-1}} \Delta \mathbf{u} + \frac{2-\gamma}{\gamma} \nabla \mathbf{u} \nabla \rho, \qquad \gamma = \frac{\tau}{n}. \tag{5.6}$$

We can see that $\tau = n$ is the threshold that determines whether the density appears on the top or the bottom in front of the leading order term. For $\tau \geqslant n$ it amplifies dissipation in thinner regions as intended in the topological model.

Concerning regularity of solutions in 1D, one can obtain an extension into the range $\alpha < 1$. In fact, the continuation criterion of Proposition 4.4 extends directly as is; in fact, in several technical places this extension is even easier due to the lower singularity order of the diffusion. The Hölder regularization result can be obtained by an adaptation of the Silvestre result [52] for forced drift-diffusion equations. The result assumes the pure fractional Laplacian as a diffusion but, as noted by the author, applies to more general kernels, even in z: $K(x,z,t) = K(x,-z,t)$. Another necessary condition to apply [52] is regularity of the drift $u \in C^{1-\alpha}$. For this we use the representation (4.27): $u = \partial_x^{-1} e - \mathscr{F} \rho$. Since $\partial_x^{-1} e \in W^{1,\infty}$, it remains to check that

$\mathscr{F}\rho \in C^{1-\alpha}$. The verification again proceeds via an optimization over the cut-off scale argument. Then, omitting constants,

$$\mathscr{F}\rho(x+\xi) - \mathscr{F}\rho(x) = \int_{|z|\geqslant|\xi|} [\ln \mathrm{d}_\rho(x+\xi+z, x+\xi) - \ln \mathrm{d}_\rho(x+z,x)] \frac{\mathrm{sgn}(z)h(z)}{|z|^\alpha} \,\mathrm{d}z$$
$$+ \int_{|z|\leqslant|\xi|} [\ln \mathrm{d}_\rho(x+\xi+z, x+\xi) - \ln \mathrm{d}_\rho(x+z,x)] \frac{\mathrm{sgn}(z)h(z)}{|z|^\alpha} \,\mathrm{d}z.$$

In the first integral, we use Taylor formula (4.28), which yields a bound by $|\xi|/|z|^{1+\alpha}$, with a uniform constant depending only on (4.14). This results in $|\xi|^{1-\alpha}$, as needed. In the latter integral, we simply observe that

$$\ln \mathrm{d}_\rho(x+\xi+z, x+\xi) - \ln \mathrm{d}_\rho(x+z,x) = \ln \frac{\mathrm{d}_\rho(x+\xi+z, x+\xi)}{\mathrm{d}_\rho(x+z,x)} \sim 1.$$

So, the order of singularity is $|z|^{-\alpha}$, which implies the bound by $|\xi|^{1-\alpha}$, as needed.

A restriction comes in the range $0<\alpha<1$, or $\alpha<\tau$ for more general models, in establishing the upper bound on the density. While the lower bound in (4.12) always holds, the extension to the upper bound reads as follows.

LEMMA 5.3. *Let* (ρ,u) *be a smooth solution of the* (τ,α)*-model, subject to initial density* ρ_0 *away from vacuum,* $0<c<\rho_0<C<\infty$. *Assume either that* (i) $\tau\leqslant\alpha$, *or else* (ii) *if* $\tau>\alpha$, *then the initial condition satisfies*

$$M_0^\tau |q_0|_\infty < \frac{R_0^{\tau-\alpha}}{\tau-\alpha}, \quad q_0 = \frac{e_0}{\rho_0}.$$

Then the density is uniformly bounded in time:

$$\rho(t,x) < C(M_0, |q_0|_\infty, \phi), \qquad x\in\mathbb{T},\ t\geqslant 0. \tag{5.7}$$

So, for $\tau>\alpha$ we need an extra smallness assumption to achieve the same result. This condition is scaling invariant; see Step 2 in the proof of Proposition 4.5. We record the generalization in the following theorem.

THEOREM 5.4. *Consider the one-dimensional system* (1.1) *on* $\mathbb{T}$ *with short-range topological kernel* (5.1) *and singularity of order* $0<\alpha<1$. *Any nonvacuous initial data* $(\rho_0,u_0)\in H^{s+\alpha}\times H^{s+1}$, $s\geqslant 3$, *satisfying the conditions of Lemma* 5.3 *admits a unique global in time solution,* (ρ,u), *in the class*

$$\rho \in C_w(\mathbb{R}^+; H^{s+\alpha}) \cap L^2_{\mathrm{loc}}(R^+; H^{s+1+\frac{\alpha}{2}}),$$
$$u \in C_w(\mathbb{R}^+; H^{s+1}) \cap L^2_{\mathrm{loc}}(R^+; H^{s+1+\frac{\alpha}{2}}).$$

Appendix A. Pointwise evaluation of topological alignment. Here we collect necessary formalities related to pointwise evaluations of the operator $\mathscr{L}_\phi$ and the commutator $\mathscr{C}_\phi$. The statements come with corresponding estimates we used throughout the text. In fact, we consider the more general class of topological kernels that we already mentioned in the previous section:

$$\phi(\mathbf{x}+\mathbf{z},\mathbf{x}) = \frac{h(|\mathbf{z}|)}{|\mathbf{z}|^{n+\alpha-\tau}} \times \frac{1}{\mathrm{d}^\tau_\rho(\mathbf{x}+\mathbf{z},\mathbf{x})}, \quad \tau>0. \tag{A.1}$$

LEMMA A.1. *For any $0 < \alpha < 2$ and $f \in C^2$ one has the natural pointwise representation formula*

$$\mathscr{L}_\phi f(\mathbf{x}) = p.v. \int_{\mathbb{T}^n} (f(\mathbf{x}+\mathbf{z}) - f(\mathbf{x}))\phi(\mathbf{x}+\mathbf{z},\mathbf{x})\, d\mathbf{z}. \tag{A.2}$$

Moreover, for any $r > 0$,

$$\mathscr{L}_\phi f(\mathbf{x}) = \int_{\mathbb{T}^n} (f(\mathbf{x}+\mathbf{z}) - f(\mathbf{x}) - \mathbf{z}\cdot\nabla f(\mathbf{x})\mathbb{1}_{|\mathbf{z}|<r}(\mathbf{z}))\phi(\mathbf{x}+\mathbf{z},\mathbf{x})\, d\mathbf{z} + b_r(\mathbf{x})\cdot\nabla f(\mathbf{x}), \tag{A.3}$$

where

$$b_r(\mathbf{x}) = p.v. \int_{|\mathbf{z}|<r} \mathbf{z}\phi(\mathbf{x}+\mathbf{z},\mathbf{x})\, d\mathbf{z}$$

satisfies

$$|b_r|_\infty \leqslant C|\nabla\rho|_\infty r^{2-\alpha}. \tag{A.4}$$

Proof. At the core of the proof is a bound on the operator given by

$$B_r\zeta(\mathbf{x}) = p.v. \int_{|\mathbf{z}|<r} \zeta(\mathbf{x}+\mathbf{z})\, \mathbf{z}\phi(\mathbf{x}+\mathbf{z},\mathbf{x})\, \mathrm{d}\mathbf{z}.$$

Clearly, $B_r 1 = b_r$. We address it more generally, as was done in preceding sections. By symmetrization,

$$\begin{aligned} B_r\zeta(\mathbf{x}) = \frac{1}{2}\int_{|\mathbf{z}|<r} \frac{\mathrm{d}^\tau_\rho(\mathbf{x}-\mathbf{z},\mathbf{x}) - \mathrm{d}^\tau_\rho(\mathbf{x}+\mathbf{z},\mathbf{x})}{\mathrm{d}^\tau_\rho(\mathbf{x}+\mathbf{z},\mathbf{x})\mathrm{d}^\tau_\rho(\mathbf{x}-\mathbf{z},\mathbf{x})|\mathbf{z}|^{n+\alpha-\tau}} \zeta(\mathbf{x}+\mathbf{z})\mathbf{z}h(\mathbf{z})\, \mathrm{d}\mathbf{z} \\ + \frac{1}{2}\int_{|\mathbf{z}|<r} \frac{\zeta(\mathbf{x}+\mathbf{z}) - \zeta(\mathbf{x}-\mathbf{z})}{\mathrm{d}^\tau_\rho(\mathbf{x}-\mathbf{z},\mathbf{x})|\mathbf{z}|^{n+\alpha-\tau}}\mathbf{z}h(\mathbf{z})\, \mathrm{d}\mathbf{z} =: I(\mathbf{x}) + J(\mathbf{x}). \end{aligned}$$

In what follows the constant C will change line to line and may depend on the underlying bounds on the density at hand, (2.2). As for J, we directly obtain

$$|J(\mathbf{x})| \leqslant C|\nabla\zeta|_\infty r^{2-\alpha}.$$

For $I(\mathbf{x})$ we first observe that

$$\begin{aligned} \mathrm{d}^\tau_\rho(\mathbf{x}+\mathbf{z},\mathbf{x}) - \mathrm{d}^\tau_\rho(\mathbf{x}-\mathbf{z},\mathbf{x}) = \frac{\tau}{n}[\mathrm{d}^n_\rho(\mathbf{x}+\mathbf{z},\mathbf{x}) - \mathrm{d}^n_\rho(\mathbf{x}-\mathbf{z},\mathbf{x})] \\ \times \int_0^1 \left[\theta\mathrm{d}^n_\rho(\mathbf{x}+\mathbf{z},\mathbf{x}) + (1-\theta)\mathrm{d}^n_\rho(\mathbf{x}-\mathbf{z},\mathbf{x})\right]^{\frac{\tau}{n}-1} \mathrm{d}\theta. \end{aligned}$$

Note that

$$|\mathrm{d}^n_\rho(\mathbf{x}+\mathbf{z},\mathbf{x}) - \mathrm{d}^n_\rho(\mathbf{x}-\mathbf{z},\mathbf{x})| = \left| \int_{\Omega(\mathbf{z},0)} (\rho(\mathbf{x}+\mathbf{w}) - \rho(\mathbf{x}-\mathbf{w}))\, \mathrm{d}\mathbf{w} \right| \leqslant |\nabla\rho|_\infty |\mathbf{z}|^{n+1},$$

and clearly,

$$\int_0^1 \left[\theta\mathrm{d}_\rho(\mathbf{x}+\mathbf{z},\mathbf{x}) + (1-\theta)\mathrm{d}_\rho(\mathbf{x}-\mathbf{z},\mathbf{x})\right]^{\frac{\tau}{n}-1} \mathrm{d}\theta \leqslant C|\mathbf{z}|^{\tau-n}.$$

Consequently,

$$|I(\mathbf{x})| \leqslant C|\nabla\rho|_\infty|\zeta|_\infty \int_{|\mathbf{z}|<r} \frac{1}{|\mathbf{z}|^{n+\alpha-2}}\,\mathrm{d}\mathbf{z} \sim |\nabla\rho|_\infty|\zeta|_\infty r^{2-\alpha}.$$

In conclusion, we obtain the bound

$$|B_r\zeta|_\infty \leqslant C\left(|\nabla\rho|_\infty|\zeta|_\infty + |\nabla\zeta|_\infty\right) r^{2-\alpha}. \tag{A.5}$$

Note that the bounds above provide a common integrable dominant for the integrands parametrized by $\mathbf{x}$. So, in addition $B_r\zeta \in C(\mathbb{T}^n)$.

The bound (A.4) now follows directly from (A.5), and we also have $b_r \in C(\mathbb{T}^n)$. With the knowledge that the drift is finite, clearly, the right-hand sides of (A.2) and (A.3) coincide. Denote them $L_\phi f(\mathbf{x})$. We now have a task to pass to the limit

$$\langle \mathscr{L}_\phi f, g_\varepsilon\rangle \to L_\phi f(\mathbf{x}_0)$$

for *every* $\mathbf{x}_0 \in \mathbb{T}^n$. Splitting the integral, we obtain

$$\begin{aligned}\langle \mathscr{L}_\phi f, g_\varepsilon\rangle = \frac{1}{2}\int_{\mathbb{T}^n\times\mathbb{T}^n} \phi(\mathbf{x},\mathbf{y})(f(\mathbf{y}) - f(\mathbf{x}) - \nabla f(\mathbf{x})(\mathbf{y}-\mathbf{x})\mathbb{1}_{|\mathbf{x}-\mathbf{y}|<r})(g_\varepsilon(\mathbf{x}) - g_\varepsilon(\mathbf{y}))\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\ + \frac{1}{2}\int_{\mathbb{T}^n\times\mathbb{T}^n} \phi(\mathbf{x},\mathbf{y})\nabla f(\mathbf{x})(\mathbf{y}-\mathbf{x})\mathbb{1}_{|\mathbf{x}-\mathbf{y}|<r})(g_\varepsilon(\mathbf{x}) - g_\varepsilon(\mathbf{y}))\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} = I + J.\end{aligned}$$

Note that $J = \frac{1}{2}\langle b_r\cdot\nabla f, g_\varepsilon\rangle + \frac{1}{2}\langle B_r\nabla f, g_\varepsilon\rangle$. By continuity of B_r proved above,

$$J \to \frac{1}{2}b_r(\mathbf{x}_0)\cdot\nabla f(\mathbf{x}_0) + \frac{1}{2}(B_r\nabla f)(\mathbf{x}_0). \tag{A.6}$$

As for I, we can unwind the symmetrization since each part of the integral is no longer singular:

$$\begin{aligned}I &= \frac{1}{2}\int_{\mathbb{T}^n\times\mathbb{T}^n} \phi(\mathbf{x},\mathbf{y})(f(\mathbf{y}) - f(\mathbf{x}) - \nabla f(\mathbf{x})(\mathbf{y}-\mathbf{x})\mathbb{1}_{|\mathbf{x}-\mathbf{y}|<r})g_\varepsilon(\mathbf{x})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\ &- \frac{1}{2}\int_{\mathbb{T}^n\times\mathbb{T}^n} \phi(\mathbf{x},\mathbf{y})(f(\mathbf{y}) - f(\mathbf{x}) - \nabla f(\mathbf{x})(\mathbf{y}-\mathbf{x})\mathbb{1}_{|\mathbf{x}-\mathbf{y}|<r})g_\varepsilon(\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}.\end{aligned}$$

Passing to the limit in each integral, we obtain

$$\begin{aligned}I &\to \frac{1}{2}\int_{\mathbb{T}^n} (f(\mathbf{y}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0)(\mathbf{y}-\mathbf{x}_0))\phi(\mathbf{x}_0,\mathbf{y})\,\mathrm{d}\mathbf{y}\\ &- \frac{1}{2}\int_{\mathbb{T}^n} (f(\mathbf{x}_0) - f(\mathbf{x}) - \nabla f(\mathbf{x})(\mathbf{x}_0-\mathbf{x}))\phi(\mathbf{x},\mathbf{x}_0)\,\mathrm{d}\mathbf{x}\\ &= \int_{\mathbb{T}^n} \phi(\mathbf{x}_0,\mathbf{y})(f(\mathbf{y}) - f(\mathbf{x}_0) - \frac{1}{2}(\nabla f(\mathbf{x}_0) + \nabla f(\mathbf{y}))(\mathbf{y}-\mathbf{x}_0)\mathbb{1}_{|\mathbf{x}_0-\mathbf{y}|<r})\,\mathrm{d}\mathbf{y}\\ &= \int_{\mathbb{T}^n} \phi(\mathbf{x}_0,\mathbf{y})(f(\mathbf{y}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0)(\mathbf{y}-\mathbf{x}_0)\mathbb{1}_{|\mathbf{x}_0-\mathbf{y}|<r})\,\mathrm{d}\mathbf{y}\\ &+ \frac{1}{2}\int_{\mathbb{T}^n} \phi(\mathbf{x}_0,\mathbf{y})(\nabla f(\mathbf{x}_0) - \nabla f(\mathbf{y}))(\mathbf{y}-\mathbf{x}_0)\mathbb{1}_{|\mathbf{x}_0-\mathbf{y}|<r}\,\mathrm{d}\mathbf{y}\\ &= L_\phi f(\mathbf{x}_0) - \frac{1}{2}b_r(\mathbf{x}_0)\cdot\nabla f(\mathbf{x}_0) - \frac{1}{2}(B_r\nabla f)(\mathbf{x}_0).\end{aligned}$$

Thus, combining with (A.6), we obtain $I + J \to L_\phi f(\mathbf{x}_0)$, which completes the proof. □

As a corollary we obtain an analogous representation formula for the commutator.

LEMMA A.2. *For any $0 < \alpha < 2$ one has the following pointwise representation:*

$$\mathscr{C}_\phi(f,\zeta)(\mathbf{x}) = p.v. \int_{\mathbb{T}^n} \phi(\mathbf{x}+\mathbf{z},\mathbf{x})\zeta(\mathbf{x}+\mathbf{z})(f(\mathbf{x}+\mathbf{z}) - f(\mathbf{x}))\, d\mathbf{z}. \tag{A.7}$$

Moreover, the following representation holds for any $r > 0$:

$$\begin{aligned}\mathscr{C}_\phi(f,\zeta)(\mathbf{x}) = &\int_{\mathbb{T}^n} \phi(\mathbf{x}+\mathbf{z},\mathbf{x})\zeta(\mathbf{x}+\mathbf{z})(f(\mathbf{x}+\mathbf{z}) - f(\mathbf{x}) - \mathbf{z}\cdot\nabla f(\mathbf{x})\mathbb{1}_{|\mathbf{z}|<r})\, d\mathbf{z} \\ &+ (\zeta(\mathbf{x})b_r(\mathbf{x}) + a_r(\mathbf{x}))\cdot\nabla f(\mathbf{x}),\end{aligned} \tag{A.8}$$

where b_r is defined as before, and

$$|a_r|_\infty \leqslant C|\nabla\zeta|_\infty r^{2-\alpha}. \tag{A.9}$$

The proof proceeds by a direct application of Lemma A.1. For the residual drift we obtain

$$a_r(\mathbf{x}) = \int_{|\mathbf{z}|<r} \phi(\mathbf{x}+\mathbf{z},\mathbf{x})(\zeta(\mathbf{x}+\mathbf{z}) - \zeta(\mathbf{x}))\mathbf{z}\, \mathrm{d}\mathbf{z}.$$

The bound (A.9) follows at once.

Appendix B. Commutator estimates. We will focus on the 1D case with $\alpha \geqslant 1$ and establish necessary commutator estimates used in the proof of Theorem 1.5. The estimates will be obtained in pointwise evaluation style, which makes them suitable for applications in both L^∞-based settings and L^2 settings. For this reason we pay special attention to dependence on the top order terms. First, we obtain a basic estimate on pointwise evaluation of the topological diffusion operator, which follows from representation formula (A.3).

LEMMA B.1. *For every smooth function f and $0 \leqslant \gamma < 1$ one has*

$$|\mathscr{L}_\phi f(x)| \lesssim r^{1-\frac{\alpha}{2}}\sqrt{D_\alpha[f'](x)} + r^{\gamma-\alpha}\|f\|_{C^\gamma} + r^{2-\alpha}|f'(x)||\rho'|_\infty \tag{B.1}$$

for all $r < R_0$ and $x \in \mathbb{T}$.

Proof. We use decomposition (A.3) with further breakdown of the integral:

$$\begin{aligned}\mathscr{L}_\phi f(x) = &\int_{|z|<r} (f(x+z) - f(x) - zf'(x))\phi\, \mathrm{d}z + f'(x)b_r(x) \\ &+ \int_{|z|>r} (f(x+z) - f(x))\ \phi\, \mathrm{d}z = I + f'(x)b_r(x) + J.\end{aligned}$$

Using that

$$|f(x+z) - f(x) - zf'(x)| = \left|\int_0^z (f'(x+w) - f'(x))\, \mathrm{d}w\right| \lesssim \sqrt{\mathrm{D}_\alpha f'(x)}|z|^{1+\frac{\alpha}{2}}, \tag{B.2}$$

we obtain

$$|I| \lesssim r^{1-\frac{\alpha}{2}}\sqrt{\mathrm{D}_\alpha f'(x)}.$$

Next, due to (A.4),

$$|b_r(x)| \lesssim |\rho'|_\infty r^{2-\alpha}.$$

And as for J, we use the Hölder continuity,

$$|J| \lesssim r^{\gamma-\alpha}\|f\|_{C^\gamma}.$$

Putting the estimates together yields (B.1). ∎

Lemma B.2. *For every smooth function f and $0 \leqslant \gamma < 1$ one has*

$$|\mathscr{L}_{\phi'} f(x)| \lesssim r^{1-\frac{\alpha}{2}} \left(|\rho'|_\infty \sqrt{D_\alpha[f'](x)} + |f'|_\infty \sqrt{D_\alpha[\rho'](x)}\right) + r^{\gamma-\alpha}\|f\|_{C^\gamma}|\rho'|_\infty + r^{2-\alpha}|f'|_\infty|\rho'|^2_\infty \tag{B.3}$$

for all $r < R_0$ and $x \in \mathbb{T}$.

Proof. Using the explicit formula for the kernel

$$\phi' = \frac{h(z)}{|z|^\alpha \mathrm{d}^2_\rho(x, x+z)} \int_{[0,z]} \rho'(x+\xi)\,\mathrm{d}\xi,$$

we obtain

$$\mathscr{L}_{\phi'} f(x) = \int_\mathbb{T} \frac{h(z)\delta_z f(x)}{|z|^\alpha \mathrm{d}^2_\rho(x,x+z)} \int_{[0,z]} [\rho'(x+\xi) - \rho'(x)]\,\mathrm{d}\xi\,\mathrm{d}z + \rho'(x)\int_\mathbb{T} \frac{h(z)}{|z|^{\alpha-1}\mathrm{d}^2_\rho(x,x+z)}\delta_z f(x)\,\mathrm{d}z = J_1 + J_2.$$

Note that J_2 is precisely one of the topological-type operators with $\tau = 2$. So, estimate (B.1) applies:

$$|J_2| \lesssim r^{1-\frac{\alpha}{2}}|\rho'|_\infty \sqrt{\mathrm{D}_\alpha[f'](x)} + r^{\gamma-\alpha}\|f\|_{C^\gamma}|\rho'|_\infty + r^{2-\alpha}|f'|_\infty|\rho'|^2_\infty.$$

As for J_1, we estimate as usual

$$\left|\int_{[0,z]} [\rho'(x+\xi) - \rho'(x)]\,\mathrm{d}\xi\right| \lesssim \sqrt{\mathrm{D}_\alpha[\rho'](x)}|z|^{1+\frac{\alpha}{2}}.$$

So, using this together with the full derivative of f in the short range $\{|z| < r\}$, and Hölder continuity of f in the long range $\{|z| > r\}$, we obtain

$$|J_1| \lesssim r^{1-\frac{\alpha}{2}}|f'|_\infty \sqrt{\mathrm{D}_\alpha[\rho'](x)} + r^{\gamma-\alpha}\|f\|_{C^\gamma}|\rho'|_\infty.$$ ∎

The statement of Lemma B.2 can be viewed as the commutator estimate of first order since

$$\mathscr{L}_{\phi'} f = (\mathscr{L}_\phi f)' - \mathscr{L}_\phi f'.$$

We will need to establish similar estimates for higher order commutators, although without the use of Hölder continuity of f.

Lemma B.3. *Let f, ρ be smooth functions and $1 \leqslant \alpha < 2$. Then for any $x \in \mathbb{T}$ the following inequalities hold: for $k \geqslant 3$*

$$\begin{aligned} |(\mathscr{L}_\phi f)^{(k)}(x) - \mathscr{L}_\phi(f^{(k)})(x)| \lesssim & \sqrt{D_\alpha[f^{(k)}](x)} + \sqrt{D_\alpha[\rho^{(k)}](x)} \\ & + \sqrt{D_\alpha[f^{(k-1)}](x)} + \sqrt{D_\alpha[\rho^{(k-1)}](x)} \\ & + |\rho^{(k)}(x)| + |f^{(k)}(x)| + 1, \end{aligned} \tag{B.4}$$

and for any $\varepsilon > 0$ and $k = 2$

$$\begin{aligned} |(\mathscr{L}_\phi f)''(x) - \mathscr{L}_\phi(f'')(x)| \lesssim & \sqrt{D_\alpha[f''](x)} + \sqrt{D_\alpha[\rho''](x)} \\ & + \varepsilon |f''|_\infty |\rho''|_\infty + |f''|_\infty + c_\varepsilon |\rho''|_\infty + c_\varepsilon, \end{aligned} \tag{B.5}$$

with $\lesssim$ meaning up to a constant factor

$$C = C(\underline{\rho}, \bar{\rho}, |\rho'|_\infty, |f'|_\infty, \dots, |f^{(k-1)}|_\infty, |\rho^{(k-1)}|_\infty).$$

Proof. According to (2.11), we have to obtain estimates on all terms

$$\mathscr{L}_{\phi^{(l)}}[f^{(k-l)}](x) \quad \text{for } 1 \leqslant l \leqslant k.$$

The kernel can be expanded using the Faà di Bruno formula

$$\phi^{(l)} = \sum_{\mathbf{j}} C_{\mathbf{j}} \frac{h(z)}{|z|^\alpha \mathrm{d}_\rho^{1+|\mathbf{j}|}(x, x+z)} \prod_{p=1}^{l} \left[\int_{[0,z]} \rho^{(p)}(x+\xi)\, \mathrm{d}\xi \right]^{j_p},$$

where $\mathbf{j} = (j_1, \dots, j_l)$ is a multi-index with weight $|\mathbf{j}| = j_1 + \dots + j_l$, and

$$1 j_1 + 2 j_2 + \dots + l j_l = l.$$

Let us take into consideration operators corresponding to the summands in the above expansion:

$$\mathscr{L}_{\mathbf{j}} f^{(k-l)}(x) = \int_{\mathbb{T}} \frac{h(z) \delta_z f^{(k-l)}(x)}{|z|^\alpha \mathrm{d}_\rho^{1+|\mathbf{j}|}(x, x+z)} \prod_{p=1}^{l} \left[\int_{[0,z]} \rho^{(p)}(x+\xi)\, \mathrm{d}\xi \right]^{j_p} \mathrm{d}z.$$

Let us consider separately one endpoint case when the index reaches its corner value $\mathbf{j} = (0, \dots, 0, 1)$. For this particular index the density receives its maximal derivative:

$$\begin{aligned} \mathscr{L}_{\mathbf{j}} f^{(k-l)}(x) &= \int_{\mathbb{T}} \frac{h(z) \delta_z f^{(k-l)}(x)}{|z|^\alpha \mathrm{d}_\rho^2(x, x+z)} \int_{[0,z]} \rho^{(l)}(x+\xi)\, \mathrm{d}\xi\, \mathrm{d}z \\ &= \int_{\mathbb{T}} \frac{h(z) \delta_z f^{(k-l)}(x)}{|z|^\alpha \mathrm{d}_\rho^2(x, x+z)} \int_{[0,z]} (\rho^{(l)}(x+\xi) - \rho^{(l)}(x))\, \mathrm{d}\xi\, \mathrm{d}z \\ &+ \rho^{(l)}(x) \int_{\mathbb{T}} \frac{h(z) \delta_z f^{(k-l)}(x)}{|z|^{\alpha-1} \mathrm{d}_\rho^2(x, x+z)}\, \mathrm{d}z = J_1 + J_2. \end{aligned}$$

The operator involved in J_2 is exactly of topological type with $\tau = 2$. So, we apply (B.1) directly with $r \sim \varepsilon$ and $\gamma = 0$:

$$|J_2| \lesssim |\rho^{(l)}(x)| \left(\varepsilon \sqrt{\mathrm{D}_\alpha[f^{(k-l+1)}](x)} + c_\varepsilon + |f^{(k-l+1)}(x)| \right).$$

Here and in the following we dismiss all the quantities depending on the lower order terms $\underline{\rho}, \bar{\rho}, |\rho'|_\infty, |f'|_\infty, \dots, |f^{(k-1)}|_\infty, |\rho^{(k-1)}|_\infty$. Using

$$\mathrm{D}_\alpha[f^{(k-l+1)}](x) \lesssim |f^{(k-l+2)}|_\infty,$$

we see that all the terms with $l = 3, \dots, k-1$ are of lower order (this only applies if $k \geqslant 4$). For $l = k$, we have

$$|J_2| \lesssim \varepsilon |\rho^{(k)}(x)||f''|_\infty + c_\varepsilon.$$

For $l = 1$, we have

$$|J_2| \lesssim \sqrt{\mathrm{D}_\alpha[f^{(k)}](x)} + |f^{(k)}(x)|.$$

For $l = 2$,

$$|J_2| \lesssim \varepsilon |\rho''|_\infty \sqrt{\mathrm{D}_\alpha[f^{(k-1)}](x)} + c_\varepsilon |\rho''|_\infty.$$

Summing up over l, we have

$$\sum_{l=1}^{k} |J_2| \lesssim \sqrt{\mathrm{D}_\alpha[f^{(k)}](x)} + |f^{(k)}(x)| + \varepsilon |\rho^{(k)}(x)||f''|_\infty + \varepsilon |\rho''|_\infty \sqrt{\mathrm{D}_\alpha[f^{(k-1)}](x)} + c_\varepsilon |\rho''|_\infty + c_\varepsilon.$$

As for the J_1 terms, for all $2 \leqslant l \leqslant k-2$, we simply estimate

$$|J_1| \lesssim |f^{(k-l+1)}|_\infty |\rho^{(l+1)}|_\infty \lesssim C.$$

For $l = 1$,

$$|J_1| \lesssim |\rho''|_\infty \int_{|z| \leqslant \varepsilon} \frac{|\delta_z f^{(k-1)}(x)|}{|z|^\alpha} \, \mathrm{d}z + c_\varepsilon |f^{(k-1)}|_\infty |\rho'|_\infty \lesssim \varepsilon^{1-\alpha/2} |\rho''|_\infty \sqrt{\mathrm{D}_\alpha[f^{(k-1)}](x)} + c_\varepsilon.$$

Resetting $\varepsilon^{1-\alpha/2} \to \varepsilon$ this term has been accounted for. For $l = k-1$,

$$|J_1| \lesssim \varepsilon |f''|_\infty \sqrt{\mathrm{D}_\alpha[\rho^{(k-1)}](x)} + c_\varepsilon.$$

Finally, for $l = k$, we have

$$|J_1| \lesssim \sqrt{\mathrm{D}_\alpha[\rho^{(k)}](x)}.$$

To summarize, the corner-case terms add up to

$$\begin{aligned}
\sum_{l=1}^{k} |\mathscr{L}_{\mathbf{j}} f^{(k-l)}(x)| \lesssim & \sqrt{\mathrm{D}_\alpha[f^{(k)}](x)} + \sqrt{\mathrm{D}_\alpha[\rho^{(k)}](x)} \\
& + \varepsilon |f''|_\infty \sqrt{\mathrm{D}_\alpha[\rho^{(k-1)}](x)} + \varepsilon |\rho''|_\infty \sqrt{\mathrm{D}_\alpha[f^{(k-1)}](x)} \\
& + \varepsilon |\rho^{(k)}(x)||f''|_\infty + |f^{(k)}(x)| + c_\varepsilon |\rho''|_\infty + c_\varepsilon.
\end{aligned} \tag{B.6}$$

Let us now consider off-corner cases, $\mathbf{j} = (j_1, \dots, j_{l-1}, 0)$, $2 \leqslant l \leqslant k$ (obviously for $l = 1$ there is only one term with $\mathbf{j} = (1)$ which is a corner case). Since $|\delta_z f^{(k-l)}| \leqslant |z||f^{(k-1)}|_\infty \lesssim C|z|$, the order of singularity of the kernel becomes $\alpha + |\mathbf{j}|$, while the order of the product in the numerator is $|\mathbf{j}|$. So, for $\alpha \geqslant 1$ this operator is still hypersingular, which means extra care is needed to find additional cancellations. Let us denote

$$a_p = \int_{[0,z]} \rho^{(p)}(x+\xi)\,\mathrm{d}\xi, \quad b_p = |z|\rho^{(p)}(x),$$

and write the product as follows:

$$\prod_{p=1}^{l-1} a_p^{j_p} = a_1^{j_1}\cdots a_{l-2}^{j_{l-2}}(a_{l-1}^{j_{l-1}} - b_{l-1}^{j_{l-1}}) + a_1^{j_1}\cdots a_{l-3}^{j_{l-3}}(a_{l-2}^{j_{l-2}} - b_{l-2}^{j_{l-2}})b_{l-1}^{j_{l-1}} + \cdots$$
$$+ (a_1^{j_1} - b_1^{j_1})b_2^{j_2}\cdots b_{l-1}^{j_{l-1}} + \prod_{p=1}^{l-1} b_p^{j_p}.$$

Now, for $p \leqslant k-2$ we simply use

$$|a_p^{j_p} - b_p^{j_p}| \leqslant |z|^{1+j_p}|\rho^{(p+1)}|_\infty |\rho^{(p)}|_\infty^{j_p-1} \lesssim |z|^{1+j_p}.$$

So, the product in this case is bounded by $\lesssim |z|^{1+|\mathbf{j}|}$, and the singularity order reduces to $\alpha - 1 < 1$. Thus, these terms are bounded by $\lesssim C$.

For $p = k-1$, if $j_{k-1} > 0$, we use

$$|a_{k-1}^{j_{k-1}} - b_{k-1}^{j_{k-1}}| \lesssim |z|^{\frac{\alpha}{2}+j_{k-1}}\sqrt{\mathrm{D}_\alpha \rho^{(k)}(x)}.$$

Thus, the order of the product is $\frac{\alpha}{2} + |\mathbf{j}|$, and the order of the operator becomes $\alpha/2 < 1$. So, this term is bounded by $\lesssim \sqrt{\mathrm{D}_\alpha \rho^{(k)}(x)}$, which has been accounted for earlier.

It remains to estimate the integral for the pure b-product:

$$\prod_{p=1}^{l-1} b_p^{j_p} = |z|^{|\mathbf{j}|}\prod_{p=1}^{l-1}(\rho^{(p)}(x))^{j_p}.$$

The product of densities is obviously subcritical and comes out of the integral. What remains is another topological operator:

$$\int_{\mathbb{T}} \frac{h(z)\delta_z f^{(k-l)}(x)}{|z|^{\alpha-|\mathbf{j}|}\mathrm{d}_\rho^{1+|\mathbf{j}|}(x,x+z)}\,\mathrm{d}z.$$

This involves the generalized kernel (A.1) with $\tau = 1 + |\mathbf{j}|$. Applying estimate (B.1) with $\gamma = 0$ and fixed absolute $r \sim 1$, we obtain the bound

$$\lesssim \sqrt{\mathrm{D}_\alpha[f^{(k-l+1)}](x)} + |f^{(k-l)}|_\infty + |f^{(k-l+1)}(x)|.$$

Recalling that we are in the range $2 \leqslant l \leqslant k$, we have

$$|f^{(k-l)}|_\infty + |f^{(k-l+1)}(x)| \lesssim 1,$$

while for $l > 2$ the dissipative term is also subcritical, and for $l = 2$, term $\sqrt{\mathrm{D}_\alpha[f^{(k-1)}](x)}$ has been accounted for.

Thus, estimate (B.4) captures all the terms we encountered. It remains to notice that for $k \geqslant 3$, the second derivative terms become of lower order, and we can set $\varepsilon \sim 1$ to obtain (B.4). For $k = 2$ we obtain (B.5). This finishes the proof. ∎

Finally, we have the needed estimates on the lower order derivatives $(\mathscr{L}_\phi f)^{(l)}$, $k \geqslant 2$, with the use of the above results. So, for any $k \geqslant 2$ and with the same convention of using $\lesssim$ up to a constant $C(\underline{\rho}, \bar{\rho}, |\rho'|_\infty, |f'|_\infty, \ldots, |f^{(k-1)}|_\infty, |\rho^{(k-1)}|_\infty)$, we deduce from Lemma B.1 with $\gamma = 0$ and $r \sim 1$,

$$|\mathscr{L}_\phi(f^{(k-1)})(x)| \lesssim \sqrt{\mathrm{D}_\alpha[f^{(k)}](x)} + |f^{(k)}(x)| + 1,$$
$$|\mathscr{L}_\phi(f^{(k-2)})(x)| \lesssim \sqrt{\mathrm{D}_\alpha[f^{(k-1)}](x)} + 1,$$
$$|\mathscr{L}_\phi(f^{(l)})(x)| \lesssim 1, \qquad 0 \leqslant l \leqslant k-3.$$

In combination with the commutator estimates established in Lemma B.3, we obtain the following lemma.

Lemma B.4. *For any smooth function f and $k \geqslant 2$, we have*

$$|(\mathscr{L}_\phi f)^{(k-1)}(x)| \lesssim \sqrt{D_\alpha[f^{(k)}](x)} + |f^{(k)}(x)| + \sqrt{D_\alpha[f^{(k-1)}](x)} + \sqrt{D_\alpha[\rho^{(k-1)}](x)} + 1, \tag{B.7}$$

$$|(\mathscr{L}_\phi f)^{(k-2)}(x)| \lesssim \sqrt{D_\alpha[f^{(k-1)}](x)} + 1, \tag{B.8}$$

$$|(\mathscr{L}_\phi f)^{(l)}(x)| \lesssim 1, \qquad 0 \leqslant l \leqslant k-3, \tag{B.9}$$

with $\lesssim$ meaning up to a constant factor

$$C = C(\underline{\rho}, \bar{\rho}, |\rho'|_\infty, |f'|_\infty, \ldots, |f^{(k-1)}|_\infty, |\rho^{(k-1)}|_\infty).$$

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