SWARMING: HYDRODYNAMIC ALIGNMENT WITH PRESSURE

EITAN TADMOR

ABSTRACT. We study the swarming behavior of hydrodynamic alignment. Alignment reflects steering towards a weighted average heading. We consider the class of so-called p-alignment hydrodynamics, based on $2p$-Laplacians, and weighted by a general family of symmetric communication kernels. The main new aspect here is the long time emergence behavior for a general class of pressure tensors *without* a closure assumption, beyond the mere requirement that they form an energy dissipative process. We refer to such pressure laws as 'entropic', and prove the flocking of p-alignment hydrodynamics, driven by singular kernels with general class of entropic pressure tensors. These results indicate the rigidity of alignment in driving long-time flocking behavior despite the lack of thermodynamic closure.

CONTENTS

1. INTRODUCTION — ALIGNMENT DYNAMICS AND ENTROPIC PRESSURE

Alignment reflects steering towards average heading, [Rey1987]. It plays an indispensable role in the process of emergence in swarming dynamics, and in particular — in flocking, herding, schooling,..., [VCBCS1995, CF2003, CKFL2005, CS2007a, CS2007b, Bal2008, Kar2008, VZ2012, MCEB2015, PT2017], as well as the formation of other self-organized clustering in human interactions and in dynamics of sensor-based networks, [Kra2000, BeN2005, BHT2009, JJ2015, RDW2018, DTW2019, Alb2019]; more can be found in [MT2014, §9], in the book series on active matter, [BDT2017/19, BCT2022], and in the recent Gibbs' lecture [Tad2022a].

We discuss alignment dynamics in two parallel descriptions. Historically, alignment models

Date: September 4, 2022.
1991 *Mathematics Subject Classification.* 35Q35, 76N10, 92D25.
Key words and phrases. alignment, fractional $2p$-Laplacian, pressure, fluctuations, flocking.
Acknowledgment. Research was supported by ONR grant N00014-2112773.

were introduced in the context of agent-based description [Aok1982, Rey1987, VCBCS1995]. In particular, our discussion is motivated by the celebrated Cucker-Smale model, [CS2007a, CS2007b], in which alignment is governed by weighted graph Laplacians. Our main focus, however, is on the corresponding hydrodynamic description, the so-called Euler alignment equations, governed by a general class of weighted p-graph Laplacians, [HT2008, CFTV2010, HHK2010, Shv2021]. In both cases — the agent-based and hydrodynamic descriptions, the weights for the protocol of alignment reflect pairwise interactions, and are quantified by proper *communication kernel*. Communication kernels are either derived empirically, deduced from higher-order principles, learned from the data, or postulated based on phenomenological arguments, e.g., [CS2007a, CDMBC2007, Bal2008, GWBL2012, JJ2015, LZTM2019, MLK2019, ST2020b]. The specific structure of such kernels, however, is not necessarily known. Instead, we ask how different classes of communication kernels affect the swarming behavior.

The passage from agent-based to hydrodynamic descriptions requires a proper notion of hydrodynamic pressure. In section 1 we introduce a class of *entropic pressures* for hydrodynamic alignment and in section 2 we extend the discussion to the larger class of hydrodynamic p-alignment. Our goal is to make a systematic study of the long-time swarming behavior of hydrodynamic alignment, portrayed in section 3, with entropy pressure laws. Specifically, we use the decay of *energy fluctuations*, discussed in section 4, in order to quantify the emergence of flocking behavior, depending on the communication kernel. Almost all available literature is devoted to the case of 'pressure-less' alignment. We review these results in section 5. The main theme here is unconditional flocking for pressure-less p-alignment, driven by *heavy-tailed* communication kernels. In section 6 we discuss hydrodynamic alignment driven by a general class of entropic pressure. The remarkable aspect here is that despite the lack of closure of such entropic pressure laws, there holds unconditional flocking of p-alignment driven by *singular, heavy-tailed* communication kernels. We are aware that the methodology developed here can be utilized with other Eulerian-based dissipative systems.

The detailed computations are outlined in appendix A, B, C and D.

1.1. Hydrodynamic description of alignment.

We study the long-time behavior of the (hydro-)dynamic description for alignment,

$$\left\{\begin{array}{l}\partial_t\rho + \nabla_{\mathbf{x}}\cdot(\rho\mathbf{u}) = 0,\\ \partial_t(\rho\mathbf{u}) + \nabla_{\mathbf{x}}\cdot(\rho\mathbf{u}\otimes\mathbf{u}+\mathbb{P}) = \mathbf{A}(\rho,\mathbf{u}),\end{array}\right. \qquad (t,\mathbf{x})\in(\mathbb{R}_t,\mathbb{R}^d). \tag{1.1a}$$

The dynamics is captured by density $\rho:\mathbb{R}_t\times\mathbb{R}^d\mapsto\mathbb{R}_+$, momentum, $\rho\mathbf{u}:\mathbb{R}_t\times\mathbb{R}^d\mapsto\mathbb{R}^d$, and pressure tensor, $\mathbb{P}:\mathbb{R}_t\times\mathbb{R}^d\mapsto\mathbb{R}^d\times\mathbb{R}^d$, subject to initial data $(\rho,\mathbf{u},\mathbb{P})_{|t=0}=(\rho_0,\mathbf{u}_0,\mathbb{P}_0)$, and is driven by an *alignment* term acting on the support $\mathcal{S}(t):=\operatorname{supp}\rho(t,\cdot)$,

$$\mathbf{A}(\rho,\mathbf{u}) := \int_{\mathcal{S}(t)} \phi(\mathbf{x},\mathbf{x}')(\mathbf{u}(t,\mathbf{x}')-\mathbf{u}(t,\mathbf{x}))\rho(t,\mathbf{x})\rho(t,\mathbf{x}')\,\mathrm{d}\mathbf{x}', \qquad \phi(\mathbf{x},\mathbf{x}')=\phi(\mathbf{x}',\mathbf{x}). \tag{1.1b}$$

The alignment term on the right reflects steering towards average heading. Here, different weighted averages are dictated by symmetric communication kernels $\phi(\cdot,\cdot)$. Prototypical examples include *metric kernels*, $\phi(\mathbf{x},\mathbf{x}')=k(|\mathbf{x}-\mathbf{x}'|)$, which go back to [CS2007a]. Other classes of symmetric kernels that either dictated by the problem or learned from the data can be found in [GWBL2012, JJ2015, LZTM2019, MLK2019], and finally we mention topologically-based kernels studied in [ST2020b], $\phi(\mathbf{x},\mathbf{x}')=k(m(C(\mathbf{x},\mathbf{x}')))$, where

$m(C(\mathbf{x},\mathbf{x}')) = \displaystyle\int_C \rho(t,\mathbf{z})\,\mathrm{d}\mathbf{z}$ is the mass enclosed in an intermediate domain $C = C(\mathbf{x},\mathbf{x}')$ with tips at $\mathbf{x}$ and $\mathbf{x}'$. The prominent role of metric kernels enters when we assume that there exists a radial kernel, $k(r)$, such that

$$\phi(\mathbf{x},\mathbf{x}') \geqslant k(|\mathbf{x}-\mathbf{x}'|). \tag{1.1c}$$

We further assume that the metric kernel $k(r)$ is decreasing with the distance r, reflecting the typical observation that the intensity of alignment decreases with the distance. In particular, we address general metric kernels $\phi(|\cdot|)$ whether decreasing or not, in terms of their *decreasing envelope* $k(r) := \min\{\phi(|\mathbf{x}|)\mid |\mathbf{x}|\leqslant r\}$. Observe that we do not place any restriction on the upper-bound of ϕ; in particular, therefore, our discussion includes the important sub-class of *singular* communication kernels $k(r) = r^{-\alpha},\ \alpha>0$, [ST2017a, DKRT2018, MMPZ2019, AC2021b].

1.2. **Entropic pressure.** System (1.1) is not closed in the sense that the pressure $\mathbb{P}$ is not specified — neither in terms of algebraic relations with $(\rho,\mathbf{u})$, nor do we specify a precise dynamics of $\mathbb{P}$. Indeed, We do not dwell here on the details of the underlying the pressure tensor. Instead, we treat a rather general class of pressure laws satisfying an essential structural (dissipative) property which, as we shall show, maintains long time flocking behavior. This brings us to the following.

Definition 1.1 (**Entropic pressure).** *We say that $\mathbb{P}$ is an entropic pressure associated with (1.1) if it has a non-negative trace, $\rho e_{\mathbb{P}} := \frac{1}{2}trace(\mathbb{P}) \geqslant 0$, which satisfies*

$$\partial_t(\rho e_{\mathbb{P}}) + \nabla_{\mathbf{x}}\cdot(\rho e_{\mathbb{P}}\mathbf{u} + \boldsymbol{q}) + \mathrm{trace}(\mathbb{P}\nabla\mathbf{u}) \leqslant -2\int_{\mathcal{S}(t)} \phi(\mathbf{x},\mathbf{x}')e_{\mathbb{P}}(t,\mathbf{x})\rho(t,\mathbf{x})\rho(t,\mathbf{x}')\,\mathrm{d}\mathbf{x}'. \tag{1.2}$$

Here $\boldsymbol{q}$ is an arbitrary C^1-flux.

Why entropic pressure? System (1.1) falls under the general category of hyperbolic balance laws [Daf2016, Chapter III], and (1.2) can be viewed as an entropy inequality associated with such balance law. To this end, we note that a formal manipulation of the mass and momentum equations, $(1.1\text{a})_1 \times \dfrac{|\mathbf{u}|^2}{2} + (1.1\text{a})_2\cdot\mathbf{u}$ yields[1]

$$\partial_t\Big(\frac{\rho}{2}|\mathbf{u}|^2\Big) + \nabla_{\mathbf{x}}\cdot\Big(\frac{\rho}{2}|\mathbf{u}|^2\mathbf{u} + \mathbb{P}\mathbf{u}\Big) - \mathrm{trace}\big(\mathbb{P}\nabla\mathbf{u}\big) = -\int_{\mathcal{S}(t)} \phi(\mathbf{x},\mathbf{x}')\rho\mathbf{u}\cdot(\mathbf{u}-\mathbf{u}')\rho'\,\mathrm{d}\mathbf{x}'. \tag{1.3}$$

Adding the entropic description of the pressure postulated in (1.2) leads to the entropic statement for the total energy, $E := \dfrac{|\mathbf{u}|^2}{2} + e_{\mathbb{P}}$,

$$\partial_t(\rho E) + \nabla_{\mathbf{x}}\cdot(\rho E\mathbf{u} + \mathbb{P}\mathbf{u} + \mathbf{q}) \leqslant -\int_{\mathcal{S}(t)} \phi(\mathbf{x},\mathbf{x}')\big(|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{u}' + 2e_{\mathbb{P}}\big)\rho\rho'\,\mathrm{d}\mathbf{x}'. \tag{1.4}$$

Thus, the notion of entropic pressure (1.2) complements the balance laws in (1.1) to form the *entropy inequality* (1.4).

To further motivate why this notion of an entropic pressure, we appeal to its underlying *kinetic formulation.* The hydrodynamics (1.1) corresponds to the large-crowd dynamics of

[1] Here and below for a quantity $\square = \square(t,\mathbf{x})$ we abbreviate $\square' := \square(t,\mathbf{x}')$

N agents with position/velocity $(\mathbf{x}_i(t), \mathbf{v}_i(t)) : \mathbb{R}_t \mapsto \mathbb{R}^d \times \mathbb{R}^d$, governed by the celebrated agent-based alignment model of Cucker & Smale [CS2007a, CS2007b]

$$\text{(1.5)} \qquad \left\{ \begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}_i(t) = \mathbf{v}_i(t), \\ & \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_i(t) = \frac{1}{N}\sum_{j=1}^N \phi_{ij}(t)(\mathbf{v}_j(t) - \mathbf{v}_i(t)), \end{aligned} \right. \qquad i = 1, 2, \dots N.$$

The alignment dynamics is driven by a weighted graph Laplacian on the right of $(1.5)_2$, dictated by the symmetric communication kernel, $\phi_{ij}(t) := \phi(\mathbf{x}_i(t), \mathbf{x}_i(t))$. The passage from the agent-based to the hydrodynamic description is realized by moments of the *empirical distribution*

$$f_N(t, \mathbf{x}, \mathbf{v}) := \frac{1}{N}\sum_{i=1}^N \delta_{\mathbf{x}_i(t)} \otimes \delta_{\mathbf{v}_i(t)}, \qquad (t, \mathbf{x}, \mathbf{v}) \in \mathbb{R}_t \times \mathbb{R}^d \times \mathbb{R}^d.$$

The large crowd limits which are assumed to exist, recover (1.1) with

$$\rho(t, \mathbf{x}) = \lim_{N\to\infty} \int_{\mathbb{R}^d} f_N(t, \mathbf{x}, \mathbf{v})\,\mathrm{d}\mathbf{v} \quad \text{and} \quad \rho\mathbf{u}(t, \mathbf{x}) = \lim_{N\to\infty} \int_{\mathbb{R}^d} \mathbf{v} f_N(t, \mathbf{x}, \mathbf{v})\,\mathrm{d}\mathbf{v}.$$

This passage from agent-based to macroscopic description is outlined in appendix A.1 below. It was justified for smooth kernels [HT2008, CFTV2010, CCR2011, FK2019, NP2021, Shv2021] and at least mildly singular kernels, [Pes2015, PS2019, MMPZ2019]. In this context, the pressure or Reynolds stress tensor corresponds to the *second-order moments*

$$\text{(1.6)} \qquad \mathbb{P}(t, \mathbf{x}) = \lim_{N\to\infty} \int_{\mathbb{R}^d} (\mathbf{v} - \mathbf{u}) \otimes (\mathbf{v} - \mathbf{u}) f_N(t, \mathbf{x}, \mathbf{v})\,\mathrm{d}\mathbf{v}.$$

We observe that the kinetic description of pressure in (1.6) is consistent with the entropic inequality postulated in (1.2). Indeed, $\rho e_{\mathbb{P}} := \frac{1}{2}\mathrm{trace}(\mathbb{P})$ is the *internal energy* which quantifies microscopic fluctuations around the bulk velocity $\mathbf{u}$,

$$\text{(1.7)} \qquad \rho e_{\mathbb{P}} = \lim_{N\to\infty} \int_{\mathbb{R}^d} \frac{1}{2}|\mathbf{v} - \mathbf{u}|^2 f_N(t, \mathbf{x}, \mathbf{v})\,\mathrm{d}\mathbf{v}.$$

This kinetic description of internal energy yields (detailed derivation is carried out in appendix A.2 below),

$$\text{(1.8)} \qquad \partial_t(\rho e_{\mathbb{P}}) + \nabla_{\mathbf{x}} \cdot (\rho e_{\mathbb{P}} \mathbf{u} + \mathbf{q}_h) + \mathrm{trace}(\mathbb{P}\nabla\mathbf{u}) = -2\int_{\mathcal{S}(t)} \phi(\mathbf{x}, \mathbf{x}') e_{\mathbb{P}}(t, \mathbf{x}) \rho\rho'\,\mathrm{d}\mathbf{x}',$$

with the so-called heat flux $\mathbf{q}_h := \lim_{N\to\infty} \frac{1}{2}\int |\mathbf{v} - \mathbf{u}|^2(\mathbf{v} - \mathbf{u}) f_N(t, \mathbf{x}, \mathbf{v})\,\mathrm{d}\mathbf{v}$. Formally, any kinetic-based pressure tensor is in particular an entropic pressure, in the sense of satisfying the *equality* (1.8). But here one encounters the familiar problem of lack closure which arises whenever one is dealing with the highest truncated $\mathbf{v}$-moments of f_N. In classical particle dynamics, the closure problem is resolved by compatibility with a preferred state of thermal equilibrium, a “Maxwellian” induced by the thermal equilibrium of the system, [Lev1996, Gol1998, Cer2003, Vil2003]. In the current setup the agent-based dynamics, however, (1.5) governs *active matter* made of ‘social particles’ which admit no universal

Maxwellian closure. Then, there are multiple reasons which led us to postulate the corresponding entropy *inequality* (1.2).

Scalar pressure. We discuss the case of scalar pressure law $\mathbb{P} = \mathbb{p}\mathbb{I}$. A large part of the existing literature on swarming *assumes* a mono-kinetic closure,

$$f_N(t, \mathbf{x}, \mathbf{v}) \stackrel{N\to\infty}{\longrightarrow} \rho(t, \mathbf{x})\delta(\mathbf{v} - \mathbf{u}(t, \mathbf{x})), \tag{1.9}$$

which is realized in terms of zero pressure, $\mathbb{p} = 0$, e.g., [HT2008, CFTV2010, FK2019, NP2021, Shv2021] and the references therein. We mention the derivation from first principles [Bia2012], the isentropic closure, $\mathbb{p} = \rho^\gamma$, of [KMT2013, KMT2015, KV2015, Cho2019, TCGW2020, Shv2022], or equations of state fitted by observation that can be found in [Sin2021] as examples for detailed thermodynamic closures for scalar pressure laws in the form of *equality* in (1.10) below.

The notion of entropic pressure covers all these entropic examples of scalar pressure laws, as it applies to a broad class of pressure laws satisfying the entropy inequality postulated in (1.2), but otherwise require no algebraic closure. Indeed, our notion of entropic pressure becomes more transparent in scalar case $\mathbb{P} = \mathbb{p}\mathbb{I}$ where the inequality postulated in (1.2) for $\mathbb{p} := \dfrac{2}{d}\rho e_{\mathbb{p}}$ reads (assuming no heat flux $\mathbf{q} = 0$),

$$\partial_t \mathbb{p} + \nabla_{\mathbf{x}} \cdot (\mathbb{p}\mathbf{u}) + \frac{2}{d}\mathbb{p}\nabla_{\mathbf{x}} \cdot \mathbf{u} \leqslant -2\mathbb{p} \int_{\mathcal{S}(t)} \phi(\mathbf{x}, \mathbf{x}')\rho(t, \mathbf{x}')\, \mathrm{d}\mathbf{x}'. \tag{1.10}$$

Formal manipulation, $(1.10)\times\rho^{-\gamma} - (1.1a)_1 \times \gamma\rho^{-\gamma-1}\mathbb{p}$ with $\gamma = 1 + \frac{2}{d}$, leads to the equivalent entropic statement for $S = \ln\left(\mathbb{p}\rho^{-\gamma}\right)$,

$$\partial_t(\rho S) + \nabla_{\mathbf{x}} \cdot (\rho\mathbf{u}S) \leqslant -2 \int_{\mathcal{S}(t)} \phi(\mathbf{x}, \mathbf{x}')\rho(t, \mathbf{x})\rho(t, \mathbf{x}')\, \mathrm{d}\mathbf{x}', \qquad S := \ln\left(\mathbb{p}\rho^{-(1+\frac{2}{d})}\right). \tag{1.11}$$

We point out that the inequality (1.11) is the *reversed* entropy inequality encountered for $-S$ in compressible Euler equations. The difference, which was already noted in [HT2008, §6], is due to different states of thermodynamic equilibria.

Entropic energy dissipation. An entropy inequality is intimately connected with the *irreversibility* of the underlying process, e.g., the enlightening discussion in [Vil2003, §2.4]. In the present context of hydrodynamics alignment, the entropy inequality (1.2), or in its equivalent form (1.4), yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{S}(t)} \rho E\, \mathrm{d}\mathbf{x} + \int_{\partial\mathcal{S}(t)} \Big(\rho E\mathbf{u}\cdot\mathbf{n} + (\mathbb{P}\mathbf{u})\cdot\mathbf{n} + \mathbf{q}\cdot\mathbf{n}\Big)\mathrm{d}S \\
&\qquad \leqslant -\iint_{\mathcal{S}(t)\times\mathcal{S}(t)} \phi(\mathbf{x}, \mathbf{x}')\big(|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{u}' + 2e_{\mathbb{p}}\big)\rho\rho'\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{x}' \\
&\qquad = -\frac{1}{2}\iint_{\mathcal{S}(t)\times\mathcal{S}(t)} \phi(\mathbf{x}, \mathbf{x}')\Big(|\mathbf{u}' - \mathbf{u}|^2 + 2e_{\mathbb{p}} + 2e'_{\mathbb{p}}\Big)\rho\rho'\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{x}' < 0,
\end{aligned} \tag{1.12}$$

which reflects the dissipativity of the total energy $\displaystyle\int \rho E\, \mathrm{d}\mathbf{x}$. Thus, the entropy inequality (1.2) complements the balance laws in (1.1) to govern the energy dissipation (1.12). This is reminiscent of P.-L. Lions' notion of dissipative solutions in the context of the Euler equations [Lio1996, §4.4].

One of the main aspects of this work is dealing with arbitrary pressure, without any specifics about the second-order closure for $\mathbb{P}$. The definition of entropic pressure in (1.2) is not concerned with the detailed balance of internal energy. Instead, its main purpose is to secure the dissipative nature of the total energy, ρE. This partially echoes Vicsek & Zaferis who argued that in the context of collective motion "*The source of energy making the motion possible ... are not relevant*" [VZ2012, §1.1]. Here, we abandon a closure in the form of thermal equality (1.8) and instead, retain the inequality postulated in (1.2), compatible with the dissipativity of internal fluctuations which we argued for in [Tad2021, p. 501]. In particular, our definition of a pressure in (1.2) can be realized in any intermediate scale between the microscopic agent-based description, (1.5), and the macroscopic hydrodynamics (1.1), and hence can be viewed as "mesoscopic". These considerations become even more pronounced when we extend our discussion to a larger class of so-called p-alignment hydrodynamics.

2. p-ALIGNMENT

We begin with the agent-based description,

$$\left\{\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}_i(t) = \mathbf{v}_i(t),\\ &\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_i(t) = \frac{1}{N}\sum_{j=1}^{N}\phi_{ij}(t)|\mathbf{v}_j(t)-\mathbf{v}_i(t)|^{2p-2}(\mathbf{v}_j(t)-\mathbf{v}_i(t)), \end{aligned}\right. \qquad i=1,2,\ldots N. \tag{2.1}$$

The case $p=1$ coincides with the Cucker-Smale model (1.5), while for $p>1$, the alignment term on the right of (2.1) corresponds to weighted graph $2p$-Laplacian[2] which is found in recent applications of neural networks [FZN2021], spectral clustering [BH2009], semi-supervised learning [ST2019],[Fu2021]. In the context of alignment dynamics it was introduced in [HHK2010, CCH2014]. We were motivated by the example of Elo rating system, [JJ2015, DTW2019], in which the alignment of scalar ratings $\{q_i\}$ is governed by odd function of local gradients (q_j-q_i), e.g., $|q_j-q_i|^{2p-2}(q_j-q_i)$.
The long time behavior of the p-alignment model with $p>1$ is distinctly different from the 'pure' alignment model when $p=1$ (and there is yet a different behavior for $0\leqslant p<1$ which we comment in remark 5.5 below). The large-crowd dynamics associated with (2.1) is captured by the corresponding hydrodynamic description

$$\left\{\begin{aligned} &\partial_t\rho+\nabla_{\mathbf{x}}\cdot(\rho\mathbf{u})=0,\\ &\partial_t(\rho\mathbf{u})+\nabla_{\mathbf{x}}\cdot(\rho\mathbf{u}\otimes\mathbf{u}+\mathbb{P})=\mathbf{A}_p(\rho,\mathbf{u}), \end{aligned}\right. \tag{2.2a}$$

with p-alignment term

$$\mathbf{A}_p(\rho,\mathbf{u}) := \int_{\mathcal{S}(t)}\phi(\mathbf{x},\mathbf{x}')|\mathbf{u}'-\mathbf{u}|^{2p-2}(\mathbf{u}'-\mathbf{u})\rho\rho'\,\mathrm{d}\mathbf{x}', \qquad p\geqslant 1. \tag{2.2b}$$

Remark 2.1 (**General p-alignment terms**). A detailed derivation of the p-alignment term $\mathbf{A}_p(\rho,\mathbf{u})$ in (2.2b) is outlined in appendix A.1. This kinetic-based derivation is compatible with the mono-kinetic closure (1.9). In fact, our line of arguments below does not require the detailed form of $\mathbf{A}_p(\rho,\mathbf{u})$, except for satisfying two 'structural' conditions. The first

[2] To simplify computations we proceed with $2p$-Laplacians rather than p-Laplacians.

condition requires that it has a zero average $\displaystyle\int_{\mathcal{S}(t)} \mathbf{A}_p(\rho,\mathbf{u})(t,\mathbf{x})\,\mathrm{d}\mathbf{x} = 0$. This clearly holds for the p-alignment (2.2b), and in fact it holds for *any* kinetic closure; see (A.4) below. The second and essential condition requires a p-alignment term which induces an entropic pressure. We discuss this notion of entropic pressure in context of p-alignment next.

We assume that $\mathbb{P}$ belongs to a class of entropic pressures, whose definition is adapted to the case of p-alignment.

Definition 2.2 (**Entropic pressure for p-alignment**)**.** *We say that $\mathbb{P}$ is a entropic pressure associated with* (2.2) *if it has a non-negative trace, $\rho e_{\mathbb{P}} := \frac{1}{2}trace(\mathbb{P}) \geqslant 0$, satisfying*

$$\partial_t(\rho e_{\mathbb{P}}) + \nabla_{\mathbf{x}}\cdot(\rho e_{\mathbb{P}}\mathbf{u} + \boldsymbol{q}) + \mathrm{trace}(\mathbb{P}\nabla\mathbf{u}) \leqslant -\frac{1}{2}\int_{\mathcal{S}(t)} \phi(\mathbf{x},\mathbf{x}')\big((2e_{\mathbb{P}})^p + (2e'_{\mathbb{P}})^p\big)\rho\rho'\,\mathrm{d}\mathbf{x}'. \tag{2.3}$$

Here $\boldsymbol{q}$ is an arbitrary C^1-flux.

Definition 2.2 is motivated by the underlying kinetic formulation, where one encounters the p-alignment quantity, see appendix A.2 below[3],

$$-\frac{1}{2}\int_{\mathcal{S}(t)} \phi(\mathbf{x},\mathbf{x}') \iint_{\mathbb{R}^d\times\mathbb{R}^d} |\mathbf{v}-\mathbf{v}'|^{2p} f_N f'_N\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{v}'\,\mathrm{d}\mathbf{x}'.$$

One cannot close the kinetic expression $\displaystyle\iint |\mathbf{v}-\mathbf{v}'|^{2p} f_N f'_N\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{v}'$ in terms of the thermodynamic quantity $e_{\mathbb{P}}$, without taking into account a more detailed thermodynamic information i.e., higher moments of the empirical distribution f_N. It is here that we abandon the detailed thermal equality in favor of the inequality which follows from polarization, $\mathbf{v}-\mathbf{v}' \equiv (\mathbf{v}-\mathbf{u}) + (\mathbf{u}-\mathbf{u}') + (\mathbf{u}'-\mathbf{v}')$,

$$\begin{aligned}
-\frac{1}{2}&\iint |\mathbf{v}-\mathbf{v}'|^{2p} f_N f'_N\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{v}' \\
&\leqslant -\frac{1}{2}\Big(\iint \Big(|\mathbf{v}-\mathbf{u}|^2 + |\mathbf{v}'-\mathbf{u}'|^2\Big) f_N f'_N\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{v}'\Big)^p (\rho\rho')^{-\frac{p}{p'}} \stackrel{N\to\infty}{\longrightarrow} -\frac{1}{2}\big((2e_{\mathbb{P}})^p + (2e'_{\mathbb{P}})^p\big)\rho\rho'.
\end{aligned}$$

This leads to the corresponding term of p-entropic pressure postulated on the right of (2.3). The special case of pure alignment, $p=1$, offers an alternative derivation where polarization implies the equality, consult (A.8) below

$$\begin{aligned}
&\iint (\mathbf{v}-\mathbf{u})\cdot(\mathbf{v}-\mathbf{v}') f_N f'_N\,\mathrm{d}\mathbf{v}'\,\mathrm{d}\mathbf{v} \\
&\quad = -\iint |\mathbf{v}-\mathbf{u}|^2 f_N f'_N\,\mathrm{d}\mathbf{v}'\,\mathrm{d}\mathbf{v} - \int (\mathbf{v}-\mathbf{u}) f_N\,\mathrm{d}\mathbf{v}\cdot\int(\mathbf{u}-\mathbf{v}')f'_N\,\mathrm{d}\mathbf{v}' \stackrel{N\to\infty}{\longrightarrow} -2e_{\mathbb{P}}\rho\rho',
\end{aligned}$$

which in turn formally yields the entropy *equality* (1.8),

$$\partial_t(\rho e_{\mathbb{P}}) + \nabla_{\mathbf{x}}\cdot(\rho e_{\mathbb{P}}\mathbf{u} + \mathbf{q}) + \mathrm{trace}(\mathbb{P}\nabla\mathbf{u}) = -2\int_{\mathcal{S}(t)} \phi(\mathbf{x},\mathbf{x}')e_{\mathbb{P}}\rho\rho'\,\mathrm{d}\mathbf{x}'. \tag{2.4}$$

[3]Here and below we abbreviate $\square' := \square(t,\mathbf{x}',\mathbf{v}')$

Thus, while for $p=1$ the inequality of entropic pressure (1.2) could be viewed as a matter of choice made in the equalities (1.8) or (2.4), for $p>1$ the entropic inequality (2.3) is a necessity in order to have a macroscopic interpretation of an entropic pressure.

Remark 2.3 (**Local vs. global flux**). We observe that the entropic statement for p-alignment (2.3) with $p=1$ is a symmetric version of the entrropic inequality of 'pure' alignment, (1.2). Apparently, the two definitions do not agree when $p=1$, but in fact, their difference is encoded in different fluxes $\mathbf{q}$. In particular, while the entropic pressure in pure alignment (1.2) is encoded in terms of a *local* heat flux, $\mathbf{q}_h$ in (A.6) below, the case of p-alignment (2.3) requires a *global* flux, $\mathbf{q}_h+\mathbf{q}_\phi$ in (A.13) below. Alternatively, we could be less 'pedantic' and combine both cases of alignment and of p-alignment under the same notion of entropic pressure inequality

$$\partial_t(\rho e_{\mathbb{P}}) + \nabla_{\mathbf{x}}\cdot(\rho e_{\mathbb{P}}\mathbf{u}+\mathbf{q}) + \mathrm{trace}(\mathbb{P}\nabla\mathbf{u}) \leqslant -2^{p-1}\int_{\mathcal{S}(t)} \phi(\mathbf{x},\mathbf{x}')e_{\mathbb{P}}^p\rho\rho'\,\mathrm{d}\mathbf{x}', \qquad p\geqslant 1.$$

This will not affect any of the follow-up results.

Of course, a general C^1 flux, $\mathbf{q}$, can also 'absorb' the convective term $\rho e_{\mathbb{P}}\mathbf{u}$; our main focus is in the global dissipative structure entailed by (2.3).

Entropic energy dissipation in p-alignment. Following the same formal manipulations as before for $p=1$, see (1.3), yield

$$\partial_t\Big(\frac{\rho}{2}|\mathbf{u}|^2\Big) + \nabla_{\mathbf{x}}\cdot\Big(\frac{\rho}{2}|\mathbf{u}|^2\mathbf{u}+\mathbb{P}\mathbf{u}\Big) - \mathrm{trace}\big(\mathbb{P}\nabla\mathbf{u}\big) \leqslant -\int_{\mathcal{S}(t)} \phi(\mathbf{x},\mathbf{x}')|\mathbf{u}-\mathbf{u}'|^{2p-2}\mathbf{u}\cdot(\mathbf{u}-\mathbf{u}')\rho\rho'\,\mathrm{d}\mathbf{x}'.$$

Adding (2.3) and integrating we find

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{S}(t)}\rho E(t,\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{\partial\mathcal{S}(t)}\Big(\rho E\mathbf{u}\cdot\mathbf{n} + (\mathbb{P}\mathbf{u})\cdot\mathbf{n} + \mathbf{q}\cdot\mathbf{n}\Big)\mathrm{d}S\\
&\qquad \leqslant -\iint_{\mathcal{S}(t)\times\mathcal{S}(t)} \phi(\mathbf{x},\mathbf{x}')\Big(|\mathbf{u}-\mathbf{u}'|^{2p-2}\big(|\mathbf{u}|^2-\mathbf{u}\cdot\mathbf{u}'\big) + \frac{1}{2}\big((2e_{\mathbb{P}})^p + (2e_{\mathbb{P}}')^p\big)\Big)\rho\rho'\,\mathrm{d}\mathbf{x}'\\
&\qquad = -\frac{1}{2}\iint_{\mathcal{S}(t)\times\mathcal{S}(t)} \phi(\mathbf{x},\mathbf{x}')\Big(|\mathbf{u}'-\mathbf{u}|^{2p} + (2e_{\mathbb{P}})^p + (2e_{\mathbb{P}}')^p\Big)\rho\rho'\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}' < 0.
\end{aligned} \tag{2.5}$$

which extends the dissipativity statement of 'pure' alignment in the case $p=1$ in (1.12).

3. Swarming

The hydrodynamic alignment (1.1) occupies a distinct 'patch' of mass,

$$\mathcal{S}(t) = \mathrm{supp}\,\rho(t,\cdot).$$

We shall refer to this patch of mass simply as a 'crowd' — a continuum of agents which encodes the large-crowd dynamics associated with (1.5). In most of the existing literature on collective dynamics, the edge of such crowd is assumed to be 'tailored' to the surrounding vacuum so that $\rho(t,\cdot)_{|\partial\mathcal{S}}=0$. Instead, we argue here for a more realistic scenario in which the density inside the crowd remains strictly bounded away from vacuum,

$$\min_{\mathbf{x}\in\mathcal{S}(t)}\rho(t,\mathbf{x})\geqslant\rho_->0, \tag{3.1}$$

while its boundary forms a shock discontinuity, moving with velocity $\mathbf{u}_{|\partial\mathcal{S}}$. A detailed discussion on the nature of boundary conditions (BCs) for such crowds is beyond the scope of this work (see [AC2021a] for the special one-dimensional case with $p=\rho$). Instead, we argue (1.1) augmented with Neumann BCs

$$\mathbf{u}\cdot\mathbf{n}_{|\partial\mathcal{S}}=0,\quad \mathbb{P}\mathbf{n}_{|\partial\mathcal{S}}=0,\quad\text{and}\quad \mathbf{q}\cdot\mathbf{n}_{|\partial\mathcal{S}}=0. \tag{3.2}$$

In particular, it follows that the total mass of the crowd, $M=M(t)$, is conserved in time,

$$M(t):=\int_{\mathcal{S}(t)}\rho(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\equiv M_0, \tag{3.3}$$

and by the symmetry of $\phi(\cdot,\cdot)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{S}(t)}\rho\mathbf{u}\,\mathrm{d}\mathbf{x}=-\int_{\partial\mathcal{S}(t)}(\mathbf{u}\otimes\mathbf{u}\cdot\mathbf{n}+\mathbb{P}\mathbf{n})\,\rho\mathrm{d}S\;-\iint_{\mathcal{S}(t)\times\mathcal{S}(t)}\phi(\mathbf{x},\mathbf{x}')|\mathbf{u}'-\mathbf{u}|^{2p-2}(\mathbf{u}'-\mathbf{u})\rho\rho'\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}'=0,$$

and hence the total momentum of the crowd, $\mathbf{m}=\mathbf{m}(t)$, is also conserved,[4]

$$\mathbf{m}(t):=\int_{\mathcal{S}(t)}\rho(t,\mathbf{x})\mathbf{u}(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\equiv\mathbf{m}_0. \tag{3.4}$$

Finally, (2.5) yields that the total energy is non-increasing

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{S}(t)}\rho E(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\leqslant-\frac{1}{2}\iint_{\mathcal{S}(t)\times\mathcal{S}(t)}\phi(\mathbf{x},\mathbf{x}')\Big(|\mathbf{u}'-\mathbf{u}|^{2p}+(2e_{\mathbb{P}})^p+(2e'_{\mathbb{P}})^p\Big)\rho\rho'\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}'. \tag{3.5}$$

In particular, we have the space-time *enstrophy* bound

$$\int_0^t\iint_{\mathcal{S}(t)\times\mathcal{S}(t)}\phi(\mathbf{x},\mathbf{x}')\Big(|\mathbf{u}'-\mathbf{u}|^{2p}+(2e_{\mathbb{P}})^p+(2e'_{\mathbb{P}})^p\Big)\rho\rho'\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}'\mathrm{d}t\leqslant C_0^2:=2\int_{\mathcal{S}(0)}\rho_0E_0\,\mathrm{d}\mathbf{x}. \tag{3.6}$$

Flocking. A characteristic feature of alignment dynamics is the emergence of coherent structure with limiting velocity $\mathbf{u}_\infty$ such that

$$\mathbf{u}(t,\mathbf{x})-\mathbf{u}_\infty(t,\mathbf{x})\stackrel{t\to\infty}{\longrightarrow}0, \tag{3.7}$$

and the corresponding limiting density, ρ_∞. This is typical in *flocking* phenomena. In the present context of the hydrodynamic alignment (1.1), the limiting behavior of the dynamics (1.1) can only approach the time-invariant mean velocity $\mathbf{u}_\infty=\overline{\mathbf{u}}:=\dfrac{\mathbf{m}_0}{M_0}$ with a limiting

[4]This is the only stage which requires the zero-average p-alignment term argued in remark 2.1,

$$\int_{\mathcal{S}(t)}\mathbf{A}_p(\rho,\mathbf{u})(t,\mathbf{x})\,\mathrm{d}\mathbf{x}=\iint_{\mathcal{S}(t)\times\mathcal{S}(t)}\phi(\mathbf{x},\mathbf{x}')|\mathbf{u}'-\mathbf{u}|^{2p-2}(\mathbf{u}'-\mathbf{u})\rho\rho'\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}'=0,$$

which in turn implies conservation of total momentum $\mathbf{m}(t)=\mathbf{m}_0$.

density carried out as a traveling wave $\rho_\infty(\mathbf{x}-\overline{\mathbf{u}}t)$, [ST2017b, §2]. The presence of additional repulsion, attraction and external forces introduce a 'richer' set of possible emerging limiting configurations, e.g., [CDMBC2007]; for example, alignment with quadratic forcing approaching an harmonic oscillator $\ddot{\mathbf{u}}_\infty(t)+a^2\mathbf{u}_\infty(t)=0$, [ST2020a, §2.4]. The precise notion of flocking convergence in (3.7) may vary. Ideally, we seek uniform convergence. A more relaxed notion of L^2_ρ-convergence becomes accessible by studying *energy fluctuations*, see the next section 4,

$$\int_{\mathcal{S}(t)} |\mathbf{u}-\mathbf{u}_\infty|^2\rho\,\mathrm{d}\mathbf{x} \stackrel{t\to\infty}{\longrightarrow} 0.$$

In practice, as we shall see below, the analysis may gain by a combination of the two.
The limiting configuration is supported on $\mathcal{S}_\infty(t) := \operatorname{supp}\rho_\infty(t,\cdot)$. For example, $\mathcal{S}_\infty(t)$ is a Dirac mass in presence additional attractive forces [ST2021, Theorem 1]. Ideally, we are interested to trace the shape of the boundary $\partial\mathcal{S}_\infty(t)$, but this seems to be out of reach in the current literature (but see [LLST2022]). In general, one expects that alignment is at least strong enough to keep the dynamics contained in a finite ball,

$$D(t)\leqslant D_+<\infty, \qquad D(t):=\max_{\mathbf{x},\mathbf{x}'\in\mathcal{S}(t)}|\mathbf{x}-\mathbf{x}'|.$$

In practice we may need to address to a more accessible notion of diameter which allows a slow time growth, $D(t)\leqslant C_D(1+t)^\gamma$ with some fixed $\gamma>0$.

4. Decay of energy fluctuations

We study the hydrodynamics of the p-alignment, (2.2), assuming it admits a strong entropic solution, (2.3); see further comments on (H*1*) in section 6.1 below.
Consider the *energy fluctuations*, [HT2008, §5], [Tad2021],

$$\delta\mathscr{E}(t) := \frac{1}{2M}\iint_{\mathcal{S}(t)\times\mathcal{S}(t)} \Big(\frac{1}{2}|\mathbf{u}(t,\mathbf{x})-\mathbf{u}(t,\mathbf{x}')|^2 + e_{\mathbb{P}}(t,\mathbf{x})+e_{\mathbb{P}}(t,\mathbf{x}')\Big)\rho(t,\mathbf{x})\rho(t,\mathbf{x}')\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}'. \tag{4.1}$$

It can be expressed in the equivalent form,[5] $\delta\mathscr{E}(t) = \displaystyle\int_{\mathcal{S}(t)} \Big(\frac{1}{2}|\mathbf{u}(t,\mathbf{x})-\overline{\mathbf{u}}(t)|^2+e_{\mathbb{P}}(t,\mathbf{x})\Big)\rho(t,\mathbf{x})\,\mathrm{d}\mathbf{x}$.

Thus, $\delta\mathscr{E}(t)$ reflects macroscopic velocity fluctuations $\displaystyle\int_{\mathcal{S}(t)}\frac{1}{2}|\mathbf{u}-\overline{\mathbf{u}}(t)|^2\rho(t,\mathbf{x})\,\mathrm{d}\mathbf{x}$ around the

[5]Specifically

$$\begin{aligned}\frac{1}{M}&\iint_{\mathcal{S}(t)\times\mathcal{S}(t)}\frac{1}{2}|\mathbf{u}(t,\mathbf{x})-\mathbf{u}(t,\mathbf{x}')|^2\rho\rho'\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}'\\ &=\frac{1}{M}\iint_{\mathcal{S}(t)\times\mathcal{S}(t)}\Big(\frac{1}{2}|\mathbf{u}(t,\mathbf{x})-\overline{\mathbf{u}}|^2+(\mathbf{u}-\overline{\mathbf{u}})\cdot(\overline{\mathbf{u}}-\mathbf{u}')+\frac{1}{2}|\mathbf{u}(t,\mathbf{x}')-\overline{\mathbf{u}}|^2\Big)\rho\rho'\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}'\\ &=\int_{\mathcal{S}(t)}|\mathbf{u}(t,\mathbf{x})-\overline{\mathbf{u}}|^2\rho(t,\mathbf{x})\,\mathrm{d}\mathbf{x}.\end{aligned}$$

mean velocity, $\overline{\mathbf{u}}(t) := \dfrac{1}{M}\displaystyle\int_{\mathcal{S}(t)} \rho\mathbf{u}(t,\cdot)\,\mathrm{d}\mathbf{x} = \dfrac{\mathbf{m}}{M}$, and in the context of kinetic formulation (1.6)–(1.7), it also reflects the microscopic velocity fluctuations, $\rho e_{\mathbb{P}} = \lim_{N\to\infty}\displaystyle\int \frac{1}{2}|\mathbf{v} - \mathbf{u}|^2 f_N(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v}$. We have the following decay bound on energy fluctuations

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\mathscr{E}(t) \leqslant -2^p M^{2-p} k(D(t))\big(\delta\mathscr{E}(t)\big)^p. \tag{4.2}$$

The derivation follows the energy inequality (3.5). Noting that $\delta\mathscr{E}(t) \equiv \displaystyle\int_{\mathcal{S}(t)} \rho E\,\mathrm{d}\mathbf{x} - \frac{1}{2M}\big|\mathbf{m}\big|^2$ with total mass and total momentum which are conserved in time, $M(t) = M_0,\ \mathbf{m}(t) = \mathbf{m}_0$, we end up with,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\delta\mathscr{E}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{S}(t)} \rho E(t,\mathbf{x})\,\mathrm{d}\mathbf{x} \\
&\leqslant -\frac{1}{2}\iint_{\mathcal{S}(t)\times\mathcal{S}(t)} \phi(\mathbf{x},\mathbf{x}')\Big(|\mathbf{u}-\mathbf{u}'|^{2p} + (2e_{\mathbb{P}})^p + (2e'_{\mathbb{P}})^p\Big)\rho\rho'\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}' \\
&\leqslant -\iint_{\mathcal{S}(t)\times\mathcal{S}(t)} \phi(\mathbf{x},\mathbf{x}')\Big(\frac{1}{2}|\mathbf{u}-\mathbf{u}'|^2 + e_{\mathbb{P}}(t,\mathbf{x}) + e_{\mathbb{P}}(t,\mathbf{x}')\Big)^p\rho\rho'\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}' \\
&\leqslant -k(D(t))\Big(\iint_{\mathcal{S}(t)\times\mathcal{S}(t)}\Big(\frac{1}{2}|\mathbf{u}-\mathbf{u}'|^2 + e_{\mathbb{P}} + e'_{\mathbb{P}}\Big)\rho\rho'\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}'\Big)^p\Big(\iint_{\mathcal{S}(t)\times\mathcal{S}(t)}\rho\rho'\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}'\Big)^{-\frac{p}{p'}} \\
&= -2^p M^{2-p} k(D(t))\big(\delta\mathscr{E}(t)\big)^p.
\end{aligned} \tag{4.3}$$

The first inequality on the right quotes (3.5); the second follows from Jensen inequality and the third from Hölder inequality and the obvious radial bound (1.1c), $\phi(\mathbf{x},\mathbf{x}') \geqslant k(D(t))$. Integration of (4.3) yields the following.

Theorem 4.1. *Let $(\rho,\mathbf{u},\mathbb{P})$ be a strong solution*[6] *of the hydrodynamic p-alignment* (2.2), *satisfying the entropy condition* (2.3), *and subject to compactly supported initial data, $(\rho_0,\mathbf{u}_0,\mathbb{P}_0)$ with $D_0 < \infty$, and boundary conditions* (3.2). *Then the* energy fluctuations $\delta\mathscr{E}(t)$ *admits the bound*

$$\delta\mathscr{E}(t) \leqslant \begin{cases} \exp\left\{-2M\displaystyle\int_0^t k\big(D(s)\big)\mathrm{d}s\right\}\delta\mathscr{E}(0), & p = 1 \\[2ex] \dfrac{1}{\left\{(p-1)2^p M^{2-p}\displaystyle\int_0^t k(D(s))\mathrm{d}s\right\}^{\frac{1}{p-1}}}, & p > 1. \end{cases} \tag{4.4}$$

[6]That is, $\big(\rho(t,\cdot),\mathbf{u}(t,\cdot),\mathbb{P}(t,\cdot)\big)$ has sufficient smoothness — say $\in L^\infty_+ \cap L^1)\big(\mathcal{S}(t)\big) \times W^{1,\infty}\big(\mathcal{S}(t)\big) \times W^{1,\infty}\big(\mathcal{S}(t)\big)$, so that (1.1) can be interpreted in a pointwise sense.

The result applies to p-alignment dynamics with general class of entropic pressure tensors satisfying (2.3) (noting that (1.2) for $p=1$ yields the same energy decay (2.5)). We refer to such solutions as 'entropic solutions'. The symmetric communication protocol ϕ in (1.1c) need not be metric nor bounded and no assumption of a uniform velocity bound is made.

We close by noting that the bound $(4.4)_2$ depends on the initial mass M but otherwise it is independent of the initial fluctuations $\delta\mathscr{E}(0)$ — a typical scenario for the Ricatti's type inequality (4.2) with $p>1$.

4.1. **Heavy-tailed kernels.** The bound (4.4) reflects a competition between the expansion rate of the diameter of the crowd, $D(t)$, and the decay rate in its communication strength, $k(r)$: their composition is required to have a non-integrable "heavy-tail" in order to enforce L^2_ρ-flocking decay. We make these considerations precise in our next statement.

Communication kernels of order $\beta \geqslant 0$. There exist constants $C_k>0, R>0$ such that

$$
\phi(\mathbf{x},\mathbf{x}') \geqslant k(|\mathbf{x}-\mathbf{x}'|) \ \text{ with } \left\{ \begin{array}{l} \displaystyle\int_{|\mathbf{x}|\leqslant R} k(|\mathbf{x}|)\,\mathrm{d}\mathbf{x} < \infty, \\[2ex] k(r) = C_k(1+r)^{-\beta}, \quad r \geqslant R. \end{array}\right. \tag{4.5}
$$

This emphasizes the fact that besides the mere requirement for intractability of ϕ near the origin — only its tail behavior matters.

Notations. We use the following two constants. We let C_R denote a constant, with different values in different contexts, depending of R as well as on the other fixed parameters $\beta,\gamma,\ldots$ and possibly $p>1$. Also, we denote the 'scaled mass'

$$
M_p := \left\{ \begin{array}{ll} 2MC_kC_D^{-\beta}, & p=1, \\[2ex] \left(2^pM^{2-p}C_kC_D^{-\beta}\right)^{-\frac{1}{p-1}}, & p>1. \end{array}\right.
$$

Corollary 4.2 (**Decay of L^2_ρ-energy fluctuations**). *Let $(\rho,\mathbf{u},\mathbb{P})$ be a strong entropy solution of the hydrodynamic p-alignment system (2.2),(2.3), $p\geqslant 1$, with communication kernel ϕ of order $\beta \geqslant 0$, (4.5). Assume that the "crowd" disperses at a rate of order $\gamma\geqslant 0$,*

$$
D(t) \leqslant C_D(1+t)^\gamma, \qquad \gamma \geqslant 0, \qquad D(t) = \max\{|\mathbf{x}-\mathbf{x}'|,\ \mathbf{x},\mathbf{x}' \in \operatorname{supp}\rho(t,\cdot)\}. \tag{4.6}
$$

If the heavy-tail condition holds in the sense that $\beta\gamma<1$, then there is long time flocking behavior such that the following decay bound holds

$$
\delta\mathscr{E}(t) \leqslant \left\{ \begin{array}{ll} C_R\, exp\big\{-M_1t^{(1-\beta\gamma)}\big\}\delta\mathscr{E}(0), & p=1 \\[2ex] C_RM_p\, t^{-\frac{1-\beta\gamma}{p-1}}, & p>1. \end{array}\right. \tag{4.7}
$$

In case of pure alignment, $p=1$, $(4.7)_1$ recovers an exponential decay of fractional order $1-\beta\gamma$, [Tad2021, Corollary 1], while for $p>1$, $(4.7)_2$ implies a Pareto-type decay of fractional order $\dfrac{1-\beta\gamma}{p-1}$. Thus, corollary 4.2 implies that for heavy-tailed kernels such that $\beta\gamma<1$, both the macroscopic and microscopic fluctuations around the mean $\overline{\mathbf{u}}(t)=\overline{\mathbf{u}}_0$ decay to zero. In particular, this shows the *trend towards equilibrium* of a kinetic-based hydrodynamics, as it decays towards mono-kinetic closure (1.9)

$$\frac{1}{2}\int_{\mathcal{S}(t)} \|\mathbb{P}(t,\mathbf{x}\|_2\,\mathrm{d}\mathbf{x} = \int_{\mathcal{S}(t)} e_{\mathbb{P}}(t,\mathbf{x})\rho(t,\mathbf{x})\,\mathrm{d}\mathbf{x} \stackrel{t\to\infty}{\longrightarrow} 0.$$

A key aspect, therefore, is to study the possible expansion of the spatial diameter with time growth of order γ (possibly depending on β), so that $\beta\gamma < 1$. This will occupy us in the rest of the work.

Remark 4.3. One can refine the statement of corollary 4.2 to include the borderline case, $\beta\gamma = 1$

5. Flocking with mono-kinetic ("pressure-less") closure

One strategy for verifying flocking is to seek a uniform bound on velocity, $u_+ := \max|\mathbf{u}(t,\cdot)| < \infty$, which in turn implies a dispersion bound on the diameter of order $\lesssim (1+t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}D(t) \leqslant \delta\mathbf{u}(t), \quad \delta\mathbf{u}(t) := \max_{\mathbf{x},\mathbf{x}'\in\mathcal{S}(t)} |\mathbf{u}(t,\mathbf{x}') - \mathbf{u}(t,\mathbf{x})| \ \ \rightsquigarrow \ \ D(t) \leqslant D_0 + 2u_+t, \tag{5.1}$$

and then appeal to corollary 4.2 with $\gamma = 1$. An instructive example for this line of argument is found in the prototype case of *mono-kinetic closure*, $\mathbb{P} = 0$,

$$\partial_t(\rho\mathbf{u}) + \nabla_{\mathbf{x}}\cdot(\rho\mathbf{u}\otimes\mathbf{u}) = \mathbf{A}_p(\rho,\mathbf{u}). \tag{5.2}$$

A main feature of the mono-kinetic closure is that the resulting system (5.2) decouples into scalar transport equations,

$$u_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}u = \int_{\mathcal{S}(t)} \phi(\mathbf{x},\mathbf{x}')|\mathbf{u}'-\mathbf{u}|^{2p-2}(u'-u)\rho'\,\mathrm{d}\mathbf{x}',$$

in which case, the coercivity of the (scalar) p-alignment term on the right implies a maximum principle, $\max|\mathbf{u}(t,\cdot)| \leqslant \max|\mathbf{u}_0|$, hence

$$D(t) \leqslant D_0 + 2u_+\cdot t, \qquad u_+ := \max|\mathbf{u}_0|.$$

Appealing to corollary 4.2 with $\gamma = 1$ implies that for heavy-tailed ϕ's of order $\beta < 1$, there exists $C_R = C(R, D_0, u_+, \beta, p)$ such that

$$\delta\mathscr{E}(t) \leqslant \begin{cases} C_R\, exp\big\{-2M(D_0+2u_+\cdot t)^{(1-\beta)}\big\}\delta\mathscr{E}(0), & p=1 \\ \\ C_R M_p\big(D_0+2u_+\cdot t\big)^{-\frac{1-\beta}{p-1}}, & p>1. \end{cases}$$

In fact, more is true — a refined argument shows that for such heavy-tailed ϕ's of order $\beta < 1$, the pressureless diameter remains uniformly bounded, $D(t) \leqslant D_+$, and hence corollary 4.2 applies with $\gamma = 0$. To this end, we split out discussion, distinguishing between the case of 'pure' alignment, $p = 1$, and the case of p-alignment $p > 1$.

5.1. **Flocking with pure alignment ($p = 1$).** We begin with the following pointwise bound of velocity fluctuations which is reproduced in section B.1 below,

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\mathbf{u}(t) \leqslant -k(D(t))M\delta\mathbf{u}(t), \qquad \delta\mathbf{u}(t) = \sup_{\mathbf{x},\mathbf{x}'\in\mathcal{S}(t)} |\mathbf{u}(\mathbf{x},t)-\mathbf{u}(\mathbf{x}',t)|. \tag{5.3}$$

In particular, $\delta\mathbf{u}(t) \leqslant \delta\mathbf{u}_0$ and hence (4.6) holds with $\gamma = 1$ in view of $D(t) \leqslant D_0 + \delta\mathbf{u}_0 \cdot t$. Consequently, for β-tailed kernels of order $\beta < 1$, (4.5), there exists a constant c_R such that

$$\int_0^t k(D(s))\mathrm{d}s \geqslant \int_0^R k(D(s))\mathrm{d}s + \int_R^{\max\{R,t\}} k(D(s))\mathrm{d}s \geqslant \frac{c_R}{(1-\beta)\delta\mathbf{u}_0}(1+\delta\mathbf{u}_0\cdot t)^{1-\beta}, \quad 0 \leqslant \beta < 1.$$

Revisiting (5.3) again yields a *decay* of pointwise velocity fluctuations of fractional exponential order, $\delta\mathbf{u}(t) \leqslant \delta\mathbf{u}_0\, exp\{-c'_R(1+\delta\mathbf{u}_0\cdot t)^{1-\beta}\}$ with $c'_R = \dfrac{M}{(1-\beta)\delta\mathbf{u}_0}c_R$, which in turn implies that the diameter remains uniformly bounded

$$\frac{\mathrm{d}}{\mathrm{d}t}D(t) \leqslant \delta\mathbf{u}_0\, e^{-c'_R(1+\delta\mathbf{u}_0\cdot t)^{1-\beta}} \quad \rightsquigarrow \quad D(t) \leqslant D_+ := \delta\mathbf{u}_0 \int_0^\infty e^{-c'_R(1+\delta\mathbf{u}_0\cdot t)^{1-\beta}}\mathrm{d}t < \infty.$$

Alternatively, one can use the deceasing Liapunov functional of [HL2009], $\delta\mathbf{u}(t)+M\displaystyle\int_{D_0}^{D(t)} k(s)\mathrm{d}s$ to conclude that any heavy-tailed kernel in the sense that $\displaystyle\int k(s)\mathrm{d}s = \infty$ implies $D(t) \leqslant D_+ < \infty$. Thus, whenever $\beta < 1$, then corollary 4.2 applies with $\gamma = 0$ and $C_D = D_+$ and one recovers the exponential decay of mono-kinetic dynamics, [CS2007a, HT2008, HL2009, CFTV2010, Shv2021].

Proposition 5.1 (Flocking for mono-kinetic alignment, $p = 1$). *Let $(\rho, \mathbf{u})$ be a strong solution of the mono-kinetic alignment system (1.1) with "heavy-tailed" communication kernel ϕ of order $0 \leqslant \beta < 1$, (4.5). There is long time flocking behavior with decay rate*

$$\int_{\mathcal{S}(t)} |\mathbf{u}(t,\mathbf{x})-\overline{\mathbf{u}}|^2\rho(t,\mathbf{x})\mathrm{d}\mathbf{x} \leqslant C_R\, e^{-M_1 t}\int_{\mathcal{S}(t)} |\mathbf{u}_0(\mathbf{x})-\overline{\mathbf{u}}|^2\rho_0(\mathbf{x})\mathrm{d}\mathbf{x}, \quad M_1 = 2MC_kD_+^{-\beta}. \tag{5.4}$$

Moreover, integration of (5.3) implies pointwise bound on the decay of velocity fluctuation

$$\max_{\mathbf{x}} |\mathbf{u}(t,\mathbf{x})-\overline{\mathbf{u}}| \leqslant C_R\, e^{-M_1 t}\max_{\mathbf{x}} |\mathbf{u}_0(\mathbf{x})-\overline{\mathbf{u}}|. \tag{5.5}$$

5.2. **Flocking with p-alignment ($p > 1$).** Our starting point is the pointwise bound of velocity fluctuations corresponding to (5.3), which is outlined in appendix B.2

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\mathbf{u}(t) \leqslant -\frac{1}{2}Mk(D(t))(\delta\mathbf{u}(t))^{2p-1}, \qquad \delta\mathbf{u}(t) = \sup_{\mathbf{x}\in\mathcal{S}(t)} |\mathbf{u}(\mathbf{x},t)-\overline{\mathbf{u}}|. \tag{5.6}$$

In particular, $\delta\mathbf{u}(t) \leqslant \delta\mathbf{u}_0$ implies $D(t) \leqslant D_0 + 2\delta\mathbf{u}_0 \cdot t$, that is, (4.6) holds with $\gamma = 1$,

$$D(t) \leqslant C_D(1+t), \qquad C_D = \max\{D_0, 2\delta\mathbf{u}_0\},$$

and corollary 4.2 implies L^2_ρ-decay rate of order $\dfrac{1-\beta}{p-1}$.

Proposition 5.2 (**Flocking for mono-kinetic alignment,** $p>1$)**.** *Let* $(\rho,\mathbf{u})$ *be a strong solution of the mono-kinetic* p*-alignment system* (2.2) *with "heavy-tailed" communication kernel* ϕ *of order* $0\leqslant\beta<1$, (4.5)*. Then there is long time flocking behavior with decay rate*

$$\delta\mathscr{E}(t)\leqslant C_R M_p(1+t)^{-\frac{1-\beta}{p-1}}, \qquad p>1, \quad 0\leqslant\beta<1. \tag{5.7}$$

We can improve these bounds, at least in the restricted range $1<p<3/2$. To this end, use an iterative argument starting with the γ-bound

$$D(t)\leqslant C_D(1+t)^{\gamma}.$$

Integrating (5.6) for $t\gtrsim R^{1/\gamma}$ where $k(D(t))\geqslant C_RC_kC_D^{-\beta}(1+t)^{-\beta\gamma}$ leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\delta\mathbf{u}(t)\big)^{2-2p}\geqslant C_R(p-1)M_1(1+t)^{-\beta\gamma}, \quad p>1,$$

where, as before, $M_1=MC_kC_D^{-\beta}$. We conclude with the flocking bound

$$\delta\mathbf{u}(t)\leqslant C_R\frac{1}{\big\{M_1(1+t)^{1-\beta\gamma}+(\delta\mathbf{u}_0)^{2-2p}\big\}^{\frac{1}{2p-2}}}\leqslant C_RM_1^{-\frac{1}{2p-2}}(1+t)^{-\frac{1-\beta\gamma}{2p-2}},$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}D(t)\leqslant 2\delta\mathbf{u}(t) \quad\rightsquigarrow\quad D(t)\leqslant D_0+C_R2\frac{2M_1^{-\frac{1}{2p-2}}}{\gamma'}(1+t)^{\gamma'}, \quad \gamma':=\frac{2p-3}{2p-2}+\frac{\beta\gamma}{2p-2}. \tag{5.8}$$

We distinguish between two cases. If $2p+\beta<3$ then after one iteration, starting with $\gamma=1$, we obtain

$$\gamma'=\frac{2p-3+\beta}{2p-2}<0.$$

If, however, $2p+\beta\geqslant 3$ and $\beta<1/2$ then $\frac{\beta}{2p-2}<1$ and, hence the fixed point iterations $\gamma\mapsto\gamma'$ form a contraction, approaching the negative value

$$\gamma_\infty=\frac{2p-3}{2p-2-\beta}<0, \qquad p<3/2.$$

In either case, the range $1<p<3/2$ and $\beta<1/2$ implies that after finitely many iterations, (5.8) holds with $\gamma<0$ and we conclude that the diameter $D(t)$ remains uniformly bounded in time, $D(t)\leqslant D_+$, that is, (4.6) holds with $\gamma=0$ and $C_D=D_+$. Corollary 4.2 implies the following refinement of proposition 5.2.

Proposition 5.3 (**Flocking for mono-kinetic alignment,** $1<p<3/2$)**.** *Let* $(\rho,\mathbf{u})$ *be a strong solution of the mono-kinetic* p*-alignment system* (2.2), $1<p<3/2$ *with "heavy-tailed" communication kernel* ϕ *of order* $0\leqslant\beta<1/2$, (4.5)*. Then there is long time flocking behavior with decay rate*

$$\delta\mathscr{E}(t)\leqslant C_RM_p(1+t)^{-\frac{1}{p-1}}, \qquad 1<p<3/2, \quad 0\leqslant\beta<1/2. \tag{5.9}$$

Thus, we have L^2_ρ-velocity fluctuations with optimal decay rate $\lesssim(1+t)^{-\frac{1}{2p-2}}$. Moreover, integration of (5.6) with $k(D(t))\geqslant C_Rk(D_+)$ implies uniform decay of velocity fluctuations at the same optimal rate

$$\max_{\mathbf{x}}|\mathbf{u}(t,\mathbf{x})-\overline{\mathbf{u}}|\leqslant C_RM_1^{-\frac{2-p}{2p-2}}(1+t)^{-\frac{1}{2p-2}}, \qquad 1<p<3/2. \tag{5.10}$$

5.3. **Agent-based description.** The hydrodynamic p-alignment with mono-kinetic closure is the continuum counterpart of the corresponding agent-based description (2.1). In particular, we have bounds on the velocity fluctuations — both the ℓ^2-energy fluctuations and uniform fluctuations, which are worked out in appendix B.3

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\mathscr{E}(t) \leqslant -2^{p-1}k(D(t))\left(\delta\mathscr{E}(t)\right)^p, \quad \delta\mathscr{E}(t) := \frac{1}{2N^2}\sum_{i,j=1}^N |\mathbf{v}_i(t)-\mathbf{v}_j(t)|^2 \tag{5.11a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\mathbf{v}(t) \leqslant -\frac{1}{2}k(D(t))\big(\delta\mathbf{v}(t)\big)^p, \quad \delta\mathbf{v}(t) := \max_i |\mathbf{v}_j(t)-\overline{\mathbf{v}}|, \ \overline{\mathbf{v}}(t) := \frac{1}{N}\sum_{j=1}^N \mathbf{v}_j(t). \tag{5.11b}$$

There is one-to-one correspondence between (5.11) and the hydrodynamic fluctuations bounds — the L^2_ρ-energy fluctuations (4.2) and uniform velocity fluctuations in (5.6).
When $p=1$, (5.11a) implies the exponential decay of heavy-tailed kernels. This should be contrasted with the case $p>1$, where the p-graph Laplacian in (2.1) implies polynomial decay. A typical scenario is summarized in the following proposition.

Proposition 5.4. *Consider the p-alignment system (2.1), with a "heavy-tailed" communication kernel of order $0 \leqslant \beta < 1$, (4.5). Then there is a uniform convergence towards the mean velocity*

$$\max|\mathbf{v}_i(t)-\overline{\mathbf{v}}| \leqslant \left\{\begin{array}{ll} C_R\, exp\big\{-C_k(1+t)^{(1-\beta)}\big\}\delta\mathscr{E}(0), & p=1, \\ \\ C_R(1+t)^{-\frac{1-\beta}{2p-2}} & p>1. \end{array}\right. \tag{5.12}$$

Remark 5.5 (**Finite-time alignment for $0 \leqslant p < 1$**). The dynamics of p-alignment with $p \geqslant 1$ is driven by gradient of velocities, $\mathbf{v}_j - \mathbf{v}_i$. For $0 \leqslant p < 1$, the dynamics emphasizes the *orientation* of velocities' gradient. The prototypical case is $p = 1/2$, in which case (2.1) reads

$$\left\{\begin{array}{l} \dfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}_i(t) = \mathbf{v}_i(t), \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_i(t) = \dfrac{1}{N}\displaystyle\sum_{j\neq i}^N \phi_{ij}(t)\frac{\mathbf{v}_j(t)-\mathbf{v}_i(t)}{|\mathbf{v}_j(t)-\mathbf{v}_i(t)|} \end{array}\right. \qquad i=1,2,\ldots N. \tag{5.13}$$

When $p=0$, $(5.13)_2$ reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_i(t) = \frac{1}{N}\sum_{j\neq i}^N \phi_{ij}(t)\frac{\mathbf{v}_j(t)-\mathbf{v}_i(t)}{|\mathbf{v}_j(t)-\mathbf{v}_i(t)|^2}, \quad i=1,2,\ldots N.$$

The balance of its energy fluctuations

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\mathscr{E}(t) = -\frac{1}{2N^2}\sum_{i,j=1}^N \phi(\mathbf{x}_i,\mathbf{x}_j) \leqslant -\frac{1}{2}k(D(t)) \ \ \rightsquigarrow \ \ \delta\mathscr{E}(t) \leqslant \delta\mathscr{E}_0 - \frac{1}{2}\int_0^t k(D(s))\mathrm{d}s,$$

proving that there is *finite-time alignment*, $\delta\mathscr{E}(t) \stackrel{t\to t_c}{\longrightarrow} 0$, for heavy-tailed kernels satisfying $\int\limits_0^t k(D(s))\mathrm{d}s \stackrel{t\to\infty}{\longrightarrow} \infty$. Finite-time alignment is typical for p-alignment in the singular range

$0 \leqslant p < 1$, [CCH2014, Theorem 2.2],

$$\delta\mathbf{v}(t) \leqslant \Big((\delta\mathbf{v}_0)^{1-p} - 2^{p-1}(1-p)\int_0^t k(D(s))\mathrm{d}s\Big)^{\frac{1}{1-p}}, \qquad 0 \leqslant p < 1.$$

In this context, at least for $0 \leqslant p \leqslant 1/2$, one encounters the need to avoid collisions,

$$|\mathbf{v}_i(t) - \mathbf{v}_j(t)| + |\mathbf{x}_i(t) - \mathbf{x}_j(t)| \neq 0, \qquad i \neq j,\ t < t_c.$$

Collision avoidance is discussed in [Mar2018] for $p \in (1/2, 3/2)$ and for the case of pure alignment, $p = 1$, with possibly singulars, $k(r) = r^{-\alpha}$, in [ACHL2021, Pes2014, CCH2014, CCMP2017].

5.4. **Flocking with matrix-valued communication kernel.** Consider the alignment dynamics

$$\partial_t(\rho\mathbf{u}) + \nabla_{\mathbf{x}}\cdot(\rho\mathbf{u}\otimes\mathbf{u}) = \int_{\mathbb{R}^d} \Phi(\mathbf{x},\mathbf{x}')(\mathbf{u}(t,\mathbf{x}') - \mathbf{u}(t,\mathbf{x}))\rho(t,\mathbf{x})\rho(t,\mathbf{x}')\,\mathrm{d}\mathbf{x}', \tag{5.14a}$$

driven by a bounded symmetric *matrix* communication kernel, $\Phi(\mathbf{x},\mathbf{x}') = \Phi(\mathbf{x}',\mathbf{x}) \in \mathbb{R}^{d\times d}$, of order $\beta \geqslant 0$

$$C_k(1 + |\mathbf{x}-\mathbf{x}'|)^{-\beta}\mathbb{I}_{d\times d} \leqslant \Phi(\mathbf{x},\mathbf{x}') \leqslant \phi_+\mathbb{I}_{d\times d}. \tag{5.14b}$$

In this case, the coupling of $\mathbf{u}$-components defies a maximum principle of $\delta\mathbf{u}(t)$ encoded in (5.3). Instead, we will show below the bound $\delta\mathbf{u}(t) \lesssim (1+t)^{1/2}$. This implies $D(t) \lesssim (1+t)^{3/2}$ and hence flocking holds for heavy-tailed kernels of order $\beta < 2/3$. To this end, we follow our argument in the discrete setup, [ST2021, Proposition 3.1], starting with the alignment dynamics

$$\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} = \int \Phi(\mathbf{x},\mathbf{x}')(\mathbf{u}'-\mathbf{u})\rho'\,\mathrm{d}\mathbf{x}',$$

which implies the *local* energy balance

$$\partial_t\frac{|\mathbf{u}|^2}{2} + \mathbf{u}\cdot\nabla_{\mathbf{x}}\frac{|\mathbf{u}|^2}{2} = \int \big\langle \mathbf{u}, \Phi(\mathbf{x},\mathbf{x}')(\mathbf{u}'-\mathbf{u})\big\rangle\rho'\,\mathrm{d}\mathbf{x}'. \tag{5.15}$$

The integrand on the right is decomposed by polarization (suppressing time dependence)

$$\begin{aligned}
&\big\langle \mathbf{u}(\mathbf{x}), \Phi(\mathbf{x},\mathbf{x}')\big(\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x})\big)\big\rangle \\
&\qquad\equiv -\frac{1}{2}\big\langle(\mathbf{u}'-\mathbf{u}), \Phi(\mathbf{x},\mathbf{x}')(\mathbf{u}'-\mathbf{u})\big\rangle - \frac{1}{2}\big\langle\mathbf{u}, \Phi(\mathbf{x},\mathbf{x}')\mathbf{u}\big\rangle + \frac{1}{2}\big\langle\mathbf{u}', \Phi(\mathbf{x},\mathbf{x}')\mathbf{u}'\big\rangle \\
&\qquad\leqslant -C_k(1+|\mathbf{x}-\mathbf{x}'|)^{-\beta}\frac{|\mathbf{u}|^2}{2} + \phi_+\frac{|\mathbf{u}'|^2}{2}, \qquad \Phi(\mathbf{x},\mathbf{x}') \leqslant \phi_+\mathbb{I}_{d\times d}.
\end{aligned}$$

In the last step we used the assumed bound on Φ having a heavy-tail of order β and satisfying a pointwise upper-bound ϕ_+. Returning to (5.15) while noting that $\displaystyle\int |\mathbf{u}'|^2\rho'\,\mathrm{d}\mathbf{x}' \leqslant C_0^2 = 2\int \rho_0 E_0$, it follows that

$$\partial_t|\mathbf{u}|^2 + \mathbf{u}\cdot\nabla_{\mathbf{x}}|\mathbf{u}|^2 \leqslant -C_k(1+D(t))^{-\beta}M|\mathbf{u}|^2 + \phi_+C_0^2. \tag{5.16}$$

By the maximum principle (— we ignore the dissipative term on the right),

$$|\mathbf{u}(t,\cdot)|^2 \leqslant \max|\mathbf{u}_0|^2 + C't, \qquad C' := \phi_+C_0^2,$$

and hence (4.6) holds with $\gamma = 3/2$, in view of

$$\frac{\mathrm{d}}{\mathrm{d}t}D(t) \leqslant 2\max|\mathbf{u}(t,\cdot)| \quad \rightsquigarrow \quad D(t) \leqslant D_0 + \frac{4}{3C'}\big(\max|\mathbf{u}_0|^2 + C't\big)^{3/2}. \tag{5.17}$$

We can now use a bootstrap argument: starting with $\gamma = 3/2$ we insert the bound $D(t) \lesssim (1+t)^\gamma$ of (5.17) into the right side of (5.16) and we have the maximum bound

$$|\mathbf{u}(t,\cdot)|^2 \leqslant \max|\mathbf{u}_0|^2 + C'(1+t)^{\beta\gamma} \ \rightsquigarrow \ D(t) \leqslant D_0 + \frac{2}{C'\gamma'}\big(\max|\mathbf{u}_0|^2 + C'(1+t)\big)^{\gamma'}, \quad \gamma' = 1 + \beta\gamma/2.$$

Iterating, $\gamma \mapsto \gamma'$, we end up with a fixed point $\gamma = \frac{2}{2-\beta}$, and with the improved bounds, still in the range of $\beta < 2/3$,

$$|\mathbf{u}(t)| \lesssim (1+t)^{\frac{\beta}{2-\beta}}, \qquad D(t) \leqslant C_D(1+t)^{\frac{2}{2-\beta}}, \quad \beta < 2/3.$$

Corollary 4.2 implies the following.

Proposition 5.6 (Flocking for matrix-based alignment). *Let $(\rho, \mathbf{u})$ be a strong solution of the hydrodynamic alignment (5.14) with "heavy-tailed" matrix communication kernel Φ of order $\beta < 2/3$. There is long time flocking behavior with fractional exponential decay rate*

$$\delta\mathscr{E}(t) \leqslant C_R\, exp\big\{-M_1 t^{\frac{2-3\beta}{2-\beta}}\big\}\delta\mathscr{E}(0). \tag{5.18}$$

6. Flocking of hydrodynamic p-alignment with entropic pressure

We consider hydrodynamic alignment (2.2) driven by the class of *singular kernels* $k_p(r) := r^{-(d+2sp)}, 0 < s < 1,\ p \geqslant 1$,

$$\partial_t(\rho\mathbf{u}) + \nabla_{\mathbf{x}}(\rho\mathbf{u}\otimes\mathbf{u} + \mathbb{P}) = p.v.\int_{\mathcal{S}(t)} \frac{|\mathbf{u}'-\mathbf{u}|^{2p-2}\big(\mathbf{u}(t,\mathbf{x}') - \mathbf{u}(t,\mathbf{x})\big)}{|\mathbf{x}'-\mathbf{x}|^{d+2sp}}\rho(t,\mathbf{x})\rho(t,\mathbf{x}')\,\mathrm{d}\mathbf{x}', \quad 0 < s < 1.$$

We emphasize that in this case of strongly singular kernels, there is no formal justification for the passage from the agent-based description (2.1) to the hydrodynamic description. In particular, the near-origin integrability sought in (4.5) is given up for the usual notion of singular integration in terms of principle value ($p.v.$). The alignment term on the right amounts to a *weighted* fractional $2p$-Laplacian, $(-\Delta)^s_{2p}$, which is properly interpreted to act on supp $\rho(t,\cdot)$; see [TGCV2021, BV2015] and the references therein.

The tail of the singular kernel, $k_p(r) = r^{-(d+2sp)}, r \gg R$, is too thin to enforce the heavy tail condition sought in corollary 4.2. Accordingly, we keep the singular 'head' and adjust it with the "heavy tail" or order β

$$\phi_{s,\beta}(\mathbf{x},\mathbf{x}') \begin{cases} = |\mathbf{x}-\mathbf{x}'|^{-(d+2sp)}, & |\mathbf{x}-\mathbf{x}'| \leqslant R \ \text{ with } \ 0 < s < 1 \\ \geqslant C_k(1+|\mathbf{x}-\mathbf{x}'|)^{-\beta}, & |\mathbf{x}-\mathbf{x}'| > R \end{cases} \tag{6.1}$$

Clearly, there exists a constant, $K = K_R$, such that $k_p(|\mathbf{x}-\mathbf{x}'|) \leqslant K_R\phi_{s,\beta}(\mathbf{x},\mathbf{x}')$ for all $(\mathbf{x},\mathbf{x}')$. Without loss of generality, we may assume that the spatial scale R is large enough, $(1+R)^\beta R^{-(d+2sp)} < C_k$, so that we may take $K_R = 1$,

$$k_p(|\mathbf{x}-\mathbf{x}'|) \leqslant \phi_{s,\beta}(\mathbf{x},\mathbf{x}'), \qquad \forall \mathbf{x},\mathbf{x}' \in \mathbb{R}^d. \tag{6.2}$$

We refer to such heavy-tailed, singular kernels as having order (s,β). If we let ϕ_β denote its tail of order β then the p-alignment dynamics now reads

$$\begin{aligned}\partial_t(\rho\mathbf{u}) &+ \nabla_{\mathbf{x}}\cdot(\rho\mathbf{u}\otimes\mathbf{u}+\mathbb{P})\\ &= p.v.\int\limits_{|\mathbf{x}'-\mathbf{x}|\leqslant R}\frac{|\mathbf{u}'-\mathbf{u}|^{2p-2}\big(\mathbf{u}'-\mathbf{u}'\big)}{|\mathbf{x}'-\mathbf{x}|^{d+2sp}}\rho\rho'\,\mathrm{d}\mathbf{x}'\\ &+\int\limits_{|\mathbf{x}'-\mathbf{x}|>R}\phi_\beta(\mathbf{x},\mathbf{x}')|\mathbf{u}'-\mathbf{u}|^{2p-2}\big(\mathbf{u}'-\mathbf{u}'\big)\rho\rho'\,\mathrm{d}\mathbf{x}',\quad \phi_\beta(\mathbf{x},\mathbf{x}')\geqslant C_k(1+|\mathbf{x}-\mathbf{x}'|)^{-\beta}.\end{aligned} \tag{6.3}$$

Remark 6.1 (**Entropic pressure with singular kernel**). In case of singular kernel $\phi_{s,\beta}$, we need to adjust the definition 2.2 of entropic pressure,

$$\partial_t(\rho e_{\mathbb{P}}) + \nabla_{\mathbf{x}}\cdot(\rho e_{\mathbb{P}}\mathbf{u}+\mathbf{q}) + \mathrm{trace}(\mathbb{P}\nabla\mathbf{u}) \leqslant -\frac{1}{2}k_p(D(t))\int\limits_{\mathcal{S}(t)}\big((2e_{\mathbb{P}})^p+(2e'_{\mathbb{P}})^p\big)\rho\rho'\,\mathrm{d}\mathbf{x}'. \tag{6.4}$$

Thus, the entropic part of the internal energy avoids the singularity of $\phi_{s,\beta}$ and emphasizes only its tail behavior. It leads to the 'adjusted' energy fluctuations bound

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\mathscr{E}(t)\leqslant -\frac{1}{2}\iint\limits_{\mathcal{S}(t)\times\mathcal{S}(t)}\big\{\phi_{s,\beta}(\mathbf{x},\mathbf{x}')|\mathbf{u}-\mathbf{u}'|^{2p}+k_p(D(t))\big((2e_{\mathbb{P}})^p+(2e_{\mathbb{P}})^p\big)\big\}\,\rho\rho'\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}', \tag{6.5}$$

which in turn, arguing along the lines of (4.3), yields (4.2); that is, the main theorem 4.1 and its corollary 4.2 survive. In particular, the enstrophy bound (3.6) holds for $\phi=\phi_{s,\beta}$. Taking into account (6.2), $\phi_{s,\beta}(\mathbf{x},\mathbf{x}')\geqslant|\mathbf{x}'-\mathbf{x}|^{-(d+2sp)}$, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\mathscr{E}(t)\leqslant -\frac{1}{2}\iint\limits_{\mathcal{S}(t)\times\mathcal{S}(t)}\frac{|\mathbf{u}(t,\mathbf{x}')-\mathbf{u}(t,\mathbf{x})|^{2p}}{|\mathbf{x}'-\mathbf{x}|^{d+2sp}}\rho\rho'\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}'. \tag{6.6}$$

The presence of pressure, let alone a pressure with an 'unknown' closure, couples the different components of velocity in a manner that defies a straightforward derivation of a uniform bound on velocity fluctuations, $\delta\mathbf{u}(t)$, along the lines of what we have done in the mono-kinetic case. Instead, we introduce a new strategy for verifying flocking in this case, in which we use an enstrophy bound associated with the singular kernel, $k_p(r)=r^{-(d+2sp)}$, in order to control the diameter $D(t)\lesssim(1+t)^\gamma$. This enables us to treat the flocking in presence of entropic pressure. The remarkable aspect here is that although the presence of pressure defies a maximum principle on the velocity field, the corresponding enstrophy bound associated with (6.3) will suffice for control of velocity fluctuations and hence flocking will follow. Thus, short-term interactions governed by kernel with a *singular head* secure the spread of velocity fluctuations, while *heavy-tailed* kernel governing the long-term interactions secure flocking.

6.1. **Enstrophy and dispersion bounds.** Throughout this section we make the following assumptions.

- (H*1*) The alignment hydrodynamics (1.1a),(6.3) admits a strong entropic solution, (6.4).
- (H*2*) The support, $\mathcal{S}(0)=\mathrm{supp}\,\rho(0,\cdot)$, has a smooth boundary satisfying a Lipschitz or a cone condition.

(H*3*) The dynamics remains uniformly bounded away from vacuum, namely — there exists $\rho_- > 0$ such that

$$\min_{\mathbf{x}\in\mathcal{S}(t)} \rho(t,\mathbf{x}) \geqslant \rho_- > 0, \quad t \geqslant 0.$$

Several comments regarding these assumptions are in order. The literature about the question of global regularity, (H*1*), is devoted mostly to mono-kinetic "pressure-less" closure; we mention the one-dimensional studies [TT2014, CCTT2016, HT2017, ST2017a, ST2017b, ST2020b, Tan2021, LS2022], the two-dimensional case [HT2017] and multi-dimensional cases [Shv2019, DMPW2019, CTT2021, Tad2022b]. Much less is known about alignment with pressure, typically when (scalar) pressure is augmented with additional process of relaxation and/or dissipation, [Cho2019, CDS2020, TCGW2020]. On the other hand, there are relatively few works on weak solutions of (1.1), [CCR2011, CFGS2017, LT2021]. As for (H*2*), we are aware of only few results on the geometric structures that emerge from alignment, [LS2019, LLST2022]. The question of uniform bound away from vacuum assume in (H*3*) plays an important role in driving global regularity [Tan2020, Shv2021, AC2021a, Tad2021]. It can be relaxed to allow mild time decay, e.g., $\rho_-(t) \gtrsim (1+t)^{-1/2}$, [ST2020b, Theorem 1.1], [Tad2021, Theorem 3], but as already noted in previous works, some sort of non-vacuous assumption is necessary.

We begin by noting that since $\phi_{s,\beta}$ dominates $k_p(r)$, (6.2), then by the non-vacuous hypothesis (H*3*), $\rho \geqslant \rho_- > 0$, we have the Sobolev bound

$$\iint\limits_{\mathcal{S}(t)\times\mathcal{S}(t)} \frac{|\mathbf{u}(t,\mathbf{x}')-\mathbf{u}(t,\mathbf{x})|^{2p}}{|\mathbf{x}'-\mathbf{x}|^{d+2sp}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}' \leqslant C_\rho^2 \iint\limits_{\mathcal{S}(t)\times\mathcal{S}(t)} \frac{|\mathbf{u}(t,\mathbf{x}')-\mathbf{u}(t,\mathbf{x})|^{2p}}{|\mathbf{x}'-\mathbf{x}|^{d+2sp}}\rho\rho'\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}', \quad C_\rho := \frac{1}{\rho_-}.$$

The space-time enstrophy bound (3.6), or more precisely — its singular version in (6.6), then yields

$$\int_0^t \|\mathbf{u}(\tau,\cdot)\|^{2p}_{\dot{W}^{s,2p}(\mathcal{S})}\mathrm{d}\tau \leqslant C_\rho^2 C_0^2, \quad \|\mathbf{u}(t,\cdot)\|^{2p}_{\dot{W}^{s,2p}(\mathcal{S})} := \iint\limits_{\mathcal{S}(t)\times\mathcal{S}(t)} \frac{|\mathbf{u}(t,\mathbf{x}')-\mathbf{u}(t,\mathbf{x})|^{2p}}{|\mathbf{x}'-\mathbf{x}|^{d+2sp}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}'. \tag{6.7}$$

The enstrophy bound (6.7) guarantees that the velocity $\mathbf{u}$ slows down the dispersion of the crowd so that its diameter $D(t)$ may not grow faster than $\lesssim (1+t)^\gamma$. Below we derive sharp bounds on the dispersion rate γ.
To this end, we note that propagation along particles paths in $(1.1a)_1$ yields, as in (5.1),

$$\frac{\mathrm{d}}{\mathrm{d}t}D(t) \leqslant \delta\mathbf{u}(t), \quad \delta\mathbf{u}(t) = \max_{\mathbf{x},\mathbf{x}'\in\mathcal{S}(t)} |\mathbf{u}(t,\mathbf{x}')-\mathbf{u}(t,\mathbf{x})|.$$

By Gagliardo-Nirenberg inequality (which we recall in appendix C below),

$$|\mathbf{u}(t,\mathbf{x})-\mathbf{u}(t,\mathbf{x}')| \leqslant C_s\|\mathbf{u}\|_{\dot{W}^{s,2p}(\mathcal{S}(t))}|\mathbf{x}-\mathbf{x}'|^{s-\theta}, \qquad \mathbf{x},\mathbf{x}'\in\mathcal{S}(t), \quad \theta := \frac{d}{2p} < s < 1. \tag{6.8}$$

This yields, $\dfrac{\mathrm{d}}{\mathrm{d}t}D(t) \leqslant \delta\mathbf{u}(t)| \leqslant C_s\|\mathbf{u}(t,\cdot)\|_{\dot{W}^{s,2p}(\mathcal{S}(t))}D^{s-\theta}(t)$, or

$$\frac{\mathrm{d}}{\mathrm{d}t}D^{1+\theta-s}(t) \leqslant C'_s\|\mathbf{u}(t,\cdot)\|_{\dot{W}^{s,2p}(\mathcal{S}(t))}, \qquad C'_s = (1+\theta-s)C_s, \tag{6.9}$$

and hence, in view of (6.7),

$$D^{1+\theta-s}(t) \leqslant D_0^{1+\theta-s} + \Big(\int_0^t \|\mathbf{u}(\tau,\cdot)\|^{2p}_{\dot{W}^{s,2p}(\mathcal{S}(t))}\mathrm{d}\tau\Big)^{\frac{1}{2p}}\Big(\int_0^t 1\mathrm{d}\tau\Big)^{\frac{1}{(2p)'}} \leqslant D_0^{1+\theta-s} + C'_s(C_\rho C_0)^{\frac{1}{p}}\, t^{\frac{1}{(2p)'}}.$$

We conclude that the crowd of multi-dimensional p-alignment dynamics, (6.3) can be dispersed at a rate no faster than

$$D(t) \leqslant C_D(1+t)^{\gamma_p}, \qquad \gamma_p = \frac{2p-1}{2p(1+\theta-s)}, \quad \theta = \frac{d}{2p} < s < 1. \tag{6.10}$$

This bound can be improved using a bootstrap argument outlined in appendix D. In particular, for $1 < p < 3/2$ we obtain a *uniform* dispersion bound which we summarize in the following key result.

Lemma 6.2 (Uniform dispersion bound for p-alignment with singular kernels). *Consider the multi-dimensional p-alignment dynamics, (6.3), $1 < p < 3/2$, with heavy-tailed, singular kernel of order (s,β), satisfying (H1)–(H3). Then we have a uniform bound*

$$D(t) \leqslant D_+, \qquad 0 \leqslant \beta < (3/2 - p)d, \quad 1 < p < 3/2. \tag{6.11}$$

Remark 6.3. Observe that since we require $d = 2p\theta < 3$, the uniform bound (6.11) is restricted to one- and two-dimensional cases.

We are unable to secure such a uniform dispersion bound for $p > 3/2$, but we can still improve the dispersion bound (6.10) as shown in remark D.2 below,

$$D(t) \leqslant C'_D(1+t)^{\gamma}, \qquad \gamma = \frac{2p\big(p - 3/2\big)}{(p-1)d - \beta}, \qquad 0 \leqslant \beta < \frac{d}{2p-1}, \quad p > 3/2.$$

6.2. **Flocking of alignment with pressure. The one-dimensional case.** The case of pure alignment $p = 1$, restricts the use of lemma 6.2 to the one-dimensional case ($d < 2p$), driven by singular kernel $k_1(r) = r^{-(1+2s)}$, $\frac{1}{2} < s < 1$, with β-tailed adjustment

$$\begin{aligned}\partial_t(\rho u) + \partial_x(\rho u^2 + \mathbb{P}) = p.v.\int_{|x'-x|\leqslant R} \frac{u(t,x') - u(t,x)}{|x-x'|^{1+2s}}\rho(t,x')\mathrm{d}x' \\ + \int_{|x'-x|>R} \phi_\beta(x,x')\big(u(t,x') - u(t,x)\big)\rho(t,x')\mathrm{d}x'.\end{aligned} \tag{6.12a}$$

The integrals on the right are restricted to the interval $\mathcal{S}(t) = [\rho_-(t), \rho_+(t)]$ supporting $\rho(t,\cdot)$, ϕ_β is a β-tailed communication kernel,

$$\phi_\beta(x,x') \geqslant C_k(1+|x-x'|)^{-\beta}, \qquad |x-x'| \geqslant R, \tag{6.12b}$$

and $\mathbb{P}$ is any scalar entropic pressure satisfying (1.2), or more precisely — its singular version (6.4),

$$\partial_t(\rho\mathbb{P}) + \partial_x(\rho\mathbb{P}u + q) + 2\mathbb{P}\partial_x u \leqslant -2\mathbb{P}D^{-(1+2s)}(t)M. \tag{6.12c}$$

By 6.10 we can apply corollary (4.2) with $\gamma_1 = \frac{1}{3-2s}$ which yields the following.

Theorem 6.4 (**One-dimensional alignment,** $p=1$)**.** *Consider the one-dimensional alignment dynamics* (6.12) *and assume (H1),(H3), hold. Let* $(\rho, u, \mathbb{P})$ *be a strong entrpoic solution with* β*-tailed singular kernel,* ϕ_β*, satisfying the heavy-tail condition*

$$\beta + 2s < 3, \qquad \beta \geqslant 0, \quad \frac{1}{2} < s < 1. \tag{6.13}$$

Then there is a large time flocking behavior with fractional exponential rate

$$\delta\mathscr{E}(t) \leqslant C_R\, exp\Big\{ -2MC_k(1+t)^{\frac{3-2s-\beta}{3-2s}}\Big\}\delta\mathscr{E}(0). \tag{6.14}$$

This extends the mono-kinetic 'pressure-less' studies in [ST2017a, ST2017b, ST2018a, DKRT2018, DMPW2019, MMPZ2019]. It is instructive to compare this result with flocking statement in the mono-kinetic closure, which is based on the uniform bound on velocity, $D(t) \lesssim (1+t)$. Theorem 6.4 allows for a *larger* class of heavy-tailed kernels since it is based on a sharper bound on the velocity fluctuations, leading to $D(t) \lesssim (1+t)^\gamma$ with $\gamma < 1$. This result can be further improved by extending the uniform dispersion bound in lemma 6.2 to the limiting case $p=1$.

6.3. **Flocking of p-alignment with pressure. The multi-dimensional case.** We consider the p-alignment dynamics (6.3) driven by singular kernel $k_p(r) = r^{-(d+2sp)}$, $\frac{d}{2p} < s < 1$. Using (6.10) we can apply corollary 4.2 with $\gamma = \gamma_p$ which yields the following.

Theorem 6.5 (**Multi-dimensional alignment,** $p>1$)**.** *Consider the multi-dimensional* p*-alignment dynamics* (6.3) *and assume (H1)–(H3) hold. Let* $(\rho, \mathbf{u}, \mathbb{P})$ *be a strong entrpoic solution,* (6.4)*, with a* β*-tailed singular kernel* $\phi_{s,\beta}$*, satisfying the heavy-tail condition*

$$\beta\gamma_p < 1, \qquad \beta \geqslant 0, \quad \gamma_p := \frac{2p-1}{2p(1+\theta-s)}, \quad \theta = \frac{d}{2p} < s < 1.$$

Then there is a large time flocking behavior with polynomial decay rate of order

$$\delta\mathscr{E}(t) \leqslant C_R M_p\, t^{-\frac{1-\beta\gamma_p}{p-1}}. \tag{6.15}$$

Remark 6.6 (**Decay of internal fluctuations**). A sufficient condition for the heavy-tailed restriction $\beta\gamma_p < 1$ sought in (6.15) is given by

$$\beta \leqslant \frac{d}{2p-1} \quad \rightsquigarrow \quad \beta\gamma_p < \beta\frac{2p-1}{d} \leqslant 1. \tag{6.16}$$

It still allows heavy-tails of order $\beta \geqslant 1$, compared with the $\beta < 1$ restriction in the mono-kinetic closure. In particular, when $\beta = \frac{d}{2p-1}$ one finds the decay of order

$$\big(\delta\mathscr{E}(t)\big)^{p-1} \lesssim t^{-(1-\beta\gamma_p)} \lesssim t^{-\frac{1-s}{1+\theta-s}}.$$

Remark 6.7. Theorem 6.5 implies the decay of both — the macroscopic velocity fluctuations $\displaystyle\int |\mathbf{u}-\overline{\mathbf{u}}|^2\rho\,\mathrm{d}\mathbf{x}$ and, in the context of kinetic formulation, the microscopic fluctuations $\displaystyle\iint |\mathbf{v}-\mathbf{u}|^2 f_N\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{x}$.

The decay bound (6.15) is not sharp, a reflection of the fact that the dispersion bound (6.10) can be improved with smaller γ_p (as noted in remark 6.3). In particular, when p is in the restricted range $1 < p < 3/2$, then corollary 4.2 applies with $\gamma = 0$ and $C_D = D_+$ which yields the following.

Theorem 6.8 (**Multi-dimensional alignment** $1 < p < 3/2$)**.** *Consider the multi-dimensional p-alignment dynamics* (6.3)*, $1 < p < 3/2$ and assume (H1)–(H3) hold. Let $(\rho, \mathbf{u}, \mathbb{P})$ be a strong entrpoic solution,* (6.4)*, with a heavy-tailed singular kernel of order (s, β). Then there is a large time flocking behavior with polynomial decay rate of order*

$$\delta\mathscr{E}(t) \leqslant C_R M_p (1+t)^{-\frac{1}{p-1}}, \qquad 0 \leqslant \beta < (3/2 - p)d, \quad \frac{d}{2p} < s < 1, \quad 1 < p < 3/2. \tag{6.17}$$

Theorem 6.8 is the analogue of the mono-kinetic "pressure-less" case in proposition 5.3. In particular, it is rather remarkable that we obtain the same optimal decay rate of order $\frac{1}{p-1}$ in the respective range $1 < p < 3/2$ for the one- and two-dimensional cases. An optimal flocking scenario with a uniform dispersion bound remains open for $d \geqslant 3$.

Appendix A. Derivation of entropic inequality in p-alignment

A.1. **From agent-based to hydrodynamic description.** We begin with the passage from the agent-based dynamics of p-alignment (2.1) to its hydrodynamic description (2.2). The large crowd dynamics is encoded in terms of their empirical distribution $f_N(t, \mathbf{x}, \mathbf{v}) := \displaystyle\frac{1}{N}\sum_{i=1}^{N} \delta_{\mathbf{x}_i(t)}(\mathbf{x}) \otimes \delta_{\mathbf{v}_i(t)}(\mathbf{v})$, which are governed by the kinetic Valsov equation in state variables $(t, \mathbf{x}, \mathbf{v}) \in \mathbb{R}_t \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\partial_t f_N + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_N + \nabla_{\mathbf{v}} \cdot Q_p(f_N, f_N) = 0, \tag{A.1}$$

and driven by interaction kernel

$$Q_p(f_N, f_N)(t, \mathbf{x}, \mathbf{v}) := \int\limits_{\mathcal{S}(t)} \int\limits_{\mathbb{R}^d} \phi(\mathbf{x}, \mathbf{x}') |\mathbf{v}' - \mathbf{v}|^{2p-2} (\mathbf{v}' - \mathbf{v}) f_N f_N' \, \mathrm{d}\mathbf{v}' \, \mathrm{d}\mathbf{x}'.$$

We distinguish between the cases of 'pure' alignment, $Q_1 = Q$, and enhanced p-alignment Q_p of order $p > 1$.

For $p = 1$, the large crowd dynamics of f_N's is captured by their first two moments which we assume to exist — the density $\rho := \displaystyle\lim_{N\to\infty} \int_{\mathbb{R}^d} f_N(t, \mathbf{x}, \mathbf{v}) \, \mathrm{d}\mathbf{v}$ and momentum $\rho\mathbf{u} := \displaystyle\lim_{N\to\infty} \int_{\mathbb{R}^d} \mathbf{v} f_N(t, \mathbf{x}, \mathbf{v}) \, \mathrm{d}\mathbf{v}$; that is,

$$\rho(\mathbf{v}' - \mathbf{u}) = \lim_{N\to\infty} \int\limits_{\mathbb{R}^d} (\mathbf{v}' - \mathbf{v}) f_N(t, \mathbf{x}, \mathbf{v}) \, \mathrm{d}\mathbf{v} \quad \text{for all} \quad \mathbf{v}' \in \mathbb{R}^d. \tag{A.2}$$

Integration of (A.1) yields the mass equation $(1.1a)_1$

$$\partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho\mathbf{u}) = 0. \tag{A.3a}$$

The first $\mathbf{v}$-moment of (A.1) yields

$$\partial_t \int_{\mathbb{R}^d} \mathbf{v} f_N \,\mathrm{d}\mathbf{v} = -\nabla_{\mathbf{x}} \cdot \int_{\mathbb{R}^d} \mathbf{v}\otimes\mathbf{v} f_N \,\mathrm{d}\mathbf{v} + \int_{\mathbb{R}^d} Q_1(f_N, f_N)\,\mathrm{d}\mathbf{v}.$$

We now treat the two terms on the right. For the first term, we decompose $\mathbf{v}\otimes\mathbf{v} \equiv -\mathbf{u}\otimes\mathbf{u} + (\mathbf{v}\otimes\mathbf{u}+\mathbf{u}\otimes\mathbf{v}) + (\mathbf{v}-\mathbf{u})\otimes(\mathbf{v}-\mathbf{u})$, where the corresponding first two moments of f_N add up to $\mathbf{u}\otimes(\rho\mathbf{u}) = \rho\mathbf{u}\otimes\mathbf{u}$, while the third yields the pressure tensor (1.6),

$$\lim_{N\to\infty}\int_{\mathbb{R}^d} \mathbf{v}\otimes\mathbf{v} f_N \,\mathrm{d}\mathbf{v} = \rho\mathbf{u}\otimes\mathbf{u} + \mathbb{P}, \qquad \mathbb{P} = \lim_{N\to\infty}\int_{\mathbb{R}^d} (\mathbf{v}-\mathbf{u})\otimes(\mathbf{v}-\mathbf{u}) f_N.$$

The second term on the right yields

$$\lim_{N\to\infty}\int_{\mathbb{R}^d} Q_1(f_N,f_N)\,\mathrm{d}\mathbf{v} = \int_{\mathcal{S}(t)} \phi(\mathbf{x},\mathbf{x}')\left(\rho'\mathbf{u}'\rho - \rho\mathbf{u}\rho'\right)\,\mathrm{d}\mathbf{x}' = \mathbf{A}(\rho,\mathbf{u}),$$

and we recover the momentum equation $(1.1\text{a})_2$

$$\partial_t(\rho\mathbf{u}) + \nabla_{\mathbf{x}}\cdot(\rho\mathbf{u}\otimes\mathbf{u} + \mathbb{P}) = \mathbf{A}(\rho,\mathbf{u}).$$

For $p>1$, we *assume* the existence of the corresponding higher moments (which are compatible with the mono-kinetic Maxwellian (1.9)),

$$\rho|\mathbf{v}'-\mathbf{u}|^{2p-2}(\mathbf{v}'-\mathbf{u}) := \lim_{N\to\infty}\int_{\mathbb{R}^d} |\mathbf{v}'-\mathbf{v}|^{2p-2}(\mathbf{v}'-\mathbf{v}) f_N(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v},$$

in which case the interaction kernel yields,

$$\lim_{N\to\infty}\int_{\mathbb{R}^d} Q_p(f_N,f_N)\,\mathrm{d}\mathbf{v} = \int_{\mathcal{S}(t)} \phi(\mathbf{x},\mathbf{x}')|\mathbf{u}'-\mathbf{u}|^{2p-2}\left(\rho'\mathbf{u}'\rho - \rho\mathbf{u}\rho'\right)\,\mathrm{d}\mathbf{x}' = \mathbf{A}_p(\rho,\mathbf{u}),$$

and we recover the momentum equation $(2.2\text{a})_2$

$$\partial_t(\rho\mathbf{u}) + \nabla_{\mathbf{x}}\cdot(\rho\mathbf{u}\otimes\mathbf{u} + \mathbb{P}) = \mathbf{A}_p(\rho,\mathbf{u}). \tag{A.3b}$$

In fact, we are not restricted here by the mono-kinetic closure assumption: for any kinetic closure we have

$$\int_{\mathcal{S}(t)}\int_{\mathbb{R}^d} Q_p(f_N,f_N)(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{x}$$
$$= \iint_{\mathcal{S}(t)\times\mathcal{S}(t)}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \phi(\mathbf{x},\mathbf{x}')|\mathbf{v}'-\mathbf{v}|^{2p-2}(\mathbf{v}'-\mathbf{v}) f_N' f_N\,\mathrm{d}\mathbf{v}'\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} = 0.$$

This follows by the anti-symmetry of the integrand on the right, and hence the zero-average condition for p-alignment sought in remark 2.1 holds,

$$\int_{\mathcal{S}(t)} \mathbf{A}_p(\rho,\mathbf{u})(t,\mathbf{x})\,\mathrm{d}\mathbf{x} = \lim_{N\to\infty}\int_{\mathcal{S}(t)}\int_{\mathbb{R}^d} Q_p(f_N,f_N)(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{x} = 0. \tag{A.4}$$

Observe that system (A.3) is not a purely hydrodynamic description, since the pressure, $\mathbb{P}$, still requires a closure of the second-order moments of f_N. Thus, the alignment dynamics in

(A.3) is left open at the mesoscale, subject to the notion of entropic pressure in definition 1.1 for $p=1$ and definition (2.2) for $p>1$.

A.2. **Entropic pressure in kinetic formulation of p-alignment.** We follow the balance of the internal energy balance as preparation for studying the large-time behavior of 'pure' hydrodynamic alignment, $p=1$, in (1.1) and hydrodynamic p-alignment, $p>1$, in (2.2). The total energy is given by the second moment which is assumed to exist

$$\rho E(t,\mathbf{x}) = \lim_{N\to\infty}\int\limits_{\mathbb{R}^d}\frac{1}{2}|\mathbf{v}|^2 f_N(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v}.$$

It is decomposed into kinetic and internal energy corresponding to the decomposition $\frac{1}{2}|\mathbf{v}|^2\equiv \frac{1}{2}|\mathbf{u}|^2+\frac{1}{2}|\mathbf{v}-\mathbf{u}|^2+\mathbf{u}\cdot(\mathbf{v}-\mathbf{u})$. Noting that $\int_{\mathbb{R}^d}(\mathbf{v}-\mathbf{u})f_N\,\mathrm{d}\mathbf{v}=0$, we find

$$\rho E = \frac{\rho}{2}|\mathbf{u}|^2+\rho e_{\mathbb{P}}, \qquad \rho e_{\mathbb{P}} := \lim_{N\to\infty}\frac{1}{2}\int\limits_{\mathbb{R}^d}|\mathbf{v}-\mathbf{u}|^2 f_N\,\mathrm{d}\mathbf{v}.$$

The balance of internal energy, $\rho e_{\mathbb{P}}$, is obtained by integrating (A.1) against $\dfrac{|\mathbf{v}-\mathbf{u}|^2}{2}$, which yields

$$\partial_t(\rho e_{\mathbb{P}})+\int\limits_{\mathbb{R}^d}\frac{|\mathbf{v}-\mathbf{u}|^2}{2}\mathbf{v}\cdot\nabla_{\mathbf{x}}f_N\,\mathrm{d}\mathbf{v} = \int\limits_{\mathbb{R}^d}(\mathbf{v}-\mathbf{u})\cdot Q_p(f_N,f_N)\,\mathrm{d}\mathbf{v}. \tag{A.5}$$

The integral on the left can be expressed as a perfect divergence of the cubic moments $\mathbf{q}_N := \frac{1}{2}\int|\mathbf{v}-\mathbf{u}|^2(\mathbf{v}-\mathbf{u})f_N\,\mathrm{d}\mathbf{v}$ (all integrals are taken over $\mathbb{R}^d$)

$$\begin{aligned}
\int\frac{|\mathbf{v}-\mathbf{u}|^2}{2}&\mathbf{v}\cdot\nabla_{\mathbf{x}}f_N\,\mathrm{d}\mathbf{v}\\
&= \nabla_{\mathbf{x}}\cdot\int\frac{|\mathbf{v}-\mathbf{u}|^2}{2}\mathbf{v}f_N\,\mathrm{d}\mathbf{v}-\int\mathbf{v}\cdot\nabla_{\mathbf{x}}\frac{|\mathbf{v}-\mathbf{u}|^2}{2}f_N\,\mathrm{d}\mathbf{v}\\
&= \nabla_{\mathbf{x}}\cdot\int\frac{|\mathbf{v}-\mathbf{u}|^2}{2}\big(\mathbf{u}+(\mathbf{v}-\mathbf{u})\big)f_N\,\mathrm{d}\mathbf{v}+\sum_{i,j}\int v_j(v_i-u_i)\frac{\partial u_i}{\partial x_j}f_N\,\mathrm{d}\mathbf{v}\\
&= \nabla_{\mathbf{x}}\cdot\Big(\int\frac{|\mathbf{v}-\mathbf{u}|^2}{2}f_N\,\mathrm{d}\mathbf{v}\,\mathbf{u}+\mathbf{q}_N\Big)+\sum_{i,j}\int(v_j-u_j)(v_i-u_i)f_N\,\mathrm{d}\mathbf{v}\frac{\partial u_i}{\partial x_j}.
\end{aligned}$$

Taking the limit we find the term $\nabla_{\mathbf{x}}\cdot(\rho e_{\mathbb{P}}\mathbf{u}+\mathbf{q}_h)+\mathrm{trace}(\mathbb{P}\nabla\mathbf{u})$, with heat-flux,

$$\mathbf{q}_h := \lim_{N\to\infty}\frac{1}{2}\int\limits_{\mathbb{R}^d}|\mathbf{v}-\mathbf{u}|^2(\mathbf{v}-\mathbf{u})f_N\,\mathrm{d}\mathbf{v}, \tag{A.6}$$

and (A.5) yields

$$\partial_t(\rho e_{\mathbb{P}})+\nabla_{\mathbf{x}}\cdot(\rho e_{\mathbb{P}}\mathbf{u}+\mathbf{q}_h)+\mathrm{trace}(\mathbb{P}\nabla\mathbf{u}) = \lim_{N\to\infty}\int\limits_{\mathbb{R}^d}(\mathbf{v}-\mathbf{u})\cdot Q_p(f_N,f_N)\,\mathrm{d}\mathbf{v}. \tag{A.7}$$

It remains to consider the moment of the alignment-based term on the right. We distinguish between the cases $p=1$ and $p>1$.
The case $p=1$. We split $\mathbf{v}-\mathbf{v}'\equiv(\mathbf{v}-\mathbf{u})+(\mathbf{u}-\mathbf{v}')$,

$$
\begin{aligned}
\int_{\mathbb{R}^d}(\mathbf{v}-\mathbf{u})\cdot Q(f_N,f_N)\,\mathrm{d}\mathbf{v} &= -\int_{\mathcal{S}(t)}\phi(\mathbf{x},\mathbf{x}')\iint_{\mathbb{R}^d\times\mathbb{R}^d}(\mathbf{v}-\mathbf{u})\cdot(\mathbf{v}-\mathbf{v}')f_Nf'_N\,\mathrm{d}\mathbf{v}'\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{x}'\\
&= -\int_{\mathcal{S}(t)}\phi(\mathbf{x},\mathbf{x}')\iint_{\mathbb{R}^d\times\mathbb{R}^d}|\mathbf{v}-\mathbf{u}|^2f_Nf'_N\,\mathrm{d}\mathbf{v}'\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{x}'\\
&\quad -\int_{\mathcal{S}(t)}\phi(\mathbf{x},\mathbf{x}')\int_{\mathbb{R}^d}(\mathbf{v}-\mathbf{u})f_N\,\mathrm{d}\mathbf{v}\cdot\int_{\mathbb{R}^d}(\mathbf{u}-\mathbf{v}')f'_N\,\mathrm{d}\mathbf{v}'\,\mathrm{d}\mathbf{x}'.
\end{aligned}\tag{A.8}
$$

The the first integral on the right ends up with $-2\displaystyle\int_{\mathcal{S}(t)}\phi(\mathbf{x},\mathbf{x}')e_{\mathbb{P}}\rho\rho'\,\mathrm{d}\mathbf{x}'$, and since the second integral on the right vanishes, (A.7) now reads [HT2008][7]

$$
\partial_t(\rho e_{\mathbb{P}})+\nabla_{\mathbf{x}}\cdot(\rho e_{\mathbb{P}}\mathbf{u}+\mathbf{q})+\mathrm{trace}\big(\mathbb{P}\nabla\mathbf{u}\big)=-2\int_{\mathcal{S}(t)}\phi(\mathbf{x},\mathbf{x}')e_{\mathbb{P}}\rho\rho'\,\mathrm{d}\mathbf{x}',\qquad \mathbf{q}=\mathbf{q}_h.\tag{A.9}
$$

Here we choose to interpret the equality (A.9) as a special case of entropic inequality (1.2), giving room to validate the formal passage to the limit in lieu of lack of formal closure.

The case $p>1$. We split $(\mathbf{v}-\mathbf{u})\cdot(\mathbf{v}-\mathbf{v}')\equiv\frac{1}{2}|\mathbf{v}'-\mathbf{v}|^2+\big(\frac{1}{2}(\mathbf{v}'+\mathbf{v})-\mathbf{u}\big)\cdot(\mathbf{v}'-\mathbf{v})$ to obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^d}(\mathbf{v}-\mathbf{u})\cdot Q_p(f_N,f_N)\,\mathrm{d}\mathbf{v}\\
&\quad= -\int_{\mathcal{S}(t)}\phi(\mathbf{x},\mathbf{x}')\iint_{\mathbb{R}^d\times\mathbb{R}^d}|\mathbf{v}'-\mathbf{v}|^{2p-2}(\mathbf{v}-\mathbf{u})\cdot(\mathbf{v}-\mathbf{v}')f_Nf'_N\,\mathrm{d}\mathbf{v}'\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{x}'\\
&\quad= -\frac{1}{2}\int_{\mathcal{S}(t)}\phi(\mathbf{x},\mathbf{x}')\iint_{\mathbb{R}^d\times\mathbb{R}^d}|\mathbf{v}'-\mathbf{v}|^{2p}f_Nf'_N\,\mathrm{d}\mathbf{v}'\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{x}'\\
&\qquad -\int_{\mathcal{S}(t)}\phi(\mathbf{x},\mathbf{x}')\iint_{\mathbb{R}^d\times\mathbb{R}^d}|\mathbf{v}'-\mathbf{v}|^{2p-2}\big(1/2(\mathbf{v}'+\mathbf{v})-\mathbf{u}\big)\cdot(\mathbf{v}'-\mathbf{v})f_Nf'_N\,\mathrm{d}\mathbf{v}'\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{x}'\\
&\quad:=\mathcal{I}_1+\mathcal{I}_2
\end{aligned}\tag{A.10}
$$

[7]This corrects a series of typos in our statement of [HT2008, Lemma 5.1]

The the internal integrand in the first term on the right of (A.10) does not exceed

$$
\begin{aligned}
-\frac{1}{2}&\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}|\mathbf{v}'-\mathbf{v}|^{2p} f_N f_N' \,\mathrm{d}\mathbf{v}'\,\mathrm{d}\mathbf{v}\\
&\leqslant -\frac{1}{2}\Big(\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}|\mathbf{v}'-\mathbf{v}|^2 f_N f_N'\,\mathrm{d}\mathbf{v}'\,\mathrm{d}\mathbf{v}\Big)^p\Big(\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} f_N f_N'\,\mathrm{d}\mathbf{v}'\,\mathrm{d}\mathbf{v}\Big)^{-\frac{p}{p'}}\\
&\leqslant -\frac{1}{2}\Big(\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}|\mathbf{v}'-\mathbf{u}'|^2 f_N f_N'\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{v}' + \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}|\mathbf{v}-\mathbf{u}|^2 f_N f_N'\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{v}'\Big)^p(\rho\rho')^{-\frac{p}{p'}}\\
&= -\frac{1}{2}(2\rho\rho' e_{\mathbb{P}} + 2\rho\rho' e_{\mathbb{P}}')^p(\rho\rho')^{-\frac{p}{p'}}\\
&\leqslant -\frac{1}{2}\big((2e_{\mathbb{P}})^p+(2e_{\mathbb{P}}')^p\big)\rho\rho'.
\end{aligned}
$$

The first passage on the right follows from Hölder inequality, the second follows from polarization $\mathbf{v}'-\mathbf{v}\equiv(\mathbf{v}'-\mathbf{u}')+(\mathbf{u}'-\mathbf{u})+(\mathbf{u}-\mathbf{v})$ and the last from Jensen's inequality. Hence

$$
\mathcal{I}_1 \leqslant -\frac{1}{2}\int\limits_{\mathcal{S}(t)}\phi(\mathbf{x},\mathbf{x}')\big((2e_{\mathbb{P}})^p+(2e_{\mathbb{P}}')^p\big)\rho\rho'. \tag{A.11}
$$

For the second term on the right of (A.10) we claim that it can be written as a complete divergence

$$
\mathcal{I}_2(\mathbf{x}) = \nabla_{\mathbf{x}}\cdot\mathbf{q}_\phi, \tag{A.12}
$$

Indeed, by anti-symmetry $(\mathbf{x},\mathbf{v})\leftrightarrow(\mathbf{x}',\mathbf{v}')$ the term $\mathcal{I}_2(\mathbf{x})$ has zero mean,

$$
\int\limits_{\mathcal{S}(t)}\mathcal{I}_2(\mathbf{x})\,\mathrm{d}\mathbf{x} = \iint\limits_{\mathcal{S}(t)\times\mathcal{S}(t)}\phi(\mathbf{x},\mathbf{x}')\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}|\mathbf{v}'-\mathbf{v}|^{2p-2}\big({}^1\!/\!{}_2(\mathbf{v}'+\mathbf{v})-\mathbf{u}\big)\cdot(\mathbf{v}'-\mathbf{v})f_N f_N'\,\mathrm{d}\mathbf{v}'\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}' = 0.
$$

Hence, there exists a solution , $\Delta\psi = \mathcal{I}_2(\mathbf{x})$, $\mathbf{x}\in\mathcal{S}(t)$ subject to Neumann boundary condition, $\frac{\partial\psi}{\partial\mathbf{n}}_{|\partial\mathcal{S}(t)}=0$, and (A.12) follows with $\mathbf{q}_\phi=\nabla\psi$. Combining (A.7) with (A.11) and (A.12) we arrive at the entrpoic inequality (2.3)

$$
\begin{aligned}
&\partial_t(\rho e_{\mathbb{P}}) + \nabla_{\mathbf{x}}\cdot(\rho e_{\mathbb{P}}\mathbf{u}+\mathbf{q}) + \mathbb{P}\nabla\mathbf{u}\\
&\qquad\leqslant -\frac{1}{2}\int\limits_{\mathcal{S}(t)}\phi(\mathbf{x},\mathbf{x}')\big((2e_{\mathbb{P}})^p+(2e_{\mathbb{P}}')^p\big)\rho\rho'\,\mathrm{d}\mathbf{x}', \quad \mathbf{q}:=\mathbf{q}_h+\mathbf{q}_\phi.
\end{aligned} \tag{A.13}
$$

Observe that while the entropic inequality (A.9) in the case $p=1$ was a matter of choice, the corresponding inequality (A.13) for $p>1$ is a matter of necessity in order to make a macroscopic interpretation.

Appendix B. Pointwise bounds on velocity fluctuations

B.1. **Pointwise fluctuations in mono-kinetic alignment.** Arguing along the lines of [HT2017, §1], we first fix an arbitrary unit vector $\mathbf{w}\in\mathbb{R}^d$ and project (5.2) onto the space spanned by $\mathbf{w}$ to get

$$(\partial_t + \mathbf{u}\cdot\nabla_{\mathbf{x}})\langle \mathbf{u}(t,\mathbf{x}),\mathbf{w}\rangle = \int_{\mathcal{S}(t)} \phi(\mathbf{x},\mathbf{x}')(\langle \mathbf{u}(t,\mathbf{x}'),\mathbf{w}\rangle - \langle \mathbf{u}(t,\mathbf{x}),\mathbf{w}\rangle)\rho(t,\mathbf{x}')\,\mathrm{d}\mathbf{x}'$$

Now we assume that $\langle \mathbf{u}(t,\mathbf{x}),\mathbf{w}\rangle$ reaches maximum and minimum values at $\mathbf{x}_+ = \mathbf{x}_+(t)$ and, respectively, $\mathbf{x}_- = \mathbf{x}_-(t)$,

$$\begin{aligned} u_+(t) &= \langle \mathbf{u}(t,\mathbf{x}_+(t)),\mathbf{w}\rangle := \sup_{\mathbf{x}\in\mathcal{S}(t)} \langle \mathbf{u}(t,\mathbf{x}),\mathbf{w}\rangle \\ u_-(t) &= \langle \mathbf{u}_-(t,\mathbf{x}_+(t)),\mathbf{w}\rangle := \inf_{\mathbf{x}\in\mathcal{S}(t)} \langle \mathbf{u}(t,\mathbf{x}),\mathbf{w}\rangle. \end{aligned}$$

To simplify notations, we temporarily suppress the $\mathbf{w}$-dependence, $u_\pm(t) = u_\pm(t;\mathbf{w})$. We abbreviate $\overline{u}(t) := \dfrac{1}{M}\displaystyle\int \rho\langle \mathbf{u}(t,\mathbf{x}'),\mathbf{w}\rangle\,\mathrm{d}\mathbf{x}'$. Since $\langle \mathbf{u}(t,\mathbf{x}'),\mathbf{w}\rangle \leqslant \langle \mathbf{u}(t,\mathbf{x}_+),\mathbf{w}\rangle$ and, by assumption, $\phi(\mathbf{x}_+,\mathbf{x}')) \geqslant k(D(t))$, we find

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}u_+(t) &= \int_{\mathcal{S}(t)} \phi(\mathbf{x}_+,\mathbf{x}')\big(\langle \mathbf{u}(t,\mathbf{x}'),\mathbf{w}\rangle - \langle \mathbf{u}(t,\mathbf{x}_+),\mathbf{w}\rangle\big)\rho(t,\mathbf{x}')\,\mathrm{d}\mathbf{x}' \\ &\leqslant k(D(t))\int_{\mathcal{S}(t)} \big(\langle \mathbf{u}(t,\mathbf{x}'),\mathbf{w}\rangle - u_+(t)\big)\rho(t,\mathbf{x}')\,\mathrm{d}\mathbf{x}' \\ &= k(D(t))M\big((\overline{u}(t) - u_+(t)\big). \end{aligned} \tag{B.1}$$

Similarly, we bound $u_-(t) := \inf_{\mathbf{x}\in\mathcal{S}}\langle \mathbf{u}(t,\mathbf{x}),\mathbf{w}\rangle$ obtaining

$$\frac{\mathrm{d}}{\mathrm{d}t}u_-(t) \geqslant k(D(t))M\big(\overline{u} - u_-(t)\big).$$

The difference of the last two bounds yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(u_+(t) - u_-(t)\big) \leqslant -k(D(t))M\big(u_+(t) - u_-(t)\big),$$

and since $\delta\mathbf{u}(t) = \sup_{\mathbf{x},\mathbf{x}'\in\mathcal{S}(t)} |\mathbf{u}(t,\mathbf{x}) - \mathbf{u}(t,\mathbf{x}')| = \sup_{|\mathbf{w}|=1}\big(u_+(t;\mathbf{w}) - u_-(t;\mathbf{w})\big)$ is the diameter of velocities projected on arbitrary unit vectors $\mathbf{w}$ we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\mathbf{u}(t) \leqslant -k(D(t))M\delta\mathbf{u}(t). \tag{B.2}$$

B.2. **Pointwise fluctuations in mono-kinetic p-alignment($p \geqslant 1$).** We extend the pointwise bound (B.2) for the general p-alignment, $p \geqslant 1$. By Galilean invariance we may assume $\mathbf{m}_0 = 0$, in which case (5.6) is simplified to the uniform bound

$$\frac{\mathrm{d}}{\mathrm{d}t}u_+(t) \leqslant -\frac{1}{2}Mk(D(t))(u_+(t))^{2p-1}, \qquad u_+(t) = \sup_{\mathbf{x}\in\mathcal{S}(t)} |\mathbf{u}(t,\mathbf{x})|. \tag{B.3}$$

Indeed, if we let $\mathbf{u}_+(t)=\mathbf{u}(t,\mathbf{x}_+(t))$ with maximal speed $u_+(t)=|\mathbf{u}_+(t)|$ along particle path $\dot{\mathbf{x}}_+(t)=\mathbf{u}(t,\mathbf{x}_+(t))$, we then find,[8]

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_+(t)=\int\limits_{\mathcal{S}(t)}\phi(\mathbf{x},\mathbf{x}')|\mathbf{u}'-\mathbf{u}|^{2p-2}(\mathbf{u}'-\mathbf{u}_+)\rho'\,\mathrm{d}\mathbf{x}'.$$

By polarization, $\mathbf{u}_+=\frac{1}{2}(\mathbf{u}_+-\mathbf{u}')+\frac{1}{2}(\mathbf{u}_++\mathbf{u}')$, we find

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}_+(t)|^2&=-\frac{1}{2}\int\limits_{\mathcal{S}(t)}\phi(\mathbf{x},\mathbf{x}')|\mathbf{u}'-\mathbf{u}_+|^{2p}\rho'\,\mathrm{d}\mathbf{x}'+\frac{1}{2}\int\limits_{\mathcal{S}(t)}\phi(\mathbf{x},\mathbf{x}')|\mathbf{u}'-\mathbf{u}_+|^{2p-2}(|\mathbf{u}'|^2-|\mathbf{u}_+|^2)\rho'\,\mathrm{d}\mathbf{x}'\\ &\leqslant-\frac{1}{2}\int\limits_{\mathcal{S}(t)}\phi(\mathbf{x},\mathbf{x}')|\mathbf{u}'-\mathbf{u}_+|^{2p}\rho'\,\mathrm{d}\mathbf{x}'\leqslant-\frac{1}{2}k(D(t))M^{-\frac{p}{p'}}\Big(\int\limits_{\mathcal{S}(t)}|\mathbf{u}'-\mathbf{u}_+|^2\rho'\,\mathrm{d}\mathbf{x}'\Big)^p\\ &\leqslant-\frac{1}{2}k(D(t))M^pM^{-\frac{p}{p'}}|\mathbf{u}_+(t)|^{2p},\end{aligned}$$

and (B.3) follows. The first inequality on the right follows from the fact that $|\mathbf{u}_+|$ is the maximal speed; the second from Hölder inequality and in the last step we use $\displaystyle\int\mathbf{u}'\rho'\,\mathrm{d}\mathbf{x}'=0$.

B.3. **Fluctuations in agent-based description.** Consider the discrete p-alignment model (2.1) and consider the energy fluctuations

$$\delta\mathscr{E}(t):=\frac{1}{2N^2}\sum_{i,j=1}^{N}|\mathbf{v}_i(t)-\mathbf{v}_j(t)|^2.$$

A straightforward computation yields (5.11a)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\delta\mathscr{E}(t)&=\frac{1}{N^2}\sum_{i,j=1}^{N}\phi_{ij}(t)|\mathbf{v}_j(t)-\mathbf{v}_i(t)|^{2p-2}\big\langle\mathbf{v}_j(t)-\mathbf{v}_i(t),\mathbf{v}_i(t)\big\rangle\\ &=-\frac{1}{2N^2}\sum_{i,j=1}^{N}\phi_{ij}(t)|\mathbf{v}_j(t)-\mathbf{v}_i(t)|^{2p}\leqslant-\frac{1}{2}\left(\frac{1}{N^2}\sum_{i,j=1}^{N}\phi_{ij}^{1/p}(t)|\mathbf{v}_j(t)-\mathbf{v}_i(t)|^2\right)^p\\ &\leqslant-2^{p-1}k(D(t))\left(\delta\mathscr{E}(t)\right)^p.\end{aligned}$$

The first equality follows since $\sum\mathbf{v}_i(t)$ is conserved in time, and the second follows from summation by parts while taking into account the assumed symmetry, $\phi_{ij}=\phi_{ji}$; next follows the Hölder inequality (for $p>1$) and finally we use the lower bound $\phi(\mathbf{x}_i(t),\mathbf{x}_j(t))\geqslant k(D(t))$. Similarly, we consider the uniform fluctuations

$$\delta\mathbf{v}(t):=\max_i|\mathbf{v}_i(t)-\overline{\mathbf{v}}|.$$

[8]the precise argument involves Rademacher lemma, see [Shv2021, Lemma 3.5].

We assume without loss of generality $\overline{\mathbf{v}}_0 = 0 \ \leadsto \overline{\mathbf{v}}(t) \equiv 0$ and it remains to bound the maximal value $\mathbf{v}_+(t) = \text{argmax}_{\mathbf{v}_i}|\mathbf{v}_i|$. Writing $\mathbf{v}_+ = \frac{1}{2}(\mathbf{v}_+ - \mathbf{v}_j) + \frac{1}{2}(\mathbf{v}_+ + \mathbf{v}_j)$ we find

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{v}_+(t)|^2 &= \frac{1}{2N}\sum_j \phi_{ij}|\mathbf{v}_+ - \mathbf{v}_j|^{2p-2}\langle \mathbf{v}_+ - \mathbf{v}_j, \mathbf{v}_j - \mathbf{v}_+\rangle + \frac{1}{2N}\sum_j \phi_{ij}|\mathbf{v}_+ - \mathbf{v}_j|^{2p-2}\big(|\mathbf{v}_j|^2 - |\mathbf{v}_+|^2\big)\rangle \\ &\leqslant -\frac{1}{2N}\sum_j \phi_{ij}|\mathbf{v}_+ - \mathbf{v}_j|^{2p} \leqslant -\frac{1}{2N}k(D(t))N^{-\frac{p}{p'}}\Big(\sum_j |\mathbf{v}_+ - \mathbf{v}_j|^2\Big)^p \\ &\leqslant -\frac{1}{2N}k(D(t))N^{-\frac{p}{p'}}N^p|\mathbf{v}_+|^{2p},\end{aligned}$$

and (5.11b) follows.

Appendix C. From enstrophy bound to Hölder regularity

For completeness, we recall here the arguments which lead to the Gagliardo-Nirenberg-Sobolev inequality, stating that for $\mathbf{u} \in W^{s,2p}(\mathcal{S})$ with 'nice' boundary satisfying (H2), we have Hölder continuity of order $s - \frac{d}{2p}$,

$$|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')| \leqslant C_s\|\mathbf{u}\|_{\dot{W}^{s,2p}(\mathcal{S})}|\mathbf{x} - \mathbf{x}'|^{s-\frac{d}{2p}}, \qquad \mathbf{x}, \mathbf{x}' \in \mathcal{S}, \quad \frac{d}{2p} < s < 1. \tag{C.1}$$

We follow [DPV2012, theorem 8.2]. As a first step we note that thanks to hypothesis (H2), $\mathbf{u}$ can be extended to $\widetilde{\mathbf{u}}$ defined over over $\mathbb{R}^d$ with comparable $W^{s,2p}$-norm, $\|\widetilde{\mathbf{u}}\|_{W^{s,2p}(\mathbb{R}^d)} \lesssim \|\mathbf{u}\|_{W^{s,2p}(\mathcal{S})}$, [DPV2012, Theorem 5.4]. We continue with the extension, $\widetilde{\mathbf{u}}$. Set $R = |\mathbf{x} - \mathbf{x}'|$, $\mathbf{x}, \mathbf{x}' \in \mathcal{S}$ and let $\langle\widetilde{\mathbf{u}}\rangle_{B_{2R}(\mathbf{z})}$ denote the average over the ball B_{2R} centered at $\mathbf{z}$,

$$\langle\widetilde{\mathbf{u}}\rangle_{B_{2R}(\mathbf{z})} := \frac{1}{|B_{2R}(\mathbf{z})|}\int\limits_{\mathbf{z}'\in B_{2R}(\mathbf{z})} \widetilde{\mathbf{u}}(\mathbf{z}')\mathrm{d}\mathbf{z}'.$$

Fix $\mathbf{x}''$ as an intermediate point in the intersection of the two balls, $B_{2R}(\mathbf{x}) \cap B_{2R}(\mathbf{x}')$ and split

$$\begin{aligned}|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')| \leqslant |\widetilde{\mathbf{u}}(\mathbf{x}) - \langle\widetilde{\mathbf{u}}\rangle_{B_{2R}(\mathbf{x})}| &+ |\langle\widetilde{\mathbf{u}}\rangle_{B_{2R}(\mathbf{x})} - \widetilde{\mathbf{u}}(\mathbf{x}'')| \\ &+ |\widetilde{\mathbf{u}}(\mathbf{x}'') - \langle\widetilde{\mathbf{u}}\rangle_{B_{2R}(\mathbf{x}')}| + |\langle\widetilde{\mathbf{u}}\rangle_{B_{2R}(\mathbf{x}')} - \widetilde{\mathbf{u}}(\mathbf{x}')|.\end{aligned} \tag{C.2}$$

By Hölder inequality, for every $\mathbf{w} \in B_{2R}(\mathbf{z})$, there holds

$$\begin{aligned}&|\langle\widetilde{\mathbf{u}}\rangle_{B_{2R}(\mathbf{z})} - \widetilde{\mathbf{u}}(\mathbf{w})| \\ &\leqslant \frac{1}{|B_{2R}(\mathbf{z})|}\int\limits_{\mathbf{z}'\in B_{2R}(\mathbf{z})} |\widetilde{\mathbf{u}}(\mathbf{w}) - \widetilde{\mathbf{u}}(\mathbf{z}')|\mathrm{d}\mathbf{z}' \leqslant C_d\|\widetilde{\mathbf{u}}\|_{W^{s,2p}(B_{2R}(\mathbf{z}))}R^{s-\frac{d}{2p}}, \quad \mathbf{w} \in B_{2R}(\mathbf{z}).\end{aligned} \tag{C.3}$$

and (C.1) follows from proper application of (C.3) to each of the terms on the right of (C.2). We close by noting that in the special 1D case, (C.1) is reduced to the inequalities of Ladyzheskaya [MRR2013] or Agmon's [Agm2010, Lemma 13.2],

$$\max_{\mathbf{x}\in\mathcal{S}} |\mathbf{u}(t, \mathbf{x})| \lesssim \|\mathbf{u}\|_{L^2(\Omega)}^{1-\frac{1}{2s}} \times \|\mathbf{u}\|_{H^s(\mathcal{S})}^{\frac{1}{2s}}, \quad {}^1\!/_2 < s < 1.$$

Appendix D. A uniform dispersion bound

Proof of lemma 6.2 (**The range** $1 < p < 3/2$). We consider the multi-dimensional p-alignment, $p > 1$, driven by heavy-tailed singular kernel of order (s, β). Assume that we have the dispersion bound

$$D(t) \leqslant C_D(1+t)^{\gamma}, \qquad \theta = \frac{d}{2p} < s < 1.$$

By (6.10) this holds with $\gamma = \gamma_p = \dfrac{2p-1}{2p(1+\theta-s)}$. We will improve this bound by a bootstrap argument. To this end we recall that corollary 4.2 implies the decay (suffice to consider $t \geqslant 1$)

$$\delta\mathscr{E}(t) \leqslant C_1(1+t)^{-\frac{1-\beta\gamma}{p-1}}, \qquad t \geqslant 1, \qquad C_1 = 2^{\frac{p-1}{1-\beta\gamma}} C_R M_p.$$

Here and below, we use the different constants $C_1, C_2, \ldots$ to trace our calculations.
For the range of β assumed in (6.11), $\beta < (3/2-p)d,\ 1 < p < 3/2$, we have[9] $2p-1 < \dfrac{1-\beta\gamma_p}{p-1}$. Fix μ such that

$$2p-1 < \mu < \frac{1-\beta\gamma_p}{p-1}.$$

The energy fluctuations bound, e.g., (6.6), yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big((1+t)^{\mu}\delta\mathscr{E}(t)\Big) &= (1+t)^{\mu}\frac{\mathrm{d}}{\mathrm{d}t}\delta\mathscr{E}(t) + \mu(1+t)^{\mu-1}\delta\mathscr{E}(t)\\ &\leqslant -\frac{\rho_-^2}{2}(1+t)^{\mu}\|\mathbf{u}(t,\cdot)\|^{2p}_{\dot{W}^{s,2p}(\mathcal{S}(t))} + C_1\mu(1+t)^{\mu'-1}, \qquad \mu' := \mu - \frac{1-\beta\gamma}{p-1} < 0,\end{aligned}$$

and hence the weighted enstrophy bound

$$\int_0^t (1+\tau)^{\mu}\|\mathbf{u}(\tau,\cdot)\|^{2p}_{W^{s,2p}}\mathrm{d}\tau \leqslant 2C_\rho^2\delta\mathscr{E}(0) + C_2, \qquad C_2 = 2C_\rho^2 C_1\mu\frac{1}{|\mu'|}. \tag{D.1}$$

We now revisit (6.9), integrating $\dfrac{\mathrm{d}}{\mathrm{d}t}D^{1+\theta-s}(t) \leqslant C'_s\|\mathbf{u}(t,\cdot)\|_{\dot{W}^{s,2p}(\mathcal{S}(t))}$ with a weighted Hölder inequality,

$$D^{1+\theta-s}(t) \leqslant D_0^{1+\theta-s} + C'_s\Big(\int_0^t (1+\tau)^{\mu}\|\mathbf{u}(\tau,\cdot)\|^{2p}_{\dot{W}^{s,2p}(\mathcal{S}(t))}\mathrm{d}\tau\Big)^{\frac{1}{2p}}\Big(\int_0^t (1+\tau)^{-\frac{\mu}{2p}(2p)'}\Big)^{\frac{1}{(2p)'}}.$$

Using (D.1) and the fact that $\dfrac{\mu}{2p-1} > 1$ we end up with the uniform bound

$$D(t) \leqslant D_+ = \big(D_0^{1+\theta-s} + C_3\big)^{\frac{1}{1+\theta-s}}, \qquad C_3 = C'_s\big(2C_\rho^2\delta\mathscr{E}(0) + C_2\big)^{\frac{1}{2p}}\Big(\frac{1}{\frac{\mu}{2p-1}-1}\Big)^{\frac{1}{(2p)'}}$$

□

[9] In fact, the precise bound enables a slightly larger range $\beta < \frac{p(3-2p)}{2p-1}d < (3/2-p)d$ but we prefer to keep it simple with the latter.

Remark D.1 (**The case** $p = 1$). It should be possible to extend the uniform dispersion bound of lemma Remark 6.2 to the limiting case of 'pure' alignment, $p = 1$. To this end one should use a proper 'exponential multiplier', instead of $(1+t)^\mu$ used above for $1 < p < 3/2$.

Remark D.2 (**The case** $p > 3/2$). When $p > 3/2$ we are unable to secure a uniform dispersion bound as in lemma 6.2, but we can still improve the dispersion rate, γ_p, using a more refined bootstrap argument. For this range of p's we have $\dfrac{1-\beta\gamma_p}{p-1} < 2p-1$. Fix μ such that $\dfrac{1-\beta\gamma_p}{p-1} < \mu < 2p-1$. In this case, $\mu' = \mu - \dfrac{1-\beta\gamma_p}{p-1}$ is positive and we have the corresponding enstrophy weighted bound

$$\int_0^t (1+\tau)^\mu \|\mathbf{u}(\tau,\cdot)\|^{2p}_{W^{s,2p}} \mathrm{d}\tau \leqslant 2C_\rho^2 \delta \mathscr{E}(0) + C_2(1+t)^{\mu'}, \qquad C_2 = \frac{2C_\rho^2 C_1 \mu}{\mu'} > 0. \tag{D.2}$$

As before, we revisit (6.9), integrating $\dfrac{\mathrm{d}}{\mathrm{d}t} D^{1+\theta-s}(t) \leqslant C'_s \|\mathbf{u}(t,\cdot)\|_{\dot{W}^{s,2p}(\mathcal{S}(t))}$ with a weighted Hölder inequality to find

$$\begin{aligned}
D^{1+\theta-s}(t) &\leqslant D_0^{1+\theta-s} + C'_s\Big(\int_0^t (1+\tau)^\mu \|\mathbf{u}(\tau,\cdot)\|^{2p}_{\dot{W}^{s,2p}(\mathcal{S}(t))}\mathrm{d}\tau\Big)^{\frac{1}{2p}} \Big(\int_0^t (1+\tau)^{-\frac{\mu}{2p}(2p)'}\Big)^{\frac{1}{(2p)'}} \\
&\leqslant D_0^{1+\theta-s} + C_3(1+t)^{\frac{\mu'}{2p}} \times (1+t)^{\left(1-\frac{\mu}{2p-1}\right)\frac{1}{(2p)'}} \\
&= D_0^{1+\theta-s} + C_3(1+t)^{\frac{1}{2p}\left((2p-1)-\frac{1-\beta\gamma}{p-1}\right)}, \qquad C_3 = C'_s C_2\Big(1-\frac{\mu}{2p-1}\Big)^{-\frac{1}{(2p)'}} > 0.
\end{aligned}$$

We conclude with a dispersion bound

$$D(t) \leqslant C'_D(1+t)^{\gamma'}, \qquad \gamma' := \frac{1}{2p(1+\theta-s)}\Big((2p-1) - \frac{1-\beta\gamma}{p-1}\Big) < \gamma_p. \tag{D.3}$$

Recall that the requirement $\beta\gamma_p < 1$ led to the β-restriction in (6.16), $\beta < \dfrac{d}{2p-1}$. Therefore,

$$\frac{\beta}{2p(1+\theta-s)(p-1)} < \frac{\beta}{d(p-1)} < \frac{1}{(2p-1)(p-1)} < 1, \qquad p > 3/2,$$

so that the fixed point iterations in (D.3), $\gamma \mapsto \gamma'$, contract towards a limiting value $\gamma = \gamma_*$

$$\gamma_* = \frac{p(2p-3)}{(p-1)2p(1+\theta-s) - \beta}, \qquad p > 3/2, \quad \beta < \frac{d}{2p-1}.$$

In particular, since $2p(1+\theta-s) > d$, then after finitely many iterations there holds

$$D(t) \leqslant C'_D(1+t)^{\gamma}, \quad \gamma = \frac{2p\big(p-3/2\big)}{(p-1)d-\beta}, \qquad \beta < \frac{d}{2p-1}, \quad p > 3/2, \tag{D.4}$$

which improves the bound of $\dfrac{2p-1}{d} > \gamma_p$.

References

[Agm2010] S. Agmon, Lectures on elliptic boundary value problems, (2010), Providence, RI: AMS.

[ACHL2021] S. Ahn, H. Choi, S.-Y. Ha, and H. Lee, On the collision avoiding initial-conguration to the Cucker-Smale type ocking models, Comm. Math. Sci., 10, (2012), 625-643.

[Alb2019] G. Albi, N. Bellomo, L. Fermo, S.-Y. Ha, J. Kim, L. Pareschi, D. Poyato, and J. Soler, Vehicular traffic, crowds, and swarms: From kinetic theory and multiscale methods to applications and research perspectives, Math. Models Methods Appl. Sci., 29(10) (2019) 1901-2005.

[AC2021a] D. Amadori and C Christoforou BV solutions for a hydrodynamic model of flocking–type with all-to-all interaction kernel ArXiv:2104.14573, (2021).

[Aok1982] I. Aoki, A simulation study on the schooling mechanism in fish, Bulletin of the Japan Society of Scientific Fisheries 48 (1982) 1081-1088.

[AC2021b] V. Arnaiz and A. Castro. Singularity formation for the fractional Euler-alignment system in 1D. Transactions of the AMS, 374(1) 2021 487-514.

[Bal2008] M. Ballerini, N. Cabibbo, R. Candelier, A. Cavagna, E. Cisbani, I. Giardina, V. Lecomte, A. Orlandi, G. Parisi, A. Procaccini, M. Viale, and V. Zdravkovic, Interaction ruling animal collective behavior depends on topological rather than metric distance: evidence from a field study, PNAS 105 (2008) 1232-1237.

[BDT2017/19] N. Bellomo, P. Degond and E. Tadmor (eds) "Active Particles", Advances in Theory, Models, and Applications, Vol 1 & 2. Modeling and Simulation in Science, Engin. Tech. Birkhuser (2017 & 2019).

[BCT2022] N. Bellomo, J. Carrillo and E. Tadmor (eds) "Active Particles", Advances in Theory, Models, and Applications, Vol 3, Modeling and Simulation in Science, Engin. Tech. Birkhuser (2022).

[BeN2005] E. Ben-Naim, Opinion dynamics: Rise and fall of political parties, Europhys. Lett., 69 (5) (2005) 671-677.

[Bia2012] W. Bialek, A. Cavagna, I. Giardina, T. Morad, E. Silvestri, M. Viale, and A. M. Walczake, Statistical mechanics for natural flocks of birds, *PNAS*, 109(13) (2012) 4786-4791.

[BH2009] T. Bühler and M. Hein, Spectral clustering based on the graph p-Laplacian in "Proceedings of the 26th annual international conference on machine learning", (2009), 81-88.

[BHT2009] V. D. Blondel, J. M. Hendricks, and J. N. Tsitsiklis, On Krauses multi-agent consensus model with state-dependent connectivity, IEEE Trans Automat. Control 54 (2009), 2586-2597.

[BV2015] M. Bonforte and J.- L. Vzquez, A priori estimates for fractional nonlinear degenerate diffusion equations on bounded domains, Arch. Rational Mech. Anal. 218 (2015) 317-362.

[CCR2011] José A. Cañizo, José A. Carrillo and Jesús Rosado, A well-posedness theory in measures for some kinetic models of collective motion M3AS 21(3) (2011) 515-539.

[CCH2014] J. A. Carrillo, Y.-P. Choi, M. Hauray, Local well-posedness of the generalized Cucker-Smale model with singular kernels. In MMCS, Mathematical modelling of complex systems, volume 47 of ESAIM Proc. Surveys (F. Abergel, M. Aiguier, D. Challet, P.-H. Cournéde, G. Faÿ & P. Lafitte , eds), pages 17-35 (2014).

[CCMP2017] J. A. Carrillo, Y.-P.Choi, P. B. Mucha, and J. Peszek, Sharp conditions to avoid collisions in singular CuckerSmale interactions, Nonlinear Anal. 37 (2017) 317-328.

[CCTT2016] J. A. Carrillo, Y.-P. Choi, E. Tadmor, and C. Tan, Critical thresholds in 1D Euler equations with non-local forces, Math. Models Methods Appl. Sci. 26(1) (2016) 185-206.

[CFGS2017] J. A. Carrillo, E. Feireisl, P. Gwiazda and A. Swierczewska-Gwiazda, Weak solutions for Euler systems with non-local interactions, J. London Math. Soc. 95(2) (2017) 705724.

[CFTV2010] J. A. Carrillo, M. Fornasier, G. Toscani, and F. Vecil, Particle, kinetic, and hydrodynamic models of swarming. in Naldi, G., Pareschi, L., Toscani, G. (eds.) Mathematical Modeling of Collective Behavior in Socio-Economic and Life Sciences, Series: Modelling and Simulation in Science and Technology, Birkhauser, (2010), 297-336.

[Cer2003] C. Cercignani The Boltzmann equation and fluid dynamics in "Handbook of Fluid Mechanics" v. II (S. Friedlander and D. Serre, eds), Elsevier, 2003, pp. 1-70.

[CTT2021] L. Chen, C. Tan and L. Tong. On the global classical solution to compressible Euler system with singular velocity alignment. M3AS 28(2) 2021 155-174.

[Cho2019] Y.-P. Choi. The global Cauchy problem for compressible Euler equations with a nonlocal dissipation. M3AS, 29(01) (2019) 185-207.

[CDMBC2007] Y.-l. Chuang, M. R. D'Orsogna, D. Marthaler, A. L. Bertozzi and L. S. Chayes State transitions and the continuum limit for a 2D interacting, self-propelled particle system *Physica D* 232 (2007) 33-47.

[CDS2020] P. Constantin, T. D Drivas, and R. Shvydkoy. Entropy hierarchies for equations of compressible fluids and self-organized dynamics. SIMA 52(3) (2020) 3073-3092.

[CF2003] I. Couzin and N. Franks *Self-organized lane formation and optimized traffic flow in army ants* Proc. R. Soc. Lond. B, **270** (2003) 139-146.

[CKFL2005] I.D. Couzin, J. Krause, N. R. Franks, and S. A. Levin, *Effective leadership and decision making in animal groups on the move* Nature **433**, (2005) 513-516.

[CS2007a] F. Cucker and S. Smale. Emergent behavior in flocks, IEEE Trans. Automat. Control, 52(5) (2007) 852-862.

[CS2007b] F. Cucker and S. Smale. On the mathematics of emergence. Japan. J. Math. 2(1):197-227, 2007.

[Daf2016] C. Dafermos, Hyperbolic Conservation Laws in Continuum Physics, Springer, 4th edition, 2016.

[DMPW2019] R. Danchin, P. B. Mucha, J. Peszek, and B. Wroblewski, Regular solutions to the fractional Euler alignment system in the Besov spaces framework, Math. Models and Methods in Appl. Sciences,, 29(1) (2019) 89-119.

[DPV2012] E. Di Nezza, G. Palatuccia and E. Valdinociac, Hitchhiker's guide to the fractional Sobolev spaces Bulletin des Sciences Mathématiques 136(5), 2012, 521-573

[DKRT2018] T. Do, A. Kiselev, L. Ryzhik, C. Tan, Global regularity for the fractional Euler alignment system, ARMA 228(1) (2018) 1-37.

[DTW2019] B. Düring, M. Torregross and M.-T. Wolfram *Boltzmann and FokkerPlanck equations modelling the Elo rating system with learning effects*, Journal of Nonlinear Science 29(3) (2019) 1095-1128.

[FK2019] A. Figalli and M.-J. Kang, A rigorous derivation from the kinetic CuckerSmale model to the pressureless Euler system with nonlocal alignment, Anal. PDE, 1293) (2019), 843-866.

[FZN2021] G. Fu, P. Zhao and Y. Bian, p-Laplacian Based Graph Neural Networks, arXiv:2111.07337 (2021).

[Fu2021] S. Fu, W. Liu, K. Zhang, Y. Zhou, D. Tao, Semi-supervised classification by graph p-laplacian convolutional networks, Information Sciences 560 (2021), 92-106.

[GWBL2012] A. Galante, S. Wisen, D. Bhaya and D. Levy, Modeling local interactions during the motion of cyanobacteria, J. of Theoretical Biology 309 (2012) 147-158.

[Gol1998] F. Golse From kinetic to macroscopic models Lecture Notes, Session L'Etat de la Recherche de la S.M.F, Université d'Orléans, 1998.

[HHK2010] S. Ha, T. Ha and J. Kim, "Emergent Behavior of a Cucker-Smale Type Particle Model With Nonlinear Velocity Couplings," in IEEE Transactions on Automatic Control, 55(7), (2010) 1679-1683.

[HL2009] S. Y Ha and J. G Liu, A simple proof of the Cucker-Smale flocking dynamics and mean-field limit, Communications in Mathematical Sciences 7(2):297–325, 2009.

[HT2008] S.-Y. Ha and E. Tadmor From particle to kinetic and hydrodynamic descriptions of flocking, Kinetic and Related Models, 1, no. 3, (2008), 415-435.

[HT2017] S. He and E. Tadmor, Global regularity of two-dimensional flocking hydrodynamics, Comptes rendus - Mathmatique Ser. I 355 (2017) 795-805.

[JJ2015] P.-E. Jabin and S. Junca, A continuous model for rating, SIAM J. Appl. Math, 75(2) (2015) 420-442.

[KV2015] M.-J. Kang and A. F. Vasseur, Asymptotic analysis of Vlasov-type equations under strong local alignment regime, Math. Models Methods Appl. Sci., 25(11) (2015) 2153-2173.

[Ka2011] Y. Katz, K. Tunstrm, C.C. Ioannou, C. Huepe and I.D. Couzin, Inferring the structure and dynamics of interactions in schooling fish, Proc. Natl. Acad. Sci. USA 108, (2011) 18720-18725.

[KMT2013] T.K. Karper, A Mellet, K Trivisa, Existence of weak solutions to kinetic flocking models, SIAM J. Math. Anal., 45 (2013) 215-243.

[KMT2015] T.K. Karper, A Mellet, K Trivisa, Hydrodynamic limit of the kinetic CuckerSmale flocking model, Mathematical Models and Methods in Applied Sciences 25 (1) (2015) 131-163.

[Kar2008] E. Karsenti, Self-organization in cell biology: a brief history Nature Reviews Molecular Cell Biology (9), 2008, 255-262.

[Kra2000] U. Krause, *A discrete nonlinear and non-autonomous model of consensus formation*, Communications in Difference Equations, (2000) 227-236.

[LS2022] D. Lear and R. Shvydkoy, Existence and stability of unidirectional flocks in hydrodynamic Euler alignment systems, Analysis & PDE, 15(1) (2022) 175-196.

[Les2020] Trevor M. Leslie, On the Lagrangian Trajectories for the One-Dimensional Euler Alignment Model without Vacuum Velocity, Comptes Rendus Mathmatique 358(4) (2020) 421-433.

[LLST2022] D. Lear, T. Leslie, R. Shvydkoy and E. Tadmor, Geometric structure of mass concentration sets for pressureless Euler alignment systems. Advances in Mathematics, 401, (2022), 108290.

[LS2019] T. Leslie and R. Shvydkoy. On the structure of limiting flocks in hydrodynamic Euler Alignment models. M3AS 29(13), 2019, 2419-2431,

[LT2021] Trevor M. Leslie and Changhui Tan, Sticky particle Cucker-Smale dynamics and the entropic selection principle for the 1D Euler-alignment system arXiv:2108.07715v1 (2021).

[Lev1996] C. D. Levermore, Moment closure hierarchies for kinetic theories, Journal of statistical Physics 83.5-6 (1996): 1021-1065.

[Lio1996] Pierre-Louis Lions Mathematical Topics in Fluid Mechanics, vol. 1. The Clarendon Press, New York, 1996

[LZTM2019] F. Lu, M. Zhong, S. Tang and M. Maggioni, Nonparametric inference of interaction laws in systems of agents from trajectory data, *Proc. Nat. Acad. Sci.*, 116 (29) (2019) 14424-14433.

[MLK2019] Z. Mao, Z. Li and G. Karniadakis, Nonlocal Flocking Dynamics: Learning the Fractional Order of PDEs from Particle Simulations, Comm. Applied Math. and Computation 1, (2019) 597-619.

[Mar2018] I. Markou, Collision-avoiding in the singular Cucker-Smale model with nonlinear velocity couplings, DCDS 38(10) (2018) 5245-5260.

[MRR2013] D.. McCormick, J. Robinson & J. Rodrigo Generalised GagliardoNirenberg Inequalities Using Weak Lebesgue Spaces and BMO, Milan Journal of Mathematics 81 (2013), 265-289.

[MMPZ2019] P. Minakowski, P. B. Mucha, J. Peszek, and E. Zatorska, Singular Cucker-Smale Dynamics, in "Active Particles", Advances in Theory, Models, and Applications, Vol **2** (N. Bellomo, P. Degond and E. Tadmor, eds.), Birkhuser (2019).

[MCEB2015] A. Morin, J.-B. Caussin, C. Eloy, and D Bartolo Collective motion with anticipation: Flocking, spinning, and swarming Phys. Rev. E 91, 012134, 2015.

[MT2014] S. Motsch and E. Tadmor, Heterophilious dynamics enhances consensus, SIAM Review 56(4) (2014) 577-621.

[NP2021] R. Natalini, T. Paul On The Mean Field limit for Cucker-Smale models, ArXiv:2011.12584

[PT2017] A Perna and G Theraulaz *When social behaviour is moulded in clay: on growth and form of social insect nests* Journal of Experimental Biology 220 (1) (2017), 83-91.

[Pes2014] J. Peszek, Existence of piecewise weak solutions of a discrete Cucker-Smale's flocking model with a singular communication weight, J. Diff. Eqs., 257, (2014), 2900-2925.

[Pes2015] J. Peszek, Discrete Cucker-Smale flocking model with a weakly singular weight, SIAM J. Math. Anal., 47(5) (2015) 3671-3686.

[PS2019] D. Poyato and J. Soler, Euler-type equations and commutators in singular and hyperbolic limits of kinetic Cucker-Smale models, M3AS 27(6) (2017) 1089-1152.

[Rey1987] C. Reynolds, Flocks, herds and schools: A distributed behavioral model, ACM SIGGRAPH 21 (1987) 25-34.

[RDW2018] K. W. Rio, G. C. Dachner, and W. H. Warren *Local interactions underlying collective motion in human crowds* Proc. R. Soc. B, **285** (2018) 20180611.

[ST2020a] R. Shu and E. Tadmor, Flocking hydrodynamics with external potentials, Archive Rational Mech. and Anal. 238 (2020) 347-381.

[ST2021] R. Shu and E. Tadmor, Anticipation breeds alignment, Archive Rational Mech. and Anal. 240 (2021) 203-241.

[Shv2019] R. Shvydkoy, Global existence and stability of nearly aligned flocks, J. Dynamics and Differential Eqs. (31) (2019) 2165-2175.

[Shv2021] R. Shvydkoy, Dynamics and Analysis of Alignment Models of Collective Behavior Neas Center Series. Birkhuser/Springer, Cham., 2021.

[Shv2022] R. Shvydkoy, Global hypocoercivity of kinetic Fokker-Planck-Alignment equations Kinetic and Related Models 15(2) (2022) 213-237.

[ST2017a] R. Shvydkoy and E. Tadmor, Eulerian dynamics with a commutator forcing, Transactions of Mathematics and its Applications 1(1) (2017) 1-26.

[ST2017b] R. Shvydkoy and E. Tadmor, Eulerian dynamics with a commutator forcing II: flocking, Discrete and Continuous Dynamical Systems-A 37(11) (2017) 5503-5520.

[ST2018a] R. Shvydkoy and E. Tadmor, Eulerian dynamics with a commutator forcing III. Fractional diffusion of order $0 < \alpha < 1$, Physica D, 376-377 (2018) 131-137.

[ST2020b] R. Shvydkoy and E. Tadmor, Topologically-based fractional diffusion and emergent dynamics with short-range interactions, SIAM J. Math. Anal. 52(6) (2020) 5792-5839.

[Sin2021] M. Sinhuber, K. van der Vaart, Y. Feng, A. M. Reynold and N. T. Ouellette, An equation of state for insect swarms, Sci Rep. (2021) 11: 3773.

[ST2019] D. Slepčev and M. Thorpe, Analysis of p-Laplacian Regularization in Semisupervised Learning, SIMA, 51(3) (2019), 2085-2120.

[Tad2021] E. Tadmor On the mathematics of swarming: emergent behavior in alignment dynamics, Notices of the AMS 68(4) (2021) 493-503.

[Tad2022a] E. Tadmor Emergent behavior in collective dynamics, AMS Josiah Willard Gibbs Lecture, http://www.ams.org/meetings/lectures/meet-gibbs-lect.

[Tad2022b] E. Tadmor Global smooth solutions and emergent behavior of multidimensional Euler alignment system, in preparation (2022).

[TT2014] E. Tadmor and C. Tan. Critical thresholds in flocking hydrodynamics with non-local alignment, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 372(2028):20130401, 22, 2014.

[Tan2020] C. Tan. On the Euler-alignment system with weakly singular communication weights. Nonlinearity, 33(4) (2020), 1907.

[Tan2021] C. Tan, Eulerian dynamics in multidimensions with radial symmetry. SIMA 53(3) (2021) 3040-3071.

[TGCV2021] F. del Teso, D. Gmez-Castro J.-L. Vzquez, Three representations of the fractional p-Laplacian: Semigroup, extension and Balakrishnan formulas Fractional Calculus and Appl. Anal., 24(4), (2021) 966-1002.

[TCGW2020] L. Tong, L. Chen, S. Göttlich and S, Wang. The global classical solution to compressible Euler system with velocity alignment. AIMS Mathematics, 5(6) (2020) 6673-6692.

[VCBCS1995] T. Vicsek, Czirók, E. Ben-Jacob, I. Cohen, and O. Schochet, Novel type of phase transition in a system of self-driven particles, Phys. Rev. Lett. 75 (1995) 1226-1229.

[VZ2012] T. Vicsek and A. Zafeiris, Collective motion, Physics Reprints, 517 (2012) 71-140.

[Vil2003] C. Villani, A review of mathematical topics in collisional kinetic theory, in “Handbook of Fluid Mechanics” v. II (S. Friedlander and D. Serre, eds), Elsevier, 2003, pp. 71-306.

Department of Mathematics and Institute for Physical Sciences & Technology (IPST), University of Maryland, College Park

Email address: tadmor@umd.edu