Review of Basic Plane Geometry

We recall some of the most important results of Euclidean plane geometry. Our treatment of course is very incomplete.

Two "things" in the plane are congruent if they have the same "shape". This is formalized in our first definition:

**Definition 0.1.** Two plane objects are **congruent** if they can be transformed into each other using a finite sequence of rigid motions: translations, rotations and reflections.

1 Angles and Lines

Lines are denoted by either a lowercase letter "a", one capital script letter "A", or using two distinct points "AB". According to our definition of congruency two lines are always congruent.

Angles are denoted by an ordered triplet of points: \( \hat{ABD} \) or \( \hat{DBA} \). Angles are measured by degrees or radians: \( m(\hat{ABD}) = 60^\circ = \frac{\Pi}{3} \). Two angles are congruent if they have the same measure (\( \hat{ABD} \equiv \hat{FBH} \) because \( m(\hat{ABD}) = m(\hat{FBH}) \)).

On the diagram above \( AC \parallel BD \), hence certain angles are congruent:
- \( \hat{FBH} \equiv \hat{ABD} \) (vertical angles)
- \( \hat{EAB} \equiv \hat{ABD} \) (alternate interior angles)
- \( \hat{GAC} \equiv \hat{ABD} \) (corresponding angles)
- \( \hat{FBH} \equiv \hat{GAC} \) (alternate exterior angles)
2 Segments

Segments are denoted by: \([AB]\) (closed segment)
\((AB)\) (half open segment)
\((AB)\) (open segment)

We use absolute value \(|\cdot|\) to denote the length of segments. Two segments are congruent if they have the same length: \([AB] \equiv [CD]\) if \(|AB| = |CD|\).

3 Triangles

Triangles are denoted using an ordered triplet of points: \(ABC_\Delta\). In the above diagramm \(ABC_\Delta\) is congruent to \(EFG_\Delta\) if one of the following holds:

(A.S.A): \([AC] \equiv [FG]\)
\(\hat{BAC} \equiv \hat{EFG}\)
\(\hat{BCA} \equiv \hat{EGF}\)

(S.A.S): \([AB] \equiv [EF]\)
\([BC] \equiv [EG]\)
\(\overrightarrow{ABC} \equiv \overrightarrow{FEG}\)

(S.S.S): \([AB] \equiv [EF]\)
\([BC] \equiv [EG]\)
\([AC] \equiv [FG]\)

Theorem 3.1. Given a triangle \(ABC_\Delta\), we have \(m(\hat{A}) + m(\hat{B}) + m(\hat{C}) = 180^\circ\)
Proof. Let $EF \parallel BC$ s.t. $A \in EF$. Then $\triangle ABC \equiv EAB$ and $\triangle ACB \equiv CAF$ because they are alternate interior angles. As complementary angles add up to $180^\circ$ we have:

$$m(\hat{EAB}) + m(\hat{BAC}) + m(\hat{CAF}) = 180^\circ.$$ 

Putting our findings together it follows that:

$$m(\hat{ABC}) + m(\hat{BAC}) + m(\hat{ACB}) = 180^\circ$$

$$\square$$

### 3.1 "Solving" a triangle

As follows from the congruency criterions, given three different measurements in a triangle one has enough information to find out all sides and angles. The following results can be used in carrying out this process:

**Theorem 3.2.** (Cosine theorem) Given a triangle $\triangle ABC$ we have

$$|BC|^2 = |AB|^2 + |AC|^2 - 2|AB||AC|\cos(\hat{BAC}).$$

**Theorem 3.3.** (Sine theorem) Given a triangle $\triangle ABC$ we have

$$\frac{|BC|}{\sin(A)} = \frac{|AC|}{\sin(B)} = \frac{|AB|}{\sin(C)} = 2R,$$

$R$ is the radius of circumscribed circle (see below).

### 3.2 Special lines in a triangle

There are many different constructions that find a special point or line associated with a triangle, satisfying some unique property. Here we mention some of the most important.

#### 3.2.1 Medians

![Diagram showing the intersection of medians in a triangle]

Figure 1: The intersection of the medians is the centroid (G)
3.2.2 Altitudes

Figure 2: The intersection of the altitudes is the orthocenter (H)

3.2.3 Angle bisectors

Figure 3: The intersection of the angle bisectors is in-center (O)

3.2.4 Perpendicular bisectors

Figure 4: The intersection of the perpendicular bisectors is the circumcenter (O)
3.3 Quantities associated with triangles

Unlike segments and angles, triangles don’t have a measure. But we can associate different quantities to arbitrary triangles.

![Figure 5: AM is an altitude](image)

The perimeter of $\Delta ABC$ is computed as $P_{\Delta ABC} = |AB| + |BC| + |CA|$. There are multiple formulas for the area:

$$\text{Area}_{\Delta ABC} = \frac{|BC||AM|}{2} = \frac{|AB||AC| \sin(\hat{BAC})}{2} = \sqrt{p(p - |AB|)(p - |AC|)(p - |BC|)},$$

where $p = \frac{P_{\Delta ABC}}{2}$ is the half-perimeter.

3.4 Special triangles

3.4.1 Right triangles

![Figure 6: $m(\hat{ABC}) = 90^\circ$](image)

Trigonometry is helpful in solving a right triangle:

$$\cos(\hat{A}) = \frac{|AB|}{|AC|}, \quad \sin(\hat{A}) = \frac{|BC|}{|AC|}, \quad \tan(\hat{A}) = \frac{|BC|}{|AB|}.$$
Theorem 3.4. (Pythagorean theorem) Given a triangle $\Delta ABC$ we have $m(\hat{ABC}) = 90^\circ$ if and only if:

$$|AC|^2 = |AB|^2 + |BC|^2.$$ 

3.4.2 Isosceles triangles

![Figure 7: $|AB| = |BC|$](image)

Theorem 3.5. Given a triangle $\Delta ABC$ we have $|AB| = |AC|$ if and only if:

$\triangle ABC \equiv \triangle ACB$.

Before we give the proof of this result (following Euclid), we mention a basic technique that will be used a lot. To conclude that two angles (or segments) are congruent, most of the time it is easier to conclude that two triangles containing these angles (or segments) are congruent.

Proof. Suppose that $|AB| = |AC|$. With the above advice in our pocket we prove first that $\Delta ABC_\Delta$ and $\Delta ACB_\Delta$ are congruent triangles (note the importance of ordered vertices!). This follows from (SAS) as $[AB] \equiv [AC]$, $\overrightarrow{BAC} \equiv \overrightarrow{CAB}$ and $[AC] \equiv [AB]$. As $\Delta ABC_\Delta \equiv \Delta ACB_\Delta$, it follows that $\overrightarrow{ABC} \equiv \overrightarrow{ACB}$, finishing one direction.

For the other direction, we assume that $\overrightarrow{ABC} \equiv \overrightarrow{ACB}$ and we want to prove that $|AB| = |AC|$. By the same analogy as above, this will follow if we prove that $\Delta ABC_\Delta \equiv \Delta ACB_\Delta$. But these triangles are congruent using (ASA) now: $\overrightarrow{ABC} \equiv \overrightarrow{ACB}$, $|BC| = |BC|$ and $\overrightarrow{ACB} \equiv \overrightarrow{ABC}$. 

As illustrated in the above figure, in an isosceles triangle $\Delta ABC$ we have the following equivalences:

$[AA']$ is a median $\iff [AA']$ is a bisector $\iff [AA']$ is an altitude $\iff [AA']$ is a perp. bisector
3.4.3 Equilateral triangles

![Figure 8: |AB| = |BC| = |AC|](image)

In an equilateral triangle we have $m(\overline{ABC}) = m(\overline{DAC}) = m(\overline{ACB}) = 60^\circ$. There is a special formula for the area: $\text{Area}_{\triangle ABC} = \frac{|AB|^2\sqrt{3}}{4}$.

3.5 Similarity

Two things are similar if they are "proportional". This is formalized in the next definition:

**Definition 3.6.** Two plane objects are similar if they coincide after applying a sequence of translations, rotations, reflections, homotheties (zoom in and zoom out).

![Triangles ABC and A'B'C' are similar](image)

Triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar ($\triangle ABC \approx \triangle A'B'C'$) if one of the following holds

- (A.A.A): $\hat{A} \equiv \hat{A'}$
  $\hat{B} \equiv \hat{B'}$
  $\hat{C} \equiv \hat{C'}$

- (S.A.S): $\hat{A} \equiv \hat{A'}$
  \[
  \frac{|AB|}{|A'B'|} = \frac{|AC|}{|A'C'|} = \frac{|BC|}{|B'C'|}
  \]

- (S.S.S): $\frac{|AB|}{|A'B'|} = \frac{|AC|}{|A'C'|} = \frac{|BC|}{|B'C'|}$
Theorem 3.7. (Thales’ theorem) Given a triangle $ABC$ with $M \in (AB), N \in (AC)$ we have

$$MN \parallel BC \iff \frac{|AM|}{|AB|} = \frac{|AN|}{|AC|}.$$ 

Given similar triangles $ABC \approx A'B'C'$, the quantity $\alpha = \frac{|AB|}{|A'B'|}$ is called the similarity ratio.

Theorem 3.8. If $ABC \approx A'B'C'$ with similarity ratio $\alpha = \frac{|AB|}{|A'B'|}$, then $\text{Area}_{ABC} = \text{Area}_{A'B'C'} \cdot \alpha^2$.

4 Circles

A circle, is the set of points in the plane that is equidistant from a fixed point. Two circles are congruent if their radius is equal. We denote a circle with center $O$ and radius $r$ by $C(O, r)$. The circumference of such circle is $2\pi r$ and its area is $\pi r^2$. 
4.1 Angles and circles

Inscribed angles corresponding to the same arc are congruent: $m(\overarc{BAC}) = m(\overarc{BA'C})$

The measure of an inscribed angle is half the measure of the corresponding interior angle: $m(\overarc{BOC}) = 2m(\overarc{BAC})$. 
4.2 Tangent to a circle

Given a point \( A \) on the circle \( C(O, |OA|) \), the line \( AB \) is tangent to \( C(O, |OA|) \) if \( C(O, |OA|) \cap AB = \{A\} \). Whether a segment is tangent to a circle can be decided rather easily according to our next theorem:

**Theorem 4.1.** Suppose \( A \) is a point on the circle \( C(O, |OA|) \). Then \( AB \) is tangent to \( C(O, |OA|) \) if and only if \( AB \) is perpendicular to \( OA \) (notation: \( AB \perp OA \)).

5 Problems

**Problem 1** (angle bisector theorem). Let \( ABC \) triangle, \( A' \in (BC) \) such that \( AA' \) is an angle bisector. Prove that

\[
\frac{|BA'|}{|CA'|} = \frac{|AB|}{|AC|}.
\]

*Hint.* Use the sine theorem to find a formula for the ratios \( \frac{|BA'|}{|AB|} : \frac{|CA'|}{|AC|} \). Now use basic trig identities to finish the argument.

When trying to prove that three lines are concurrent there is only one "strategy". Pick two lines and show that the their intersection point is on the third line.

**Problem 2** (perpendicular bisectors are concurrent). Let \( ABC \) a triangle. Show that the perpendicular bisectors are concurrent. Also argue that the intersection point of the angle bisectors is the center of the circumscribed circle.
Solution. Let $B'O$ and $A'O$ be perpendicular bisectors of $AC$ and $BC$ in the above figure. Let $C'$ be the midpoint of $[AB]$. To conclude that $O$ is on the perpendicular bisector of $AB$ we have to show that $OC'$ is perpendicular to $AB$.

We first show that $|OA| = |OC'|$. This is true because $OB'A_\Delta \equiv OB'C_\Delta$ (A.S.A. $|OB'| = |OB'|$, $m(\overline{OB'A}) = m(\overline{OB'C}) = 90^\circ$, $|AB'| = |B'C|$).

One can similarly conclude that $|OB| = |OC'|$ (because $A'B_\Delta \equiv OA'C_\Delta$ A.S.A.).

We have proved that $|OB| = |OC'| = |OA|$. This implies that $OAB_\Delta$ is an isosceles triangle. $OC'$ is a median, but in an isosceles triangle the median of the base is also an altitude, and this concludes that $OC' \perp AB$.

From $|OB| = |OC|= |OA|$ it follows that the points $A,B,C$ are on the circle $C(O, |OA|)$. 

Problem 3 (altitudes are concurrent). Let $ABC_\Delta$ a triangle. Show that the altitudes are concurrent.

Solution. We take up the altitudes $AA'$, $BB'$, $CC'$ according to the figure above. Also the points $P, Q, R$ are constructed so that $PR \parallel AC$, $RQ \parallel BC$, $AB \parallel PQ$ and $A \in \overline{(RQ)}$, $B \in \overline{(PR)}$, $C \in \overline{(PQ)}$.

Now $CBRA$ is a parallelogram because $BC \parallel RA$ and $PR \parallel AC$. As in parallelogram the sides facing eachother are congruent we have $|BC| = |RA|$. One can similarly conclude that $|BC| = |AQ|$ ($CBAQ$ is a parallelogram). Hence $|RA| = |AQ|$, which implies that $AA'$ is the perpendicular bisector of $[RQ]$.

The same reasoning implies that $BB'$, $CC'$ is the perpendicular bisector of $[RP], [PQ]$ respectively. By the previous problem $AA', BB', CC'$ are concurrent, and their intersection is the circum-center of $PRQ_\Delta$. 

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