Spherical Geometry
MATH430

In these notes we summarize some results about the geometry of the sphere to complement the textbook. Most notions we had on the plane (points, lines, angles, triangles etc.) make sense in spherical geometry, but one has to be careful about defining them. Some classical theorems from the plane however are no longer true in spherical geometry. For example, the sum of the angles of a triangle on a sphere is always greater than $180^\circ$. Also there is no notion of parallelism.

We will also prove Euler’s theorem which says that in a convex polyhedron, if you count the number of its vertices, subtract the number of its edges, and add the number of its faces you will always get 2. Perhaps surprisingly, in our proof of Euler’s theorem we will use spherical geometry.

The main property of segments on the plane is that any segment $[AB]$ is the shortest path between the points $A, B$. In general this is how one defines segments/lines in every geometry (sometimes called geodesic lines, or geodesics). Here we will do this the other way around. We will ”declare” straight lines on the sphere to be great circles, i.e. circles one obtains after cutting the sphere with planes passing through the origin.

If we want to geometrically construct the straight segment between two points $A, B$ on the sphere, we just need to cut the sphere with the plane $ABO$:

Now that we know what the straight segments is between $A$ and $B$, how should we compute the (spherical) distance between $A$ and $B$? As it is geometrically justified, this should be the length of the arc $AB$ as shown in the figure above. This is computed by the following formula:

$$d_s(A, B) = |\widehat{AB}| = m(\widehat{AOB})R,$$
where $R$ is the radius of the sphere. Here we already notice something that has no analog in the plane: one computes distances on the sphere by measuring central angles.

Suppose $C$ is another point on the sphere and we would like to define the spherical angle $\widehat{ABC}$ formed by the straight line segments $\widehat{AB}$ and $\widehat{BC}$. Remember that we introduced straight lines by cutting the sphere with planes passing through the origin. By definition, the \textbf{spherical angle $\widehat{ABC}$} is the angle formed by the defining planes $AOB$ and $AOC$ of the arcs $\widehat{AB}$ and $\widehat{BC}$, as shown on the above figure.

For any two points there is a unique great circle passing through them, unless they are opposite. If they are opposite (i.e. belong to a diameter) there are infinitely many such great circles.

\textbf{Angles of triangles and polygons}

Once we have defined the straight lines on a sphere, the definition of a triangle(polygon) carries over from the plane: the vertices are points on a sphere and the sides are arc segments of great circles.

The polygon with smallest number of vertices in the plane is a triangle. On a sphere it is a two angle or \textbf{spherical sector}. On a sphere if you issue two straight lines (remember that straight lines for us are arcs of great circles) from a point $A$ they will intersect at another point $A'$, the point opposite to $A$, as illustrated below:

![Diagram of a spherical sector](image)

According to the next result, the area of a spherical sector is not hard to compute:

\textbf{Theorem 1.} Let $\text{Sect}(AA')$ be a spherical sector as described in the above figure. Let $\alpha$ be the angle between the segments defining the sector. Then the area of the sector is equal to $2\alpha R^2$, where $R$ is the radius of the sphere.

\textit{Proof.} Clearly, the area of the sector (we denote it $A_\alpha$) is directly proportional to the angle $\alpha$. If $\alpha = 2\pi$ then the sector area would be equal to the area of the whole sphere which is $A_{2\pi} = 4\pi R^2$. By proportionality we obtain that $A_\alpha = 2\alpha R^2$. $\square$

Now we are able to calculate the area of a spherical triangle:

\textbf{Theorem 2.} Let $ABC$ be a spherical triangle with angles $\alpha, \beta, \gamma$. Then

$$\alpha + \beta + \gamma = \pi + \frac{A_{ABC}}{R^2}$$

where $A_{ABC}$ is the area of the triangle and $R$ is the radius of the sphere. In particular, the sum of the angles is always greater than $\pi$. 

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**Proof.** If we continue the sides of the triangle they will meet at other three points \( A', B', C' \) opposite to \( A, B, C \) as shown on the figure below:

As we argue now, the triangle \( A'B'C'_\Delta \) is congruent to the triangle \( ABC \) by \((SSS)\). Here we note that the congruence criteria for triangles in the plane carries over into spherical geometry. We only have to show that \(|AC| = |A'C'|\), as \(|AB| = |A'B'|\) and \(|BC| = |B'C'|\) will follow similarly.

To see that \(|AC| = |A'C'|\) notice that these arcs are cut out from their defining great circle by two planes that go through \( O \). Hence we have the following figure:

Consequently \(|AC| = |A'C'|\), because they are arcs that correspond to the same central angle.

Now we observe that the surface of the sphere is covered by \( ABC_\Delta \) and \( A'B'C'_\Delta \) and three non-overlapping spherical sectors, each corresponding to an external angle of \( ABC_\Delta \) as suggested by first figure of the proof. As these segments are not overlapping, we can write:

\[
A_{ABC_\Delta} + A_{A'B'C'_\Delta} + A_{\pi - \alpha} + A_{\pi - \beta} + A_{\pi - \gamma} = 4\pi R^2.
\]

Since \( A_{ABC_\Delta} = A_{A'B'C'_\Delta} \), by the previous theorem we can write:

\[
2A_{ABC} = 4\pi R^2 - 2(\pi - \alpha)R^2 - 2(\pi - \beta)R^2 - 2(\pi - \gamma)R^2,
\]

thus after dividing with \( 2R^2 \) we obtain:

\[
\frac{A_{ABC}}{R^2} = -\pi + \alpha + \beta + \gamma,
\]

which is what we wanted to prove.

We remark the corresponding result about external angles in a triangle. We will generalize this shortly in order to prove Euler’s theorem:
Remark 1. Let $\hat{\alpha} = \pi - \alpha, \hat{\beta} = \pi - \beta, \hat{\gamma} = \pi - \gamma$ be the external angles of a spherical triangle $ABC$. Then

$$\hat{\alpha} + \hat{\beta} + \hat{\gamma} = 2\pi - \frac{A_{ABC}}{R^2}.$$ 

This follows directly from the previous theorem.

Theorem 3. Let $P = A_1A_2\ldots A_n$ be a spherical polygon. Let $\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n$ be its external angles. Then

$$\hat{\alpha}_1 + \hat{\alpha}_2 + \ldots + \hat{\alpha}_n = 2\pi - \frac{A_P}{R^2},$$

where $A_P$ is the area of $P$ and $R$ is the radius of the sphere.

Proof. The proof is by induction on $n$. By the previous remark the theorem holds for $n = 3$. Consider the polygon $P' = A_1A_2\ldots A_{n-1}$ obtained from $P$ by removing the last vertex $A_n$. Then external angles $\hat{\alpha}'_i$ of $P'$ are the same as of $P$ except for $\hat{\alpha}'_1$ and $\hat{\alpha}'_{n-1}$.

Looking at the figure below, it is easy to see that

$$\hat{\alpha}'_1 = \hat{\alpha}_1 + \overrightarrow{A_1}, \quad \hat{\alpha}'_i = \hat{\alpha}_i \ (2 \leq i \leq n - 2), \quad \hat{\alpha}'_{n-1} = \hat{\alpha}_{n-1} + \overrightarrow{A_{n-1}},$$

where $A_1$ and $\overrightarrow{A_{n-1}}$ are the corresponding angles of the triangle $A_1A_{n-1}A_n$:

![Diagram](image)

By the inductive assumption

$$\hat{\alpha}'_1 + \ldots + \hat{\alpha}'_{n-1} = 2\pi - \frac{A_{P'}}{R^2}.$$ 

Therefore

$$(\hat{\alpha}_1 + \overrightarrow{A_1}) + \ldots + (\hat{\alpha}_{n-1} + \overrightarrow{A_{n-1}}) + \hat{\alpha}_n = 2\pi - \frac{A_{P'}}{R^2} + \hat{\alpha}_n.$$ 

Since $\hat{\alpha}_n = \pi - \overrightarrow{A_n}$ we get

$$\hat{\alpha}_1 + \ldots + \hat{\alpha}_n = 2\pi - \frac{A_{P'}}{R^2} - (\overrightarrow{A_1} + \overrightarrow{A_{n-1}} + \overrightarrow{A_n} - \pi).$$ 

By the previous theorem we have $\overrightarrow{A_1} + \overrightarrow{A_{n-1}} + \overrightarrow{A_n} = \pi + A_{A_1A_{n-1}A_n}/R^2$, thus we obtain

$$\hat{\alpha}_1 + \ldots + \hat{\alpha}_n = 2\pi - \frac{A_{P'}}{R^2} - \frac{A_{A_1A_{n-1}A_n}}{R^2} = 2\pi - \frac{A_P}{R^2}.$$

$\square$
A polyhedron (sometimes called polytope) is a solid in three dimensions with flat faces, straight edges and sharp corners or vertices. A polyhedron is said to be convex if the line segment joining any two points of the polyhedron is contained in the interior or surface. Examples of convex polyhedra are the platonic solids: tetrahedron, cube, octahedron, dodecahedron, icosahedron.

We will prove Euler’s theorem about convex polyhedra:

**Theorem 4.** Let $\Delta$ be a convex polyhedron. Let $V, E$ and $F$ denote the number of its vertices, edges and faces respectively. Then

$$V - E + F = 2.$$ 

**Proof.** First we place the polyhedron $\Delta$ inside a sphere centered at the origin. Project $\Delta$ from the center $O$ on the surface of the sphere. You can think of this as if the polyhedron is made of rubber and you pump air into it until it takes the shape of the sphere. See also the diagram below for the case of the “spherical cube”:

Now the sphere is covered by $F$ spherical polygons $P_1, ..., P_F$ that come from the faces of $\Delta$. Thus the sum of the areas of these polygons is $4\pi$. (We assume that the radius of the sphere is 1.) Using the previous Theorem we obtain

$$4\pi = A_{P_1} + ... + A_{P_F} = 2\pi F - \sum \hat{\alpha}_i,$$

where on the right hand side we have the sum of all external angles of all polygons on the sphere. Reorganizing terms, we arrive at:

$$2\pi F - 4\pi = \sum \hat{\alpha}_i. \quad (1)$$

Let’s take a look at the sum on the right hand side from a different point of view. At each vertex $v$ the sum of internal angles is $2\pi$:

$$\alpha_1 + ... + \alpha_{E_v} = 2\pi,$$

where $E_v$ denotes the number of edges meeting at the vertex $v$. But $\alpha_i = \pi - \hat{\alpha}_i$, thus for each vertex $v$

$$\hat{\alpha}_1 + ... + \hat{\alpha}_{E_v} = \pi E_v - 2\pi.$$

Therefore, the sum of all external angles of all polygons can be expressed:

$$\sum \hat{\alpha}_i = \sum_{v \text{ is a vertex of } \Delta} (\pi E_v - 2\pi) = \pi(\sum_{v \text{ is a vertex of } \Delta} E_v) - 2\pi V. \quad (2)$$
Since every edge connects exactly two vertices $\sum_v E_v = 2E$. Finally, from (1) and (2) we obtain

$$2\pi F - 4\pi = 2\pi E - 2\pi V.$$ 

After dividing by $2\pi$ we arrive at:

$$V - E + F = 2.$$

### Solving the spherical triangle

As in the case of the Euclidean plane, given enough information about a triangle we want to develop tools that allow to determine everything about the triangle in question. As expected, for this we need to find analogs of the cosine and sine theorems for the sphere.

Before we do this let us recall how one projects vectors in space. Given $u, v \in \mathbb{R}^3$, the projection of $v$ onto $u$ (denoted by $\text{proj}_u v \in \mathbb{R}^3$) is determined by two properties: $\text{proj}_u v$ is parallel to $u$ and the vector $\text{proj}_u v - v$ is perpendicular to $u$. This is illustrated in the figure below:

![Projection of a vector](image.png)

We recall the following formula that allows to compute $\text{proj}_u v$:

$$\text{proj}_u v = \frac{\langle u, v \rangle}{|u|^2} u.$$ 

Indeed, it is clear that with this choice that $\text{proj}_u v \parallel u$ and $(\text{proj}_u v - v) \perp u$ because:

$$\langle \text{proj}_u v - v, u \rangle = \frac{\langle u, v \rangle}{|u|^2} |u|^2 - \langle v, u \rangle = \frac{|u|^2}{|u|^2} \langle u, u \rangle - \langle v, u \rangle = \langle v, u \rangle - \langle v, u \rangle = 0.$$

In solving spherical triangles the following theorem is the most important result:

**Theorem 5** (Spherical law of cosines). Suppose $S(0, 1)$ is the sphere with radius $1$ and center at $0$. Suppose $ABC$ is a spherical triangle with sides $a, b, c$. Then we have

$$\cos(c) = \cos(a) \cos(b) + \sin(a) \sin(b) \cos(\hat{ACB}).$$
Proof. Let $\overrightarrow{OA}, \overrightarrow{OB}$, and $\overrightarrow{OC}$ denote the unit vectors from the center of the sphere to those corners of the triangle. Then, the lengths (angles) of the sides are given by the inner/dot products:

$$\cos(a) = \cos(a) ||\overrightarrow{OB}||\overrightarrow{OC} = \langle \overrightarrow{OB}, \overrightarrow{OC} \rangle,$$

$$\cos(b) = \langle \overrightarrow{OA}, \overrightarrow{OC} \rangle,$$

$$\cos(c) = \langle \overrightarrow{OA}, \overrightarrow{OB} \rangle.$$ 

The angle $\hat{ACB}$ is the angle between the planes $AOC$ and $AOB$. Hence to find this angle we need the tangent vectors $t_a$ and $t_b$, pointing from $\overrightarrow{OC}$, along the directions of the sides $AC$ and $BC$ respectively. The vector $t_a$ is the unit vector perpendicular to $\overrightarrow{OC}$ in the $AOC$ plane, whose direction is given by $\overrightarrow{OA} - \text{proj}_{\overrightarrow{OC}} \overrightarrow{OA}$. This means:

$$t_a = \frac{\overrightarrow{OA} - \text{proj}_{\overrightarrow{OC}} \overrightarrow{OA}}{|\overrightarrow{OA} - \text{proj}_{\overrightarrow{OC}} \overrightarrow{OA}|}.$$ 

By the formula for projections we have:

$$\text{proj}_{\overrightarrow{OC}} \overrightarrow{OA} = \langle \overrightarrow{OA}, \overrightarrow{OC} \rangle \overrightarrow{OC} = \cos(b) \overrightarrow{OC},$$

and also

$$|\overrightarrow{OA} - \text{proj}_{\overrightarrow{OC}} \overrightarrow{OA}|^2 = \langle \overrightarrow{OA} - \text{proj}_{\overrightarrow{OC}} \overrightarrow{OA}, \overrightarrow{OA} - \text{proj}_{\overrightarrow{OC}} \overrightarrow{OA} \rangle$$

$$= \langle \overrightarrow{OA} - \cos(b) \overrightarrow{OC}, \overrightarrow{OA} - \cos(b) \overrightarrow{OC} \rangle$$

$$= \langle \overrightarrow{OA}, \overrightarrow{OA} \rangle - 2 \cos(b) \langle \overrightarrow{OA}, \overrightarrow{OC} \rangle + \cos^2(b) \langle \overrightarrow{OC}, \overrightarrow{OC} \rangle$$

$$= 1 - \cos^2(b) = \sin^2(b).$$

Putting the last three identities together we obtain

$$t_a = \frac{\overrightarrow{OA} - \cos(b) \overrightarrow{OC}}{\sin(b)}.$$

Similarly,

$$t_b = \frac{\overrightarrow{OB} - \cos(a) \overrightarrow{OC}}{\sin(a)}.$$

Then, a small computation similar to what we did above gives that the angle $\hat{ACB}$ is given by:

$$\cos(\hat{ACB}) = \langle t_a, t_b \rangle = \frac{\cos(c) - \cos(a) \cos(b)}{\sin(a) \sin(b)}$$

from which the theorem immediately follows. \hfill \Box
An immediate corollary of this result is the spherical Pythagorean theorem:

**Corollary 1 (Spherical Pythagorean theorem).** Suppose $ABC\Delta$ is a spherical right triangle on the unit sphere $ABC\Delta$ with $m(\widehat{ACB}) = 90^\circ$. Then we have
\[
cos(c) = cos(a) \cos(b).
\]

A less obvious consequence of the spherical cosine theorem is the following

**Theorem 6 (Spherical trig identites).** Suppose $ABC\Delta$ is a spherical right triangle on the unit sphere with $m(\widehat{ACB}) = 90^\circ$. Then we have
\[
\sin(\widehat{ABC}) = \sin(b) \sin(c),
\]
\[
\sin(\widehat{BAC}) = \sin(a) \sin(c).
\]

**Proof.** We only prove the first formula. The second formula is proved similarly. As $m(\widehat{ABC}) = 90^\circ$, the Pythagorean formula says that
\[
cos(c) = cos(a) \cos(b).
\]

We write up the cosine theorem for the angle $\widehat{ABC}$:
\[
\cos(b) = cos(a) \cos(c) + \sin(a) \sin(c) \cos(\widehat{ABC}).
\]

From the first identity we have $\cos(a) = \cos(c) / \cos(b)$ and $\sin(a) = \sqrt{1 - \cos^2(c) / \cos^2(b)}$. We plug this into to the second identity to obtain:
\[
\cos(b) = \frac{\cos^2(c)}{\cos(b)} + \sin(c) \sqrt{1 - \frac{\cos^2(c)}{\cos^2(b)}} \cos(\widehat{ABC}).
\]

After reorganizing terms we obtain:
\[
\cos(\widehat{ABC}) = \frac{\cos(b)}{\sin(c)} \sqrt{1 - \frac{\cos^2(c)}{\cos^2(b)}} = \sqrt{\frac{\cos^2(b) - \cos^2(c)}{\sin^2(c)}} \]
\[
= \sqrt{\frac{\sin^2(c) - \sin^2(b)}{\sin^2(c)}} = \sqrt{1 - \frac{\sin^2(b)}{\sin^2(c)}}.
\]

Using the formula $\sin^2(\widehat{ABC}) + \cos^2(\widehat{ABC}) = 1$ we obtain that
\[
\sin(\widehat{ABC}) = \frac{\sin(b)}{\sin(c)}.
\]

Finally we prove the spherical version of the sine theorem:
Theorem 7 (Spherical sine theorem). Suppose $ABC_\Delta$ is a spherical triangle on $S(0, 1)$. Then
\[
\frac{\sin \hat{ABC}}{\sin(b)} = \frac{\sin \hat{BAC}}{\sin(a)} = \frac{\sin \hat{ACB}}{\sin(c)}. 
\]

Proof. The proof is an imitation of the argument yielding the regular sine theorem from the plane. Draw the spherical altitude $AH$ in the triangle $ABC_\Delta$. As there exists a plane passing through $O$ (the center of the sphere) perpendicular to the plane $BOC$, the altitude $AH$ indeed exists.

Then $ABH_\Delta$ is a right triangle so we can apply the spherical trig identity to conclude:
\[
\sin(\hat{ABC}) = \frac{\sin(|AH|)}{\sin(|AB|)}. 
\]
Similarly, $ACH_\Delta$ is a right triangle hence:
\[
\sin(\hat{ACB}) = \frac{\sin(|AH|)}{\sin(|AC|)}. 
\]
After expressing $\sin(|AH|)$ two different ways using the above identities we obtain
\[
\sin(\hat{ACB})\sin(|AC|) = \sin(\hat{ABC})\sin(|AB|),
\]
hence:
\[
\frac{\sin(\hat{ACB})}{\sin(c)} = \frac{\sin(\hat{ABC})}{\sin(b)}.
\]
The other equalities of the theorem are proved similarly.

Isometries of the Sphere

In this section we argue that the three reflections theorem of the plane holds for the sphere as well: every isometry of the sphere can be written as a composition of one, two or three isometries. A more detailed treatment of isometries of the sphere can be found in Sections 7.4 and 7.5 of the textbook.

Given two points $A, B$ on the unit sphere $S(0, 1)$ we introduced the spherical distance $d_s(A, B) = m(\widehat{AOB})$, which is nothing but the length of the arc $[\widehat{AB}]$. An isometry of the unit sphere is just a function $F : S(0, 1) \to S(0, 1)$ that maintains distance:
\[
d_s(F(A), F(B)) = d_s(A, B), \text{ for all } A, B \in S(0, 1).
\]
It turns out there is another natural distance that one can introduce on the sphere different from the spherical distance. One can look at the sphere as the set of points $P(x, y, z)$ in $\mathbb{R}^3$ that satisfy the identity

$$x^2 + y^2 + z^2 = 1.$$ 

Hence we can think of $A, B \in \mathbb{S}(0, 1) \subset \mathbb{R}^3$ as points having coordinates $(A_1, A_2, A_3)$, $(B_1, B_2, B_3)$. As such, we can introduce the **chordal distance**:

$$d_c(A, B) = \sqrt{(A_1 - B_1)^2 + (A_2 - B_2)^2 + (A_3 - B_3)^2}.$$ 

which is just the length of the straight segment $[AB]$ in the space $\mathbb{R}^3$.

**Theorem 8.** Suppose $A, B \in \mathbb{S}(0, 1)$ then the following formulas hold between the chordal and spherical distances:

$$d_c(A, B) = 2 \sin \left( \frac{d_s(A, B)}{2} \right),$$

$$d_s(A, B) = 2 \arcsin \left( \frac{d_c(A, B)}{2} \right).$$

**Proof.** As detailed above, $d_s(A, B)$ is the length of the arc $[AB]$ and $d_c(A, B)$ is the length of the straight segment $[AB]$, as illustrated on the above figure.

As suggested in the figure below, we draw the altitude $[OH]$ in the isosceles triangle $\triangle AOB$.

As in an isosceles triangle altitudes are also medians and angle bisectors as well, we obtain:

$$\sin(\widehat{AOH}) = \frac{|AH|}{|AO|} = |AH|,$$

as $|AO| = 1$ (the radius of the sphere is 1). Because $|AH| = d_c(A, B)/2$ and $m(\widehat{AOH}) = d_s(A, B)/2$ the first identity of the theorem follows.

As $\arcsin$ is the inverse of $\sin$, the second identity of the theorem is easily seen to follow from the first. 

\[ \square \]
An important corollary of this result is the following:

**Corollary 2.** Given $A, B \in S(0, 1)$ the set of equidistant points from $A$ and $B$ is the great circle that prependiculary bisects the spherical segment $[AB]$.

**Proof.** Suppose $A, B \in S(0, 1)$ have coordinates $(A_1, A_2, A_3)$ and $(B_1, B_2, B_3)$. Let us forget for a second that these points lie on the sphere. The set of equidistant points in $\mathbb{R}^3$ from $A$ and $B$ is the plane $P$ perpendicular to the straight segment $[AB]$. As the origin $O$ is equidistant from $A$ and $B$ it is also contained in $P$. Hence $P$ cuts out a great circle $C = P \cap S(0, 1)$ that is perpendicular to the spherical segment $AB$. If $D \in C$ then by definition of $P$ we have $$d_s(A, D) = d_s(B, D).$$

By the second formula in the previous theorem this implies that $$d_s(A, D) = d_s(B, D).$$

Hence $D$ is equidistant from $A$ and $B$ with respect to the spherical distance $d_s$. This implies that points of $C$ are equidistant from $A$ and $B$.

As all of the implications in our argument where equivalences it follows that all points equidistant from $A$ and $B$ lie on the great circle $C$. □

**Theorem 9.** Suppose $A, B, C$ are points on $S(0, 1)$ that don't lie on a great circle. Then the distances from $A, B, C$ uniquely determine any point on $S(0, 1)$

**Proof.** We prove this result using contraposition. Suppose $D, D' \in S(0, 1)$ are distinct points such that $$d_s(A, D) = d_s(A, D'),$$ $$d_s(B, D) = d_s(B, D'),$$ $$d_s(C, D) = d_s(C, D').$$

According to the previous corollary, this means that $A, B, C$ lie on the equidistant great circle between $D, D'$ which is a contradiction. □

Given a plane $P \subset \mathbb{R}^3$ that passes through the origin, recall that reflection across this plane $r_P : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is an isometry of $\mathbb{R}^3$. Examples of reflections are reflections to the coordinate axes:

$$R_{xy}(a, b, c) = (a, b, -c),$$
$$R_{yz}(a, b, c) = (-a, b, c),$$
$$R_{xz}(a, b, c) = (a, -b, c).$$

**Spherical reflections** are just restrictions of reflections of space (described above) to the sphere $S(0, 1)$. Using the previous results, one can easily prove that these are indeed isometries of the sphere with respect to the spherical distance $d_s$.

One can similarly introduce spherical rotations as well but we will not spend time on them here and we refer to Section 7.5 of the textbook for more information. As it is the case in the plane, it turns out that every spherical isometry can be constructed using only reflections:

**Theorem 10.** Every isometry $F : S(0, 1) \rightarrow S(0, 1)$ of the sphere is a composition of one,two or three spherical reflections.
Proof. The proof follows very closely the argument of the three reflections theorem from the plane. All the ingredients needed are present as they where in case of theorem for plane isometries. Because of this we only sketch the ideas and leave the reader to fill in the obvious details.

Suppose \( A, B, C \in \mathbb{S}(0, 1) \) don’t lie on the same great circle. According to the previous theorem the isometry \( F \) is uniquely determined by the values \( F(A), F(B), F(C) \in \mathbb{S}(0, 1) \). Hence to finish the proof, we just need to find reflections \( r_{P_1}, r_{P_2}, r_{P_3} : \mathbb{S}(0, 1) \to \mathbb{S}(0, 1) \) such that

\[
\begin{align*}
    r_{P_1}(r_{P_2}(r_{P_3}(A))) &= F(A), \\
    r_{P_1}(r_{P_2}(r_{P_3}(B))) &= F(B), \\
    r_{P_1}(r_{P_2}(r_{P_3}(C))) &= F(C).
\end{align*}
\]

As in the case of the plane, this is enough to conclude that \( F = r_{P_1} \circ r_{P_2} \circ r_{P_3} \). These reflections are constructed exactly the same way as they where in the planar case and we encourage the reader to verify that all the steps of the construction go through in the spherical case as well. \( \square \)